Journal of Fractional Calculus and Applications,
Vol. 6(1) Jan. 2015, pp. 51-61.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON EXISTENCE AND UNIQUENESS OF SOLUTION FOR FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we investigate existence and uniqueness of solution for a nonlinear fractional differential equation with boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) + f(u(t)) = 0, \quad 1 < \alpha < 2, \\ u(\xi) = 0, \quad D^p u(1) - \mu u(\eta) = 0, \quad 0 < p < 1. \end{cases}$$

where $\eta < \xi$. The differential operator is Caputo fractional derivative. We use Shauder fixed point theorem and Banach contraction principle. We impose some growth conditions on the nonlinear function f.

1. Introduction

Recently fractional calculus has gained much popularity and importance in science and engineering. It has already been proved by experiments that most of the situations associated with complex systems have nonlocal dynamics possessing long memory in time. The fractional order derivatives and integrals have some of these characteristics which has the capabilities of modelling various complex phenomena and the fractional modelling is considered to be a powerful tool for the swift development of fractional calculus. The concentration of many researchers have been attracted in a verity of research fields due to the development and applications of fractional calculus such as engineering, mathematics, physics, chemistry, etc [1, 2, 3, 4, 5, 6]. The researchers have studied and developed various features of fractional differential equations but the theory of existence and uniqueness of solutions of fractional order differential equations is being considered the most significant area of research and the researchers in the field of mathematics also struggles to develop theory for applied sciences which can play an important role in this area of mathematics. The community of mathematicians showed a lot of interest in this field of research, particularly, to study the boundary value problems for fractional order differential equations, we refer the readers to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

Here we refer some boundary value problems which motivated us for the present work. In [24], Bai and Lu investigated the existence and multiplicity of positive

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2000 *Mathematics Subject Classification.* 34A12, 34G20, 34K05.
Key words and phrases. Fractional differential equation, Caputo's fractional derivative, Fixed point theorems.
Submitted April 19, 2014.

solutions for fractional differential equation

$$\begin{cases} D_{0+}^{q} u(t) + f(t, u(t))) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < q \leq 2$ and D_{0+}^{q} is the Riemann-Liouville fractional derivative.

By an application of Green's function in [25], X. Xu et.al studied multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{\sigma} u(t) = f(t, u(t))) = 0, & 0 < t < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number and D_{0+}^{σ} is the standard Riemann-Liouville differentiation.

By means of Guo-Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem in [26], S. Zhang studied the existence and multiplicity of positive solutions for nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) + u'(0) = 0, & u(1) + u'(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number and D_{0+}^{α} is the standard Caputo's fractional derivative.

By means of upper and lower solution method and fixed point theorems S. Liang and J. Zhang in [27], studied positive solution of fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number and D_{0+}^{α} is standard Riemenn-Liouville fractional derivative.

In this paper we study existence and uniqueness of solution for fractional differential equation.

$$\begin{cases} {}^{c}D_{0+}^{\alpha} u(t) + f(t, u(t), u'(t)) = 0 & 1 < \alpha < 2 \\ u(\xi) = 0, \quad D^{p} u(1) - \mu u(\eta) = 0 & 0 < p < 1 \end{cases} \tag{1}$$

where $0 < \eta < \xi < 1$, $0 < \mu < 1$, $f \in C([0,1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ and D^{α} is Caputo's fractional derivative of order α.

We recall some basic definitions and results. For $\alpha > 0$, choose $n = [\alpha] + 1$ in case α in not an integer and $n = \alpha$ in case α is an integer. The fractional order integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the integral converges. For a function $f \in C^{n}[0,1]$, the Caputo fractional derivative of order α is define by

$$(D^{\alpha}) f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

The following Lemmas gives some properties of fractional integrals.
Lemma 1 [2] For $\alpha, \beta > 0$, the following relation hold:

$$D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha-1}, \beta > n \text{ and } D^{\alpha} t^{k} = 0, k = 0, 1, 2, ..., n-1.$$

Lemma 2 [2] Fort $\beta \geq \alpha > 0$ and $f \in L_1[a,b]$, the following

$$D^{\alpha} I_{a+}^{\beta} f(t) = I_{a+}^{\beta-\alpha} f(t) \text{ holds almost everywhere on } [a,b]$$

and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.
Lemma 3 Let $\alpha > 0$ then

$$I^{\alpha}\, D_0^{\alpha} u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}, \text{ for } c_i \in R. \tag{2}$$

Lemma 4 [2] For $g(t) \in C(0,1)$, the homogenous fractional order differential equation $D_{0+}^{\alpha} g(t) = 0$ has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},\, c_i \in R, i = 1, 2, 3, ..., n. \tag{3}$$

We use the following notations for our convenience,

$$G_1(t,s) = -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1},$$

$$G_2(t,s) = \frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1} + \frac{t-\xi}{\Delta_2}\left(\frac{\mu}{\alpha}(\xi-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right) + \frac{1}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1},$$

$$G_3(t,s) = \frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1} + \frac{t-\xi}{\Delta_2}\frac{\mu}{\alpha}(\xi-s)^{\alpha-1} + \frac{1}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}$$

and

$$G_4(t,s) = \frac{1}{\Gamma(\alpha-p)}(1-s)^{\alpha-p-1}.$$

Lemma 5 Let $f \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, the BVP for fractional differential equation

$$\begin{cases} {}^{c}D_{0+}^{\alpha} u(t) + f(u(t)) = 0, \quad 1 < \alpha < 2, \\ u(\xi) = 0, \quad D^{p} u(1) - \mu u(\eta) = 0, \quad 0 < p < 1, \end{cases} \tag{4}$$

has a solution given by

$$u(t) = \int_0^1 G(t,s) f(u(s)) ds, \tag{5}$$

where

$$G(t,s) = \begin{cases} G_1(t,s) + G_2(t,s) & 0 \leq s \leq \eta \leq \xi \leq t \leq 1\ , \\ G_2(t,s), & 0 \leq t \leq s \leq \eta \leq \xi \leq 1\ , \\ G_3(t,s) & 0 \leq \eta \leq t \leq s \leq \xi \leq 1\ , \\ G_4(t,s) & 0 \leq \eta \leq \xi \leq t \leq s \leq 1, \\ G_1(t,s) + G_3(t,s) & 0 \leq \eta \leq s \leq \xi \leq t \leq 1\ , \\ G_1(t,s) + G_4(t,s) & 0 \leq \eta \leq \xi \leq s \leq t \leq 1\ . \end{cases}$$

Proof. Applying the integral operator I_0^α on the equation (4) and by the help of lemma 1, we get the following

$$u(t) = -I_0^\alpha y(t) + c_1 + c_2 t \tag{6}$$

Applying p order derivative on (6) we have

$${}^cD_{0+}^p u(t) = -I^{\alpha-p} y(t) + c_2 \frac{t^{1-p}}{\Gamma(2-p)} \tag{7}$$

by the boundary conditions in (4) we have

$$c_1 = I^\alpha y(\xi) - \frac{\xi}{\Delta}(\mu I^\alpha(y(\xi) - y(\eta)) + I^{\alpha-p} y(1))$$

and

$$c_2 = \frac{1}{\Delta}(I^{\alpha-p} y(1) + \mu I^\alpha(y(\xi) - y(\eta)))$$

where $\Delta = \frac{1}{\Gamma(2-p)} - \eta\mu + \xi\mu > 0$. Substituting the values of c_1, c_2 in (6), we have

$$u(t) = -I^\alpha y(t) + I^\alpha y(\xi) + \frac{t-\xi}{\Delta}(\mu I^\alpha(y(\xi) - y(\eta)) + I^{\alpha-p} y(1)) \tag{8}$$

which can be written as

$$\begin{aligned} u(t) =& -\frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^\xi \frac{y(s)}{(\xi-s)^{1-\alpha}} ds \\ &+ \frac{t-\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi \frac{y(s)}{(\xi-s)^{1-\alpha}} ds - \int_0^\eta \frac{y(s)}{(\eta-s)^{1-\alpha}} ds) \\ &+ \frac{1}{\Gamma(\alpha-p)} \int_0^1 \frac{y(s)}{(1-s)^{1-\alpha+p}} ds) = \int_0^1 G(t,s) f(u(s)) ds. \end{aligned} \tag{9}$$

2. MAIN RESULTS

We consider the space $E = \{u(t) \in C[0,1] : u'(t) \in C[0,1]\}$ with the norm defined by

$$\|u\|_1 = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)|. \tag{10}$$

E is a Banach space [20]. For convenience, use the following notations

$$\begin{aligned} p_1 =& (\int_0^1 G(t,s) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^1 (\xi-s)^{\alpha-1} \\ &+ \int_0^\eta (\eta-s)^{\alpha-1}) + \frac{1}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1}))m(s), \end{aligned} \tag{11}$$

$$\begin{aligned} p_2 =& \frac{t^\alpha + \xi^\alpha}{\Gamma(\alpha+1)} + \frac{t+\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha + \eta^\alpha) + \frac{1}{\Gamma(\alpha-p+1)}) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &+ \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha + \eta^\alpha) + \frac{1}{\Gamma(\alpha-p+1)}), \end{aligned} \tag{12}$$

$$\lambda_1 = \frac{t^\alpha + \xi^\alpha}{\Gamma\alpha+1} + \frac{t-\xi}{\Delta_2}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha + \eta^\alpha) + \frac{1}{\Gamma\alpha-p+1}), \tag{13}$$

$$\lambda_2 = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\eta^\alpha + \xi^\alpha), \tag{14}$$

and $\varpi = \lambda_1+\lambda_2$. Choose $\mathcal{R} \geq \max\{2p_1, (2kp_2)^{\frac{1}{1-\delta}},\}$, where $k_1 = \max_{t\in[0,1]} a_1(t)$, $k_2 = \max_{t\in[0,1]} a_2(t)$ and consider a closed bounded subset $U = \{u(t) \in E : \|u\|_1 \leq \mathcal{R}, t \in [0,1]\}$ of E.

Assume that the following growth conditions hold:

(A1) $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous.

(A2) There exists a nonnegative function $m(t) \in L_1(I)$ such that

$$|f(u(t))| \leq m(t) + a|u(t)|^{\delta},$$

where $a \in \mathbb{R}$ are nonnegative constants and $0 < \delta < 1$.

(A3) There exists a nonnegative function $m(t) \in L_1(I)$ such that

$$|f(u(t))| \leq m(t) + a|u|^{\delta},$$

where $a \in \mathbb{R}$ are nonnegative constants and $\delta > 1$.

(A4) There exists constant $\zeta > 0$, such that

$$|f(u(t)) - f(v(t))| \leq \zeta(|u(t) - v(t)|),$$

for each $t \in I$ and u, v, are real valued functions of t.

Lemma 6 Assume that $(A1)$ holds then the function $u(t) \in E$ is the solution of the fractional boundary value problem (1) if and only if $\mathcal{T}u(t) = u(t)$, for all $t \in [0,1]$.

Proof. Let $u(t)$ be solution of (1) and

$$v(t) = \int_0^1 G(t,s)f(u(s))ds$$

by (8) we have

$$v(t) = -I^{\alpha}y(t) + I^{\alpha}y(\xi) + \frac{t-\xi}{\Delta}(\mu I^{\alpha}(y(\xi) - y(\eta)) + I^{\alpha-p}y(1)). \tag{15}$$

Applying ${}^cD^{\alpha}_{0+}$ on (15) and using lemma (1), we have

$$\begin{aligned} {}^cD^{\alpha}_{0+}v(t) =& {}^cD^{\alpha}_{0+}(-I^{\alpha}y(t) + I^{\alpha}y(\xi) + \frac{t-\xi}{\Delta}(\mu I^{\alpha}(y(\xi) - y(\eta)) + I^{\alpha-p}y(1))) \\ =& -f(u(t)) \end{aligned} \tag{16}$$

and it is easy to check the boundary conditions. □

Theorem 7 Assume that $(A1)$, $(A2)$ hold. Then the problem (1) has a solution.

Proof. Define an operator $\mathcal{T} : E \to E$ by

$$\mathcal{T}(u(t)) = \int_0^1 G(t,s)f(u(s))ds,\ t \in [0,1]. \tag{17}$$

By the continuity of f and G, we claim the continuity of $\mathcal{T}$. Here we show that $\mathcal{T}: U \to U$. Let $u(t) \in U$, we have

$$\begin{aligned}
|\mathcal{T}u(t)| &= |\int_0^1 G(t,s)y(s)ds| \le |\int_0^1 G(t,s)(m(t)+k|u|^\delta)ds| \\
&\le \int_0^1 G(t,s)m(t)ds + k|u|^\delta \int_0^1 G(t,s)ds \\
&\le \int_0^1 G(t,s)m(t)ds + k|u|^\delta\{\int_0^t (G_1(t,s)+G_2(t,s))ds + \int_t^\eta G_2(t,s)ds \\
&+ \int_\eta^\xi G_3(t,s)ds + \int_\xi^1 G_4(t,s)ds\} \\
&\le \int_0^1 G(t,s)m(t)ds + k|u|^\delta\{\frac{t^\alpha+\xi^\alpha}{\Gamma(\alpha+1)} + \frac{t+\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) \\
&+ \frac{1}{\Gamma(\alpha-p+1)})\}
\end{aligned} \tag{18}$$

from (17), we have

$$(\mathcal{T}u)'(t) = -I^{\alpha-1}y(t) + \frac{1}{\Delta}\big(\mu I^\alpha(y(\xi)-y(\eta)) + I^{\alpha-p}y(1)\big). \tag{19}$$

and

$$\begin{aligned}
|\mathcal{T}'u(t)| &= |\frac{d}{dt}\int_0^1 G(t,s)f(u(s))ds| \\
&\le \frac{1}{\Gamma(\alpha-1)}(\int_0^t (t-s)^{\alpha-2}m(s)ds + k|u|^\delta \int_0^t (t-s)^{\alpha-2}ds) \\
&+ \frac{1}{\Delta}\{\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi (\xi-s)^{\alpha-1}m(s)ds + k|u|^\delta \int_0^\xi (\xi-s)^{\alpha-1}ds \\
&+ \int_0^\eta (\eta-s)^{\alpha-1}m(s)ds + k|u|^\delta \int_0^\eta (\eta-s)^{\alpha-1}ds) \\
&+ \frac{1}{\Gamma(\alpha-p)}(\int_0^1 (1-s)^{\alpha-p-1}m(s)ds + k|u|^\delta \int_0^1 (1-s)^{\alpha-p-1}ds)\} \\
&\le (\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi (\xi-s)^{\alpha-1} + \int_0^\eta (\eta-s)^{\alpha-1}) \\
&+ \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}))m(s)ds + k|u|^\delta(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) \\
&+ \frac{1}{\Gamma(\alpha-p+1)})
\end{aligned} \tag{20}$$

from (18) and (20), we have

$$
\begin{aligned}
\|\mathcal{T}u(t)\|_1 \leq & \int_0^1 G(t,s)m(s)ds + k|u|^\delta\{\frac{t^\alpha+\xi^\alpha}{\Gamma(\alpha+1)} + \frac{t+\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) \\
& + \frac{1}{\Gamma(\alpha-p+1)})\} + (\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi (\xi-s)^{\alpha-1} \\
& + \int_0^\eta (\eta-s)^{\alpha-1}) + \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1})))m(s)ds + k|u|^\delta(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\
& + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) + \frac{1}{\Gamma(\alpha-p+1)}) \\
& = (\int_0^1 G(t,s) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^1 (\xi-s)^{\alpha-1} \\
& + \int_0^\eta (\eta-s)^{\alpha-1}) + \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1})))m(s)ds + k|u|^\delta(\frac{t^\alpha+\xi^\alpha}{\Gamma(\alpha+1)} \\
& + \frac{t+\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) + \frac{1}{\Gamma(\alpha-p+1)}) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\
& + \frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha) + \frac{1}{\Gamma(\alpha-p+1)})
\end{aligned} \tag{21}
$$

by the use of (11), (12) and (21), we have

$$\|\mathcal{T}u(t)\|_1 \leq p_1 + k|\mathcal{R}|^\delta p_2 \leq \frac{\mathcal{R}}{2} + \frac{\mathcal{R}}{2} = \mathcal{R}.$$

which implies that $\mathcal{T}: U \to U$. Now we show that $\mathcal{T}$ is completely continuous operator. Let $t > \tau$ and $\mathcal{M} = \max\{|f(u(t))| : t \in [0,1], u \in U\}$, we have

$$
\begin{aligned}
|\mathcal{T}u(t) - \mathcal{T}u(\tau)| &= |\int_0^1 G(t,s)f(u(s))ds - \int_0^1 G(\tau,s)f(u(s))ds| \\
&\leq \mathcal{M}|\int_0^1 G(t,s)ds - \int_0^1 G(\tau,s)ds| \\
&= \mathcal{M}(\frac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{\alpha-1}ds - \int_0^\tau (\tau-s)^{\alpha-1}ds) \\
&+ \frac{t-\tau}{\Delta}\{\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi (\xi-s)^{\alpha-1}ds) \\
&- \int_0^\eta (\eta-s)^{\alpha-1}ds + \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}ds\}) \\
&= \mathcal{M}(\frac{t^\alpha-\tau^\alpha}{\Gamma(\alpha+1)} + \frac{t-\tau}{\Delta}(\frac{1}{\Gamma(\alpha-p+1)} + \frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha-\eta^\alpha)))
\end{aligned}
$$

and

$$|\mathcal{T}'u(t) - \mathcal{T}'u(\tau)| = |-I^{\alpha-1}(f(u(t)) - f(u(\tau)))| \leq \mathcal{M}(\frac{t^{\alpha-1}-\tau^{\alpha-1}}{\Gamma(\alpha)})$$

thus

$$\begin{aligned}\|\mathcal{T}u(t)-\mathcal{T}u(\tau)\|_1 \leq \mathcal{M}(&\frac{t^\alpha-\tau^\alpha}{\Gamma(\alpha+1)}+\frac{t-\tau}{\Delta}(\frac{1}{\Gamma(\alpha-p+1)}+\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha-\eta^\alpha)))\\ &+\frac{\mathcal{M}}{\Gamma(\alpha)}(t^{\alpha-1}-\tau^{\alpha-1}).\end{aligned} \tag{22}$$

since the functions$t^{\alpha-1},\tau^{\alpha-1},t^\alpha,\tau^\alpha$, are uniformly continuous on the interval [0,1], it follows that $\mathcal{T}$ is equicontinuous and by Arzela-Ascoli theorem $\mathcal{T}$ is completely continuous. By Schauder fixed point theorem $\mathcal{T}$ has a fixed point.
Lemma 8 Assume that $(A1)$, $(A3)$ hold. Then the boundary value problem (1) has a solution.

Proof. The proof is similar like theorem 2, so we exclude the proof. □

Theorem 9 Assume that $(A1)$, $(A4)$ hold. If $\zeta\varpi<1$ then the problem (1) has a unique solution.

Proof. By the help of our supposition $(A4)$, we have the following estimates By the help of our supposition $(A4)$, we have the following estimates

$$\begin{aligned}|\mathcal{T}(u(t))-\mathcal{T}(v(t))| &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(u(s))-f(v(s))|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}|f(u(s))-f(v(s))|ds\\ &+\frac{t-\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi(\xi-s)^{\alpha-1}|f(u(s))-f(v(s))|ds\\ &+\int_0^\eta(\eta-s)^{\alpha-1}|f(u(s))-f(v(s))|ds)\\ &+\frac{1}{\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f(u(s))-f(v(s))|ds)\\ &\leq\frac{t^\alpha+\xi^\alpha}{\Gamma(\alpha+1)}\zeta(|u-v|)+\frac{t-\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha)\zeta(|u-v|)\\ &+\frac{1}{\Gamma(\alpha-p+1)}\zeta(|u-v|))\\ &\leq(\frac{t^\alpha+\xi^\alpha}{\Gamma(\alpha+1)}+\frac{t-\xi}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha)+\frac{1}{\Gamma(\alpha-p+1)})\zeta(|u-v|)\end{aligned} \tag{23}$$

using (13) and (23), we have

$$|\mathcal{T}(u(t))-\mathcal{T}(v(t))|\leq\lambda_1\zeta(|u-v|) \tag{24}$$

by the help of (19), we have

$$\begin{aligned}
|\mathcal{T}'(u(t))-\mathcal{T}'(v(t))| &= |-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}|f(u(s))-f(v(s))|ds \\
&+\frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha)}(\int_0^\xi (\xi-s)^{\alpha-1}|f(u(s))-f(v(s))|ds \\
&-\int_0^\eta (\eta-s)^{\alpha-1}|f(u(s))-f(v(s))|ds) \\
&+\frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}|f(u(s))-f(v(s))|ds)| \\
&\le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\zeta(|u-v|)+\frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\xi^\alpha+\eta^\alpha)+\frac{1}{\Gamma(\alpha-p+1)})\zeta(|u-v|) \\
&\le \zeta(|u-v|)(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Delta}(\frac{\mu}{\Gamma(\alpha+1)}(\eta^\alpha+\xi^\alpha))+\frac{1}{\Gamma(\alpha-p+1)})
\end{aligned} \tag{25}$$

using (14) and (25), we have

$$|\mathcal{T}'(u(t))-\mathcal{T}'(v(t))| = \lambda_2\zeta(|u-v|) \tag{26}$$

thus by the help of (24) and (26), we have

$$\begin{aligned}
\|\mathcal{T}(u)-\mathcal{T}(v)\|_1 &= \max_{t\in[0,1]}|\mathcal{T}(u)-\mathcal{T}(v)|+\max_{t\in[0,1]}|\mathcal{T}'(u)-\mathcal{T}'(v)| \\
&\le \lambda_1\zeta(|u-v|)+\lambda_2\zeta(|u-v|) \\
&= \varpi\zeta\|u-v\|_1
\end{aligned} \tag{27}$$

Thus, by contraction mapping principle the boundary value problem (1) has a unique solution.

Example 1

$$\begin{aligned}
lcllD_t^{\frac{3}{2}}u(t) &= \frac{u(t)}{35(5+7|u(t)|)} \\
u(\tfrac{1}{2})=0, \quad D^{\frac{1}{3}}u(1) &= \tfrac{1}{10}u(\tfrac{1}{3})
\end{aligned} \tag{28}$$

For the unique solution of problem (28), we apply theorem (2) with

$$f(u(t)) = \frac{u(t)}{35(5+7|u(t)|)},$$

$t\in[0,1]$, $u(t),\in[0,\infty)$, $\alpha=\frac{3}{2}$, $p=\frac{1}{3}$, $\mu=\frac{1}{10}$, $\eta=\frac{1}{3}$ and for $u=u(t), v=v(t)$ we have that $|f(u(t))-f(v(t))|\le\frac{1}{5}\{|u-v|\}$ and thus condition (A4), is satisfied. By computation we have $\varpi=2.6250$ and $\zeta\varpi=.5250<1$. Thus by theorem (2), the problem (28) has a unique solution.

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