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SECOND ORDER HANKEL DETERMINANTS FOR CLASS OF BOUNDED TURNING FUNCTIONS DEFINED BY SALAGEAN DIFFERENTIAL OPERATOR

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Abstract. In this paper, a brief study of certain properties of bounded turning functions is carried out. Furthermore, bound to the famous Fekete - Szego functional $H_2(1) = |a_3 - ta_2^2|$, with t real and the Second Hankel Determinant $H_2(2) = |a_2a_4 - a_3^2|$ for functions of bounded turning of order β associated with Salagean differential operator are obtained using a succinct mathematical approach.

1. Introduction

Let $D \subset C, f : D \longrightarrow C$ holomorphic. Let $U = \{z \in C : |z| < 1\}$ be the open unit disk and A denote the class of holomorphic functions:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + a_2 z^2 + a_3 z^3 + \dots, \tag{1}$$

normalized by $f(0) = 0$ and $f'(0) = 1$.

Also, consider the class P of Caratheodory functions:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = 1 + c_1 z + c_2 z^2 + \dots \tag{2}$$

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defined in $U = \{z \in C : |z| < 1\}$ which are also holomorphic with $Rep(z) > 0$. The $q-th$ Hankel determinant of f for $q \geq 1$ and $n \geq 1$ is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1)$$

This determinant has been investigated by several authors in the literature ranging from rate of growth of $H_q(n)$ as $n \to \infty$ for the function S with a bounded boundary [1], to the determination of precise bounds on $H_q(n)$ for specific q and n for some favored classes of functions [4, 5].

In this present work, we consider the 2^{nd} - order Hankel determinant denoted by $H_q(n)$ in the cases of $q = 2$, $n = 1$ and $q = 2$, $n = 2$, given by

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1a_3 - a_2^2$$

For $f(z) \in A$ and choosing $a_1 = 1$ such that, the following is obtained

$$H_2(1) = a_3 - a_2^2$$

By applying triangle inequality, we arrived at

$$|H_2(1)| \leq \left|a_3 - a_2^2\right| \tag{3}$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2| \tag{4}$$

We can observe that the right hand side of the inequality in (3) is the well - known Fekete-Szego functional for the second Hankel determinant $H_2(1) = \left|a_3 - a_2^2\right|$. Fekete-Szego further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ with μ real and $f \in S$. For R (the class of bounded turning functions), best possible (sharp) bound $2/3$ was given in [2] (with R corresponding to $n = \alpha = 1, \beta = 0$ in the class of function $T_n^\alpha(\beta)$ studied) while for S^* (Class of Starlike functions) and C (Class of convex functions), best possible (sharp) bounds 1 and $1/3$, respectively, were reported in [6]. Further, best possible (sharp) bounds for the functional $|a_2a_4 - a_3^2|$ in (4) were report for various subclasses of univalent and multivalent holomorphic functions by several authors in the literature.

In this paper, our focus is on finding the bounds for the functional $\left|a_3 - ta_2^2\right|$, with t real and $\left|a_2a_4 - a_3^2\right|$ for $BT(m, \beta), \beta < 1, m \in \{0, 1, 2, \dots\}$, the class of bounded turning functions of order β defined by Salagean differential operator. It follows a method of classical analysis introduced by Libera and Zlotkiewicz [7, 8]. The same has been applied by many authors in the literature [4, 5].

In 1983, Salagean defined the following operator:

$$D^0 f(z) = f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

$$D^1 f(z) = zD\left(D^0 f(z)\right) = zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k$$

$$D^2 f(z) = zD\left(D^1 f(z)\right) = zf''(z) = z + \sum_{k=2}^{\infty} k^2 a_k z^k$$

$$\vdots$$

$$D^m f(z) = zD\left(D^{m-1} f(z)\right) = z + \sum_{k=2}^{\infty} k^m a_k z^k \tag{5}$$

Obviously, from (5), we can deduce that

$$[D^m f(z)]' = 1 + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1} \tag{6}$$

Using (6), we define the following class of bounded turning functions of order β, $\beta < 1$, $m \in \{0, 1, 2, \dots\}$.

Definition 1 The class $BT(m, \beta)$ is said to be bounded turning of order β if

$$Re\left[\frac{[D^m f(z)]' - \beta}{1-\beta}\right] > 0$$

That is,

$$Re\left[\frac{1 + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1} - \beta}{1-\beta}\right] > 0$$

$$Re\left[\frac{(1-\beta) + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1}}{1-\beta}\right] > 0 \tag{7}$$

where $f(z) \in A$, $\beta < 1$, $m \in \{0, 1, 2, \dots\}$. Observe that with $\beta = 0$ in (7), we obtain (6) whose anti-derivative was reported in [11]. We denote by $BT(m, \beta)$ the class of functions in S which are bounded turning of order β, in $U = \{z \in C : |z| < 1\}$ which satisfy (7).

Definition 2 A function $f(z) \in S$ is said to be bounded turning of order β in $U = \{z \in C : |z| < 1\}$, if and only if

$$Re\left\{[D^m f(z)]'\right\} > \beta, z \in U \tag{8}$$

for fixed β, $\beta < 1$, $m \in \{0, 1, 2, \dots\}$.

2. PRELINARY RESULTS

Before we proceed into the main results, the following Lemmata shall be necessary.

Lemma 1[3] The power series for

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

converges in the open unit disk $U = \{z \in C : |z| < 1\}$ to a function in P if and only if the *Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{k-1} \\ c_{-2} & c_{-1} & 2 & \ldots & c_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-k} & c_{-k+1} & c_{-k+2} \cdots & & 2 \end{vmatrix}, \quad k = 1, 2, 3, \ldots$$

and $c_{-k} = \bar{c}_k$, are all non - negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0 \left(e^{it_k} z\right)$$

$\rho_k > 0, t_k$ real and $t_k \neq t_j$, for $k \neq j$, where

$$\rho_0(z) = \frac{1+z}{1-z}$$

in this case $D_n > 0$ for $n < (m-1) \wedge D_n = 0$ for $n \geq m$.
The necessary and sufficient condition in Lemma 1 is due to Caratheodory and Toeplitz. It may be assumed without restriction that $c_1 > 0$.
Using Lemma 1, for $n = 2 \wedge n = 3$, we have the following:

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = \left[8 + 2Re\{c_1^2 c_2\} - 2\left|c_2\right|^2 - 4\left|c_1\right|^2\right] \geq 0,$$

$$= c_2 \begin{vmatrix} \bar{c}_1 & 2 \\ \bar{c}_2 & \bar{c}_1 \end{vmatrix} - c_1 \begin{vmatrix} 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 \end{vmatrix} + 2 \begin{vmatrix} 2 & c_1 \\ \bar{c}_1 & 2 \end{vmatrix},$$

$$= c_2(\bar{c_1}^2 - 2\bar{c_2}) - c_1(2\bar{c_1} - c_1\bar{c_2}) + 2(4 - c_1\bar{c_1})$$

Taking rid of all the negative signs on $c's$ on the right hand-side of D_2 and expanding the brackets, we get

$$= c_2(c_1^2 - 2c_2) - c_1(2c_1 - c_1c_2) + 2(4 - c_1c_1)$$

$$= c_2c_1^2 - 2c_2^2 - 2c_1^2 + c_1^2c_2 + 8 - 2c_1^2$$

Collecting like terms, we get

$$= c_2c_1^2 + c_1^2c_2 - 2c_2^2 - 2c_1^2 - 2c_1^2 + 8$$

$$= 2c_2c_1^2 - 2c_2^2 - 4c_1^2 + 8$$

$$= 8 + 2Re\{c_1^2c_2\} - 2\left|c_2\right|^2 - 4\left|c_1\right|^2 \geq 0$$

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = \left[8 + 2Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2\right] \geq 0,$$

equivalent to

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \text{ for some } x, |x| \leq 1. \tag{9}$$

and

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0,$$

equivalent to

$$\left|\left(4c_3 - 4c_1c_2 + c_1^3\right)\left(4 - c_1^2\right) + c_1\left(2c_2 - c_1^2\right)^2\right| \leq 2\left(4 - c_1^2\right)^2 - 2\left|\left(2c_2 - c_1^2\right)\right|^2. \tag{10}$$

By simplifying (9) and (10), we obtained the following:

$4c_3 = \left\{c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z\right\}$ for some z, with $|z| \leq 1$

. (11)

Lemma (2([9], [10]) if $p \in P$, then $|p_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.

3. MAIN RESULTS

Theorem 1 If $f(z) \in BT(m, \beta)$, the class of bounded turning functions of order β, with $\beta < 1$, $m \in \{0, 1, 2, \dots\}$, then

$$\left|a_3 - ta_2^2\right| \leq \frac{4(1-\beta)}{2 \times 3^{m+1}}$$

Proof. From (8), we say that

$$[D^m f(z)]' - \beta > 0$$

For the function $f(z) \in BT(m, \beta), \exists$ a holomorphic function $p \in P$ in $U = \{z \in C : |z| < 1\}$ with $p(0) = 1$ and $Rep(z) > 0$ such that

$$\frac{[D^m f(z)]' - \beta}{1 - \beta} = p(z) \Leftrightarrow [D^m f(z)]' - \beta = (1 - \beta)p(z) \tag{12}$$

Using the series representation for $[D^m f(z)]'$ and $p(z)$ in (6) and (2), we have

$$\left\{1 + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1}\right\} - \beta = (1 - \beta)\left\{1 + \sum_{k=1}^{\infty} c_k z^k\right\}$$

$$(1 - \beta) + \left\{\sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1}\right\} = (1 - \beta)\left\{1 + \sum_{k=1}^{\infty} c_k z^k\right\}.$$

Simplifying, we have

$$(1-\beta)+2^{m+1}a_2z+3^{m+1}a_3z^2+4^{m+1}a_4z^3+5^{m+1}a_5z^4+6^{m+1}a_6z^5+\ldots=(1-\beta)\left(1+c_1z+c_2z^2+c_3z^3+c_4z^4+c_5z^5+\ldots\right)$$

$$(1-\beta)+2^{m+1}a_2z+3^{m+1}a_3z^2+4^{m+1}a_4z^3+5^{m+1}a_5z^4+6^{m+1}a_6z^5+\ldots=(1-\beta)+(1-\beta)c_1z+(1-\beta)c_2z^2+(1-\beta)c_3z^3+(1-\beta)c_4z^4+(1-\beta)c_5z^5+(1-\beta)c_6z^6$$

$$2^{m+1}a_2z+3^{m+1}a_3z^2+4^{m+1}a_4z^3+5^{m+1}a_5z^4+6^{m+1}a_6z^5+\ldots=(1-\beta)c_1z+(1-\beta)c_2z^2+(1-\beta)c_3z^3+(1-\beta)c_4z^4+(1-\beta)c_5z^5+(1-\beta)c_6z^6+\ldots \tag{13}$$

By equating the coefficients of the like powers of z's in (13), we have

$$2^{m+1}a_2z=(1-\beta)c_1 \Longrightarrow a_2=\frac{(1-\beta)c_1}{2^{m+1}} \tag{14}$$

$$3^{m+1}a_3=(1-\beta)c_2 \Longrightarrow a_3=\frac{(1-\beta)c_2}{3^{m+1}} \tag{15}$$

$$4^{m+1}a_4=(1-\beta)c_3 \Longrightarrow a_4=\frac{(1-\beta)c_3}{4^{m+1}} \tag{16}$$

$$5^{m+1}a_5=(1-\beta)c_4 \Longrightarrow a_5=\frac{(1-\beta)c_4}{5^{m+1}} \tag{17}$$

$$6^{m+1}a_6=(1-\beta)c_5 \Longrightarrow a_6=\frac{(1-\beta)c_5}{6^{m+1}} \tag{18}$$

In general, we can see that

$$|a_k|\le\frac{(1-\beta)c_{k-1}}{k^{m+1}} \tag{19}$$

Substituting the values of a_2 and a_3 from (14) and (15) in the functional $\mid a_3-ta_2^2\mid$ for the function $BT(m,\beta)$, we have

$$\mid a_3-ta_2^2\mid\le\left|\frac{(1-\beta)c_2}{3^{m+1}}-t\left(\frac{(1-\beta)c_1}{2^{m+1}}\right)^2\right|$$

$$\le\left|\frac{(1-\beta)c_2}{3^{m+1}}-t\left(\frac{(1-\beta)^2c_1^2}{2^{2(m+1)}}\right)\right|$$

Using Lemma 1, we know that

$$
\begin{aligned}
| a_3 - ta_2^2 | &\leq \left| \frac{(1-\beta)}{3^{m+1}} \left[\frac{1}{2} \left\{ c_1^2 + x(4-c_1^2) \right\} \right] - t \left(\frac{(1-\beta)^2 c_1^2}{2^{2(m+1)}} \right) \right| \\
| a_3 - ta_2^2 | &\leq \left| \frac{(1-\beta)\{c_1^2 + x(4-c_1^2)\}}{2 \times 3^{m+1}} - t \left(\frac{(1-\beta)^2 c_1^2}{2^{2(m+1)}} \right) \right| \\
| a_3 - ta_2^2 | &\leq \left| \frac{(1-\beta)\{c_1^2 + x(4-c_1^2)\}}{2 \times 3^{m+1}} - t \left(\frac{(1-\beta)(1-\beta) c_1^2}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{(1-\beta)\{c_1^2 + x(4-c_1^2)\}}{2 \times 3^{m+1}} - t \left(\frac{(1-\beta-\beta+\beta^2) c_1^2}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{(1-\beta)c_1^2 + x(4-c_1^2)(1-\beta)}{2 \times 3^{m+1}} - t \left(\frac{(1-2\beta+\beta^2) c_1^2}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - t \left(\frac{(c_1^2 - 2\beta c_1^2 + \beta^2 c_1^2)}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - \left(\frac{t(c_1^2 - 2\beta c_1^2 + \beta^2 c_1^2)}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - \left(\frac{tc_1^2 - 2t\beta c_1^2 + t\beta^2 c_1^2)}{2^{2(m+1)}} \right) \right| \\
&\leq \left| \frac{c_1^2}{2 \times 3^{m+1}} - \frac{\beta c_1^2}{2 \times 3^{m+1}} + \frac{4x}{2 \times 3^{m+1}} - \frac{4x\beta}{2 \times 3^{m+1}} - \frac{xc_1^2}{2 \times 3^{m+1}} + \frac{xc_1^2\beta}{2 \times 3^{m+1}} - \frac{tc_1^2}{2^{2m+2}} \right. \\
&\quad \left. + \frac{2t\beta c_1^2}{2^{2m+2}} - \frac{t\beta^2 c_1^2}{2^{2m+2}} \right| \\
&\leq \left| \frac{4x}{2 \times 3^{m+1}} - \frac{4x\beta}{2 \times 3^{m+1}} + \frac{c_1^2}{2 \times 3^{m+1}} - \frac{xc_1^2}{2 \times 3^{m+1}} - \frac{\beta c_1^2}{2 \times 3^{m+1}} + \frac{xc_1^2\beta}{2 \times 3^{m+1}} - \frac{tc_1^2}{2^{2m+2}} + \frac{2t\beta c_1^2}{2^{2m+2}} - \frac{t\beta^2 c_1^2}{2^{2m+2}} \right| \\
| a_3 - ta_2^2 | &\leq \left| \frac{4 \mid x \mid}{2 \times 3^{m+1}} - \frac{4\beta \mid x \mid}{2 \times 3^{m+1}} - \frac{\mid x \mid c_1^2}{2 \times 3^{m+1}} + \frac{\mid x \mid c_1^2\beta}{2 \times 3^{m+1}} + \frac{c_1^2}{2 \times 3^{m+1}} - \frac{\beta c_1^2}{2 \times 3^{m+1}} - \frac{tc_1^2}{2^{2m+2}} + \frac{2t\beta c_1^2}{2^{2m+2}} - \frac{t\beta^2 c_1^2}{2^{2m+2}} \right|
\end{aligned}
$$

Let $\mid x \mid = t \in [0,1], c_1 = c \in [0,2]$ and applying the triangle inequality, above equation reduce to

$$| a_3 - ta_2^2 | \leq \frac{4t}{2 \times 3^{m+1}} + \frac{4t\beta}{2 \times 3^{m+1}} + \frac{tc^2}{2 \times 3^{m+1}} + \frac{tc^2\beta}{2 \times 3^{m+1}} + \frac{c^2}{2 \times 3^{m+1}} + \frac{\beta c^2}{2 \times 3^{m+1}} + \frac{tc^2}{2^{2m+2}} + \frac{2t\beta c^2}{2^{2m+2}} + \frac{t\beta^2 c^2}{2^{2m+2}}$$

$$| a_3 - ta_2^2 | \leq \frac{1}{2 \times 3^{m+1}} \left[4t + 4t\beta + tc^2 + t\beta c^2 + \beta c^2 + c^2 \right] + \frac{1}{2^{2m+2}} \left[tc^2 + 2t\beta c^2 + t\beta^2 c^2 \right]$$

Suppose that

$$F(c,t) := \frac{1}{2 \times 3^{m+1}} \left[4t + 4t\beta + tc^2 + t\beta c^2 + \beta c^2 + c^2 \right] + \frac{1}{2^{2m+2}} \left[tc^2 + 2t\beta c^2 + t\beta^2 c^2 \right]$$

Then we get

$$\frac{\partial F}{\partial t} = \frac{1}{2 \times 3^{m+1}} \left[4 + 4\beta + c^2 + \beta c^2 \right] + \frac{1}{2^{2m+2}} \left[c^2 + 2\beta c^2 + \beta^2 c^2 \right] \geq 0$$

This shows that $F(c,t)$ is an increasing function on the closed interval $[0,1]$ about t. Therefore, the function $F(c,t)$ can get the maximum value at $t=1$, that is,

$$\begin{aligned} maxF(c,t) = \quad F(c,1) &= \frac{1}{2\times 3^{m+1}}\left[4+4\beta+c^2+\beta c^2+\beta c^2+c^2\right]+\frac{1}{2^{2m+2}}\left[c^2+2\beta c^2+\beta^2c^2\right] \\ &= \frac{1}{2\times 3^{m+1}}\left[4+4\beta+2c^2+2\beta c^2\right]+\frac{1}{2^{2m+2}}\left[c^2+2\beta c^2+\beta^2c^2\right] \end{aligned}$$

Next, let

$$\begin{aligned} G(c) :=& \frac{1}{2\times 3^{m+1}}\left[4+4\beta+2c^2+2\beta c^2\right]+\frac{1}{2^{2m+2}}\left[c^2+2\beta c^2+\beta^2c^2\right] \\ :=& \frac{4}{2\times 3^{m+1}}+\frac{4\beta}{2\times 3^{m+1}}+\frac{2c^2}{2\times 3^{m+1}}+\frac{2\beta c^2}{2\times 3^{m+1}}+\frac{c^2}{2^{2m+2}}+\frac{2\beta c^2}{2^{2m+2}}+\frac{\beta^2c^2}{2^{2m+2}} \\ G(c) :=& \frac{4+4\beta}{2\times 3^{m+1}}+\left[\frac{1}{2^{2m+2}}+\frac{2}{2\times 3^{m+1}}+\frac{2\beta}{2\times 3^{m+1}}+\frac{2\beta}{2^{2m+2}}+\frac{\beta^2}{2^{2m+2}}\right]c^2 \end{aligned}$$

Then we easily see that the function $G(c)$ have maximum value at $c=0$, also which is

$$\begin{aligned} \mid a_3-ta_2^2 \mid\leq G(c) :=& \frac{4+4\beta}{2\times 3^{m+1}}+\left[\frac{1}{2^{2m+2}}+\frac{2}{2\times 3^{m+1}}+\frac{2\beta}{2\times 3^{m+1}}+\frac{2\beta}{2^{2m+2}}+\frac{\beta^2}{2^{2m+2}}\right]\times 0^2 \\ =& \frac{4+4\beta}{2\times 3^{m+1}}+\left[\frac{1}{2^{2m+2}}+\frac{2}{2\times 3^{m+1}}+\frac{2\beta}{2\times 3^{m+1}}+\frac{2\beta}{2^{2m+2}}+\frac{\beta^2}{2^{2m+2}}\right]\times 0 \\ =& \frac{4+4\beta}{2\times 3^{m+1}}+0 \\ =& \frac{4+4\beta}{2\times 3^{m+1}} \\ & \mid a_3-ta_2^2 \mid\leq \frac{4(1-\beta)}{2\times 3^{m+1}} \end{aligned}$$

□

Theorem 2 Let $f(z)\in BT(m,\beta)$, with $\beta<1$, $m\in\{0,1,2,...\}$. Then

$$|a_2a_4-a_3^2|\leq\frac{4(1-\beta)^2}{3^{2m+2}}$$

Proof. Recall from (14), (15) and (16) we have that

$$a_2=\frac{(1-\beta)c_1}{2^{m+1}},\quad a_3=\frac{(1-\beta)c_2}{3^{m+1}},\quad a_4=\frac{(1-\beta)c_3}{4^{m+1}} \tag{20}$$

Substituting the values of a_2, a_3 and a_4 from (20) in the functional $|a_2a_4 - a_3^2|$ for the function $f(z) \in BT(m, \beta)$, and simplifying, we get

$$
\begin{aligned}
|a_2a_4 - a_3^2| =& \left| \left(\frac{(1-\beta)c_1}{2^{m+1}}\right)\left(\frac{(1-\beta)c_3}{4^{m+1}}\right) - \left(\frac{(1-\beta)c_2}{3^{m+1}}\right)^2 \right| \\
=& \left| \left(\frac{(1-\beta)^2c_1c_3}{2^{m+1}\times 2^{2(m+1)}}\right) - \left(\frac{(1-\beta)c_2}{3^{m+1}}\right)^2 \right| \\
=& \left| \left(\frac{(1-\beta)^2c_1c_3}{2^{(m+1)+(2m+2)}}\right) - \frac{(1-\beta)^2c_2^2}{3^{2(m+1)}} \right| \\
=& (1-\beta)^2 \left| \frac{c_1c_3}{2^{3m+3}} - \frac{c_2^2}{3^{2(m+1)}} \right| \\
=& (1-\beta)^2 \left| \frac{1}{2^{3(m+1)}}c_1c_3 - \frac{1}{3^{2(m+1)}}c_2^2 \right|
\end{aligned}
$$

$$
|a_2a_4 - a_3^2| = \quad (1-\beta)^2|w_1c_1c_3 + w_2c_2^2| \tag{21}
$$

where

$$
w_1 = \frac{1}{2^{3(m+1)}}; \quad w_2 = -\frac{1}{3^{2(m+1)}} \tag{22}
$$

Substituting the values of c_2 and c_3 from (9) and (11) respectively from Lemma 1 on the right-hand side of (21),

$$
\begin{aligned}
|w_1c_1c_3 + w_2c_2^2| =& \Big| w_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1\left(4-c_1^2\right)x - c_1\left(4-c_1^2\right)x^2 + 2\left(4-c_1^2\right)\left(1-|x|^2\right)z\} \\
& + w_1 \times \left[\frac{1}{2}\{c_1^2 + x\left(4-c_1^2\right)\}\right]^2 \Big| \\
|w_1c_1c_3 + w_2c_2^2| =& \Big| \frac{1}{4}\{w_1c_1c_1^3 + 2w_1c_1^2\left(4-c_1^2\right)x - w_1c_1^2\left(4-c_1^2\right)x^2 + 2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\} \\
& + \frac{1}{4}\left[w_2c_1^4 + w_2\left(4-c_1^2\right)^2x^2\right] \Big|
\end{aligned}
$$

$$\begin{aligned}
\left|w_1c_1c_3+w_2c_2^2\right| =& \left|\frac{1}{4}\{w_1c_1c_1^3+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\}\right.\\
&\left.+\frac{1}{4}\left[w_2c_1^4+w_2\left(4-c_1^2\right)^2x^2\right]\right|\\
=&\frac{1}{4}\left|\{w_1c_1c_1^3+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\}\right.\\
&\left.+w_2c_1^4+w_2\left(4-c_1^2\right)^2x^2\right|\\
=&\frac{1}{4}\left|w_1c_1c_1^3+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\right.\\
&\left.+w_2c_1^4+w_2\left(4-c_1^2\right)^2x^2\right|\\
=&\frac{1}{4}\left|w_1c_1c_1^3+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\right.\\
&\left.+w_2c_1^4+w_2\left(4-c_1^2\right)^2x^2\right|\\
4\left|w_1c_1c_3+w_2c_2^2\right| =&\left|w_1c_1^4+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z\right.\\
&\left.+w_2c_1^4+w_2\left(4-c_1^2\right)^2x^2\right|\\
=&\left|w_1c_1^4+w_2c_1^4+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2+2w_1c_1\left(4-c_1^2\right)\right.\\
&\left.-2w_1c_1\left(4-c_1^2\right)|x|^2z+w_2\left(4-c_1^2\right)^2x^2\right|\\
=&\left|\left(w_1+w_2\right)c_1^4+2w_1c_1\left(4-c_1^2\right)+2w_1c_1^2\left(4-c_1^2\right)x-w_1c_1^2\left(4-c_1^2\right)x^2\right.\\
&\left.-2w_1c_1\left(4-c_1^2\right)|x|^2z+w_2\left(4-c_1^2\right)^2x^2)\right|
\end{aligned}$$

Using the triangle inequality and the fact that $|z|<1$, and simplifying, we get

$$\begin{aligned}
4\left|w_1c_1c_3+w_2c_2^2\right| \leq&\left|\left(w_1+w_2\right)c_1^4+2w_1c_1\left(4-c_1^2\right)+2w_1c_1^2\left(4-c_1^2\right)|x|\right.\\
&\left.-w_1c_1^2\left(4-c_1^2\right)|x|^2-2w_1c_1\left(4-c_1^2\right)|x|^2+w_2\left(4-c_1^2\right)^2|x|^2)\right|
\end{aligned}$$

$$\begin{aligned}
4\left|w_1c_1c_3+w_2c_2^2\right| \leq&\left|\left(w_1+w_2\right)c_1^4+2w_1c_1\left(4-c_1^2\right)+2w_1c_1^2\left(4-c_1^2\right)|x|\right.\\
&\left.-w_1c_1\left(c_1-2\right)\left(4-c_1^2\right)|x|^2+w_2\left(4-c_1^2\right)\left(4-c_1^2\right)|x|^2)\right|
\end{aligned}\tag{23}$$

Using the values of w_1, w_2 given in (22), we can write

$$\begin{aligned} w_1 + w_2 =& \frac{1}{2^{3(m+1)}} + \left(-\frac{1}{3^{2(m+1)}}\right) \\ =& \frac{1}{2^{3(m+1)}} - \frac{1}{3^{2(m+1)}} \\ =& \frac{3^{2(m+1)} - 2^{3(m+1)}}{2^{3(m+1)} \times 3^{2(m+1)}} \\ w_1 + w_2 =& \frac{9^{m+1} - 8^{m+1}}{2^{3(m+1)} \times 3^{2(m+1)}} \end{aligned}$$

and

$$\begin{aligned} 2w_1 =& 2\left(\frac{1}{2^{3(m+1)}}\right) \\ =& \frac{2}{2^{3(m+1)}} \\ =& 2^1 \times 2^{-3(m+1)} \\ =& 2^1 \times 2^{-3m-3} \\ =& 2^{-3m-3+1} \\ =& 2^{-3m-2} \\ =& 2^{-(3m+2)} \\ 2w_1 =& \frac{1}{2^{3m+2}} \end{aligned}$$

Therefore,

$$w_1 + w_2 = \frac{9^{m+1} - 8^{m+1}}{2^{3(m+1)} \times 3^{2(m+1)}}; \qquad 2w_1 = \frac{1}{2^{3m+2}} \tag{24}$$

Substituting the values from (24) on the right–hand side of (23), we have

$$\begin{aligned} 4\left|w_1c_1c_3 + w_2c_2^2\right| \leq& \Bigg| \frac{(9^{m+1} - 8^{m+1})}{2^{3(m+1)} \times 3^{2(m+1)}} c_1^4 + \frac{1}{2^{(3m+2)}} c_1(4 - c_1^2) + \frac{1}{2^{(3m+2)}} c_1^2(4 - c_1^2)\,|x| - \frac{1}{2^{3(m+1)}} c_1^2(4 - c_1^2)\mid x\mid^2 \\ & - \frac{1}{2^{(3m+2)}} c_1(4 - c_1^2)\mid x\mid^2 - \frac{1}{3^{2(m+1)}}\left(4 - c_1^2\right)\left(4 - c_1^2\right)\mid x\mid^2 \Bigg| \\ \leq& \Bigg| \frac{(9^{m+1} - 8^{m+1})}{2^{3(m+1)} \times 3^{2(m+1)}} c_1^4 + \frac{1}{2^{(3m+2)}} c_1(4 - c_1^2) + \frac{1}{2^{(3m+2)}} c_1^2\left(4 - c_1^2\right)\mid x\mid \\ & - \frac{1}{2^{3(m+1)}} c_1(c_1 - 2)(4 - c_1^2)\mid x\mid^2 - \frac{1}{3^{2(m+1)}}\left(4 - c_1^2\right)\left(4 - c_1^2\right)\mid x\mid^2 \Bigg| \end{aligned}$$

Since $c_1 = c \in [0,2]$, noting that $c_1 + a = c_1 - a$, where $a \geq 0$ and replacing $\mid x \mid$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned} 4\left|w_1c_1c_3 + w_2c_2^2\right| \leq & \Big[\frac{(9^{m+1}-8^{m+1})}{2^{3(m+1)}\times 3^{2(m+1)}}c^4 + \frac{1}{2^{(3m+2)}}(4-c^2)c + \frac{1}{2^{(3m+2)}}c^2(4-c^2)\mu \\ & - \frac{1}{2^{3(m+1)}}(c+2)c(4-c^2)\mu^2 - \frac{1}{3^{2(m+1)}}(4-c^2)(4-c^2)\mu^2\Big] \\ \leq & \Big[\frac{(9^{m+1}-8^{m+1})}{2^{3m+3}\times 3^{2m+2}}c^4 + \frac{1}{2^{3m+2}}(4-c^2)c + \frac{1}{2^{3m+2}}c^2(4-c^2)\mu \\ & + \frac{1}{2^{3m+3}}(c+2)c(4-c^2)\mu^2 + \frac{1}{3^{2m+2}}(4-c^2)(4-c^2)\mu^2\Big] \\ = & F(c,\mu), 0 \leq \mu =\mid x \mid\leq 1 \quad and \quad 0 \leq c \leq 2 \end{aligned} \tag{25}$$

where

$$\begin{aligned} F(c,\mu) = & \Big[\frac{(9^{m+1}-8^{m+1})}{2^{3m+3}\times 3^{2m+2}}c^4 + \frac{1}{2^{3m+2}}(4-c^2)c + \frac{1}{2^{3m+2}}c^2(4-c^2)\mu \\ & + \frac{1}{2^{3m+3}}(c+2)c(4-c^2)\mu^2 + \frac{1}{3^{2m+2}}(4-c^2)(4-c^2)\mu^2\Big] \end{aligned} \tag{26}$$

Further, we maximize the function $F(c,\mu)$ on the region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (26) partially with respect to μ, we get

$$\begin{aligned} \frac{\partial F}{\partial \mu} =& \frac{1}{2^{3m+2}}\left(4-c^2\right)c^2 + \frac{2}{2^{3m+3}}\left(c+2\right)c\left(4-c^2\right)\mu + \frac{2}{3^{2m+2}}\left(4-c^2\right)\left(4-c^2\right)\mu \\ \frac{\partial F}{\partial \mu} =& \frac{1}{2^{3m+2}}\left(4-c^2\right)c^2 + 2\times 2^{-(3m+3)}\left(c+2\right)c\left(4-c^2\right)\mu + \frac{2}{3^{2m+2}}\left(4-c^2\right)\left(4-c^2\right)\mu \\ \frac{\partial F}{\partial \mu} =& \frac{1}{2^{3m+2}}\left(4-c^2\right)c^2 + 2^{-3m-3+1}\left(c+2\right)c\left(4-c^2\right)\mu + \frac{2}{3^{2m+2}}\left(4-c^2\right)\left(4-c^2\right)\mu \\ \frac{\partial F}{\partial \mu} =& \frac{1}{2^{3m+2}}\left(4-c^2\right)c^2 + 2^{-(3m+2)}\left(c+2\right)c\left(4-c^2\right)\mu + \frac{2}{3^{2m+2}}\left(4-c^2\right)\left(4-c^2\right)\mu \\ \frac{\partial F}{\partial \mu} =& \frac{1}{2^{3m+2}}\left(4-c^2\right)c^2 + \frac{1}{2^{3m+2}}\left(c+2\right)c\left(4-c^2\right)\mu + \frac{2}{3^{2m+2}}\left(4-c^2\right)\left(4-c^2\right)\mu \\ \frac{\partial F}{\partial \mu} =& \left[\frac{c^2}{2^{3m+2}} + \frac{c(c+2)\mu}{2^{3m+2}} + \frac{2(4-c^2)\mu}{3^{2m+2}}\right]\times(4-c^2) \end{aligned}$$

$$\frac{\partial F}{\partial \mu} = \left[\frac{c^2 + c(c+2)\mu}{2^{3m+2}} + \frac{2(4-c^2)\mu}{3^{2m+2}}\right]\times(4-c^2) \tag{27}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and for β, m with $\beta < 1, m \in \{0,1,2,\ldots\}$, from (27), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes am increasing function of μ and hence it cannot have a maximum at any point in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = \quad G(c) \tag{28}$$

In view of (28), simplifying the relation (26), we have

$$
\begin{aligned}
G(c) =& \frac{(9^{m+1}-8^{m+1})}{2^{3m+3}\times 3^{2m+2}}c^4 + \frac{1}{2^{3m+2}}(4-c^2)c + \frac{1}{2^{3m+2}}c^2(4-c^2)\mu \\
&+ \frac{1}{2^{3m+3}}(c+2)c(4-c^2)\mu^2 + \frac{1}{3^{2m+2}}(4-c^2)(4-c^2)\mu^2 \\
G(c) =& \frac{(9^{m+1}-8^{m+1})}{2^{3m+3}\times 3^{2m+2}}c^4 + \frac{1}{2^{3m+2}}(4-c^2)c + \frac{1}{2^{3m+2}}c^2(4-c^2)\times 1 \\
&+ \frac{1}{2^{3m+3}}(c+2)c(4-c^2)\times 1^2 + \frac{1}{3^{2m+2}}(4-c^2)(4-c^2)\times 1^2 \\
G(c) =& \frac{(9^{m+1}-8^{m+1})}{2^{3m+3}\times 3^{2m+2}}c^4 + \frac{1}{2^{3m+2}}(4-c^2)c + \frac{1}{2^{3m+2}}c^2(4-c^2) \\
&+ \frac{1}{2^{3m+3}}(c+2)c(4-c^2) + \frac{1}{3^{2m+2}}(4-c^2)(4-c^2) \\
G(c) =& \left(\frac{1}{2^{3m+3}} - \frac{1}{3^{2m+2}}\right)c^4 + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{1}{2^{3m+3}}\left[(c^2+2c)(4-c^2)\right] \\
&+ \frac{1}{3^{2m+2}}\left[16-4c^2-4c^2+c^4\right] \\
G(c) =& \frac{c^4}{2^{3m+3}} - \frac{c^4}{3^{2m+2}} + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{1}{2^{3m+3}}\left[4c^2-c^4+8c-2c^3)\right] \\
&+ \frac{1}{3^{2m+2}}\left[c^4-8c^2+16\right] \\
G(c) =& \frac{c^4}{2^{3m+3}} - \frac{c^4}{3^{2m+2}} + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{4c^2}{2^{3m+3}} - \frac{c^4}{2^{3m+3}} + \frac{8c}{2^{3m+3}} \\
&- \frac{2c^3}{2^{3m+3}} + \frac{c^4}{3^{2m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}}
\end{aligned}
$$

Collecting like terms, we

$$
\begin{aligned}
G(c) =& \frac{c^4}{2^{3m+3}} + \frac{4c^2}{2^{3m+3}} - \frac{c^4}{2^{3m+3}} + \frac{8c}{2^{3m+3}} - \frac{2c^3}{2^{3m+3}} + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} \\
&- \frac{c^4}{3^{2m+2}} + \frac{c^4}{3^{2m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}} \\
G(c) =& \left[\frac{c^4}{2^{3m+3}} + \frac{4c^2}{2^{3m+3}} - \frac{c^4}{2^{3m+3}} + \frac{8c}{2^{3m+3}} - \frac{2c^3}{2^{3m+3}}\right] + \left[\frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}}\right] \\
&+ \left[\frac{16}{3^{2m+2}} - \frac{8c^2}{3^{2m+2}}\right] \\
G(c) =& \left[\frac{c^4}{2^{3m+3}} - \frac{c^4}{2^{3m+3}} - \frac{2c^3}{2^{3m+3}} + \frac{4c^2}{2^{3m+3}} + \frac{8c}{2^{3m+3}}\right] + \left[\frac{4c}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}}\right] \\
&+ \left[\frac{16}{3^{2m+2}} - \frac{8c^2}{3^{2m+2}}\right] \\
G(c) =& \left[\frac{8c}{2^{3m+3}} + \frac{4c^2}{2^{3m+3}} - \frac{2c^3}{2^{3m+3}}\right] + \left[\frac{4c}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}}\right] + \left[\frac{16}{3^{2m+2}} - \frac{8c^2}{3^{2m+2}}\right]
\end{aligned}
$$

$$G(c) = \frac{1}{2^{3m+3}}\left[8c + 4c^2 - 2c^3\right] + \frac{1}{2^{3m+2}}\left[4c + 4c^2 - c^3 - c^4\right] + \frac{1}{3^{2m+2}}\left[16 - 8c^2\right]$$

$$G(c) = \frac{8c}{2^{3m+3}} + \frac{4c^2}{2^{3m+3}} - \frac{2c^3}{2^{3m+3}} + \frac{4c}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = \left(2^3 \times 2^{-(3m+3)}\right)c + \left(2^2 \times 2^{-(3m+3)}\right)c^2 - \frac{2c^3}{2^{3m+3}} + \left(2^2 \times 2^{-(3m+2)}\right)c + \left(2^2 \times 2^{-(3m+2)}\right)c^2 - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = 2^{-3m}c + 2^{-3m-1}c^2 - \frac{2c^3}{2^{3m+3}} + 2^{-3m}c + 2^{-3m}c^2 - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = \frac{c}{2^{3m}} + \frac{c^2}{2^{3m+1}} - \frac{2c^3}{2^{3m+3}} + \frac{c}{2^{3m}} + \frac{c^2}{2^{3m}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} - \frac{8c^2}{3^{2m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = \frac{c}{2^{3m}} + \frac{c}{2^{3m}} + \frac{c^2}{2^{3m}} + \frac{c^2}{2^{3m+1}} - \frac{8c^2}{3^{2m+2}} - \frac{2c^3}{2^{3m+3}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = \frac{2c}{2^{3m}} + \frac{c^2}{2^{3m}} + \frac{c^2}{2^{3m+1}} - \frac{8c^2}{3^{2m+2}} - \frac{2c^3}{2^{3m+3}} - \frac{c^3}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{16}{3^{2m+2}}$$

$$G(c) = \frac{16}{3^{2m+2}} + \frac{2c}{2^{3m}} + \left[\frac{c^2}{2^{3m}} + \frac{c^2}{2^{3m+1}} - \frac{8c^2}{3^{2m+2}}\right] - \left[\frac{2c^3}{2^{3m+3}} + \frac{c^3}{2^{3m+2}}\right] - \frac{c^4}{2^{3m+2}}$$

$$G(c) = \frac{16}{3^{2m+2}} + \frac{2c}{2^{3m}} + \left[\frac{1}{2^{3m}} + \frac{1}{2^{3m+1}} - \frac{8}{3^{2m+2}}\right]c^2 - \left[\frac{2}{2^{3m+3}} + \frac{1}{2^{3m+2}}\right]c^3 - \frac{c^4}{2^{3m+2}} \quad (29)$$

$$G'(c) = \frac{2}{2^{3m}} + 2\left[\frac{1}{2^{3m}} + \frac{1}{2^{3m+1}} - \frac{8}{3^{2m+2}}\right]c - 3\left[\frac{2}{2^{3m+3}} + \frac{1}{2^{3m+2}}\right]c^2 - \frac{4c^3}{2^{3m+2}}$$

$$G'(c) = \frac{2}{2^{3m}} + \left[\frac{2}{2^{3m}} + \frac{2}{2^{3m+1}} - \frac{16}{3^{2m+2}}\right]c - \left[\frac{6}{2^{3m+3}} + \frac{3}{2^{3m+2}}\right]c^2 - \frac{4c^3}{2^{3m+2}}$$

$$G'(c) = \frac{2}{2^{3m}} + \left[\frac{2}{2^{3m}} + 2 \times 2^{-(3m+1)} - \frac{16}{3^{2m+2}}\right]c - \left[\frac{6}{2^{3m+3}} + \frac{3}{2^{3m+2}}\right]c^2 - \frac{4c^3}{2^{3m+2}}$$

$$G'(c) = \frac{2}{2^{3m}} + \left[\frac{2}{2^{3m}} + 2^{-3m} - \frac{16}{3^{2m+2}}\right]c - \left[\frac{6}{2^{3m+3}} + \frac{3}{2^{3m+2}}\right]c^2 - \frac{4c^3}{2^{3m+2}}$$

$$G'(c) = \frac{2}{2^{3m}} + \left[\frac{2}{2^{3m}} + \frac{1}{2^{3m}} - \frac{16}{3^{2m+2}}\right]c - \left[\frac{6}{2^{3m+3}} + \frac{3}{2^{3m+2}}\right]c^2 - \frac{4c^3}{2^{3m+2}} \quad (30)$$

$$G''(c) = \left[\frac{2}{2^{3m}} + \frac{1}{2^{3m}} - \frac{16}{3^{2m+2}}\right] - 2\left[\frac{6}{2^{3m+3}} + \frac{3}{2^{3m+2}}\right]c - \frac{12c^2}{2^{3m+2}}$$

$$G''(c) = \frac{2}{2^{3m}} + \frac{1}{2^{3m}} - \frac{16}{3^{2m+2}} - \left[\frac{12}{2^{3m+3}} + \frac{6}{2^{3m+2}}\right]c - \frac{12c^2}{2^{3m+2}}$$

$$G''(c) = \frac{3}{2^{3m}} - \frac{16}{3^{2m+2}} - \frac{12c}{2^{3m+3}} - \frac{6c}{2^{3m+2}} - \frac{12c^2}{2^{3m+2}}$$

$$G''(c) = \frac{3}{2^{3m}} - \frac{16}{3^{2m+2}} - 6c\left[\frac{1}{2^{3m+2}} + \frac{2}{2^{3m+3}}\right] - \frac{12c^2}{2^{3m+2}} \quad (31)$$

For optimum value of $G(c)$, consider $G'(c) = 0$, then the root is $c = 0$, with $m \in \{0, 1, 2, \ldots\}$. After a simple calculation, we can deduce that $G''(c) > 0$, which means that the function $G(c)$ can take the minimum value at $c = 0$, $m \in \{0, 1, 2, \ldots\}$, also which is

$$| a_2a_4 - a_3^2 | \leq G(0) = \frac{16}{3^{2m+2}} + \frac{2\times 0}{2^{3m}} + \left[\frac{1}{2^{3m}} + \frac{1}{2^{3m+1}} - \frac{8}{3^{2m+2}}\right] \times 0^2 - \left[\frac{2}{2^{3m+3}} + \frac{1}{2^{3m+2}}\right] \times 0^3 - \frac{0^4}{2^{3m+2}}$$

$$| a_2a_4 - a_3^2 | \leq G(0) = \frac{16}{3^{2m+2}} + 0 + 0 - 0 - 0$$

$$| a_2a_4 - a_3^2 | \leq G(0) = \frac{16}{3^{2m+2}} \tag{32}$$

Simplifying the expression (25) and (32), we get

$$| w_1c_1c_3 + w_2c_2^2 | \leq \frac{1}{4} \times \left(\frac{16}{3^{2m+2}}\right)$$

$$| w_1c_1c_3 + w_2c_2^2 | \leq \frac{\left(\frac{16}{3^{2m+2}}\right)}{4}$$

$$| w_1c_1c_3 + w_2c_2^2 | \leq \left(\frac{16}{3^{2m+2}}\right) \div \frac{4}{1}$$

$$| w_1c_1c_3 + w_2c_2^2 | \leq \left(\frac{16}{3^{2m+2}}\right) \times \frac{1}{4}$$

$$| w_1c_1c_3 + w_2c_2^2 | \leq \frac{4}{3^{2m+2}} \tag{33}$$

From the relation (21) and (33), upon simplification, we obtain

$$| a_2a_4 - a_3^2 | \leq (1-\beta)^2 \times \frac{4}{3^{2m+2}}$$

$$| a_2a_4 - a_3^2 | \leq \frac{4(1-\beta)^2}{3^{2m+2}}$$

□

Conclusion: In this paper, we mainly determined the bound for the well-known Fekete-Szego functional $| a_3 - ta_2^2 |$, with t real and the Second Hankel Determinant$| a_2a_4 - a_3^2 |$ for functions of bounded turning of order β associated with Salagean differential operator.

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