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SOME INEQUALITIES OF HADAMARD TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE H-CONVEX VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we establish some Hadamard type inequalities involving Riemann-Liouville fractional integrals for mappings whose second derivatives are h-convex.

1. Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a < b$ we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

This remarkable results is well known in the literature as the Hermite-Hadamard inequality.

In 1978, Breckner in [1] introduced an s-convex function as a generalization of a convex function. Such a function is defined in the following way: a function $f : [0, \infty) \to R$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

hold for all $x, y \in [0, \infty]$, $t \in [0, 1]$ and for fixed $s \in [0, 1]$.

Dragomir and Fitzpatrick [3] proved the following variant of Hermite-Hadamard inequality for s-convex functions:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{3}$$

In 2007, Varošanec in [9] introduced a large class of non-negative functions, the so-called h-convex functions. This class contains several well-known classes of functions such as non-negative convex functions. This class is defined in the

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following way: a non-negative function $f : I \longrightarrow R, \emptyset \neq I \subset R$ is an interval, is called h-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{4}$$

holds for all $x, y \in I$, $t \in (0,1)$, where $h : J \longrightarrow R$ is non-negative function, $h \not\equiv 0$ and J is an interval, $(0,1) \subseteq J$.

In 2008, Sarikaya, Saglam and Yildirim [7] proved that for h-convex function the following variant of the Hermite-Hadamard inequality is fulfilled:

$$\frac{1}{2h\left(\frac{1}{2}\right)} f(\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)d(x) \leq [f(a)+f(b)] \cdot \int_0^1 h(t)dt \tag{5}$$

For recent results, refinement, generalizations and new Hermite-Hadamard type inequalities see [2, 4, 5, 6].

In 2013, Sarikaya, Set, Yaldiz and Basak [8] establish the following Hermite-Hadamard inequalities for Riemann-Liouville fractional integral

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{6}$$

where f is convex function and the symbols $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integral of the order $a \geq 0$ that are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \qquad (a < x), \tag{7}$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \qquad (x < b), \tag{8}$$

respectively. Here $\Gamma(\cdot)$ is the gamma function.

The aim of this paper is to establish Hermite-Hadamard inequalities for Riemann-Liouville fractional integral for mappings whose second derivatives are h-convex.

2. Main Results

To prove our main results, we consider the following lemma.

Lemma 1 Let $f : I \longrightarrow R$ be a differentiable mapping in the interior I° where $a, b \in I$ with $a < b$. If $f^{''} \in L[a,b]$ (the space of integrable functions), then the following equality holds:

$$\begin{aligned}(\alpha+1)\Gamma(\alpha+1)&\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - 2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\\ &= \left(\frac{b-a}{2}\right)^{\alpha+2}\left[\int_0^1 t^{\alpha+1}f^{''}\left(t\frac{a+b}{2} + (1-t)a\right)dt\right.\\ &\qquad \left. + \int_0^1 (1-t)^{\alpha+1}f^{''}\left(tb + (1-t)\frac{a+b}{2}\right)dt\right] \end{aligned} \tag{9}$$

Proof. By integration by parts and by the change of the variables, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha+1} f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)dt = \frac{2}{b-a}f^{'}\left(\frac{a+b}{2}\right) \\
&\quad -\frac{2(\alpha+1)}{b-a}\int_0^1 t^{\alpha} f^{'}\left(t\frac{a+b}{2}+(1-t)a\right)dt = \frac{2}{b-a}f^{'}\left(\frac{a+b}{2}\right) \\
&\quad -\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{4\alpha(\alpha+1)}{(b-a)^2}\int_0^1 t^{\alpha-1} f\left(t\frac{a+b}{2}+(1-t)a\right)dt \\
&= \frac{2}{b-a}f^{'}\left(\frac{a+b}{2}\right)-\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\alpha(\alpha+1)}{(b-a)^{\alpha+2}}\int_a^{\frac{a+b}{2}}(u-a)^{\alpha-1}f(u)du \\
&= \frac{2}{b-a}f^{'}\left(\frac{a+b}{2}\right)-\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\alpha(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)
\end{aligned} \tag{10}$$

Similarly, by integration by parts and by the change of the variables, we have

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha+1} f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)dt = \frac{-2}{b-a}f^{'}\left(\frac{a+b}{2}\right) \\
&\quad +\frac{2(\alpha+1)}{b-a}\int_0^1 (1-t)^{\alpha} f^{'}\left(tb+(1-t)\frac{a+b}{2}\right)dt = \frac{-2}{b-a}f^{'}\left(\frac{a+b}{2}\right) \\
&\quad -\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{4\alpha(\alpha+1)}{(b-a)^2}\int_0^1 (1-t)^{\alpha-1} f\left(tb+(1-t)\frac{a+b}{2}\right)dt \\
&= \frac{-2}{b-a}f^{'}\left(\frac{a+b}{2}\right)-\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\alpha(\alpha+1)}{(b-a)^{\alpha+2}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)du \\
&= \frac{-2}{b-a}f^{'}\left(\frac{a+b}{2}\right)-\frac{4(\alpha+1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\alpha(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)
\end{aligned} \tag{11}$$

From (10) and (11), we get (9). This completes the proof.

Theorem 1. Let $f : I \subset [0, \infty) \to R$ be a differentiable mapping on I° such that $f^{''} \in L[a, b]$, where $a, b \in I$, with $a < b$. If $|f^{''}|$ is h-convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
&\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \left(\frac{b-a}{2}\right)^{\alpha+2}\left[2\left|f^{''}\left(\frac{a+b}{2}\right)\right|\int_0^1 t^{\alpha+1}h(t)dt+(|f^{''}(a)|\right. \\
&\qquad \left. +|f^{''}(b)|)\int_0^1 (1-t)^{\alpha+1}h(t)dt\right]
\end{aligned} \tag{12}$$

Proof. From Lemma 1, using the h-convexity of $|f^{''}|$, we have

$$\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\le\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\int_0^1 t^{\alpha+1}\left|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.$$
$$\left.+\int_0^1(1-t)^{\alpha+1}\left|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$
$$\le\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\int_0^1 t^{\alpha+1}\left(h(t)\left|f^{''}\left(\frac{a+b}{2}\right)\right|+h(1-t)|f^{''}(a)|\right)dt\right.$$
$$\left.+\int_0^1(1-t)^{\alpha+1}\left(h(t)|f^{''}(b)|+h(1-t)\left|f^{''}\left(\frac{a+b}{2}\right)\right|dt\right]$$
$$=\left(\frac{b-a}{2}\right)^{\alpha+2}\left[2\left|f^{''}\left(\frac{a+b}{2}\right)\right|\int_0^1 t^{\alpha+1}h(t)dt+\right.$$
$$\left.(|f^{''}(a)|+|f^{''}(b)|)\int_0^1(1-t)^{\alpha-1}h(t)d(t)\right] \quad (13)$$

this proves inequality (12) and thus the proof is completed.

Corrolary 1. If in Theorem 1 we take $h(t)=t$ then the inequality (12) reduces to the following inequality for the convex function:

$$\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\le\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\left|f^{''}\left(\frac{a+b}{2}\right)\right|\frac{2}{\alpha+3}+(|f^{''}(a)|+|f^{''}(b)|)\frac{2-\alpha}{2\alpha}\right]. \quad (14)$$

Corrolary 2. If in Theorem 1 we take $h(t)=t^s$ then the inequality (12) reduces to the following inequality for the s-convex function:

$$\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\le\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\left|f^{''}\left(\frac{a+b}{2}\right)\right|\frac{2}{\alpha+s+2}+(|f^{''}(a)|+|f^{''}(b)|)\frac{\Gamma(s+1)\Gamma(\alpha)}{\Gamma(\alpha+s+1)}\right]. \quad (15)$$

Theorem 2. Let $f:I\subset[0,\infty)\longrightarrow R$ be a differentiable mapping on I° such that $f^{''}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f^{''}|^q$ is h-convex on $[a,b]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ then the following inequality holds:

$$\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\le\left(\frac{b-a}{2}\right)^{\alpha+2}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^q+|f^{''}(a)|^q\right)^{\frac{1}{q}}+\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^q+|f^{''}(b)|^q\right)^{\frac{1}{q}}\right] \quad (16)$$

Proof. From Lemma 1 and the Hőlder inequality, we have

$$\begin{aligned}
&\left|(a+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|\\
&\leq\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\int_0^1 t^{\alpha+1}\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.\\
&\quad\left.+\int_0^1(1-t)^{\alpha+1}\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\
&\leq\left(\frac{b-a}{2}\right)^{\alpha+2}\left[\left(\int_0^1 t^{\alpha p+p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1(1-t)^{\alpha p+p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Because $|f''|^q$ is h-convex, we have

$$\begin{aligned}
&\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\\
&\leq\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(t)dt+|f''(a)|^q\int_0^1 h(1-t)dt\\
&=\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q\right]\int_0^1 h(t)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\\
&\leq\left|f''(b)\right|^q\int_0^1 h(t)dt+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(1-t)dt\\
&=\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]\int_0^1 h(t)dt
\end{aligned}$$

Using the fact

$$\int_0^1 t^{\alpha p+p}dt=\frac{1}{\alpha p+p+1}$$

and

$$\int_0^1(1-t)^{\alpha p+p}dt=\frac{1}{\alpha p+p+1}$$

and using the last two inequalities we obtain (16). This completes the proof of the theorem.

Corrolary 3. If in Theorem 2 we take $h(t)=t$ then the inequality (16) reduces to the following inequality for the convex function:

$$\begin{aligned}&\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|\\&\leq\left(\frac{b-a}{2}\right)^{\alpha+2}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\\&\times\left[\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}+|f^{''}(a)|^{q}\right)^{\frac{1}{q}}+\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}+|f^{''}(b)|^{q}\right)^{\frac{1}{q}}\right]. \quad (17)\end{aligned}$$

Corrolary 4. If in Theorem 2 we take $h(t)=t^{s}$ then the inequality (16) reduces to the following inequality for the s-convex function:

$$\begin{aligned}&\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|\\&\leq\left(\frac{b-a}{2}\right)^{\alpha+2}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\&\times\left[\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}+|f^{''}(a)|^{q}\right)^{\frac{1}{q}}+\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}+|f^{''}(b)|^{q}\right)^{\frac{1}{q}}\right]. \quad (18)\end{aligned}$$

Theorem 3. Let $f:I\in[0,\infty)\longrightarrow R$ be a differentiable mapping on I° such that $f^{''}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f^{''q}$, $q\geq 1$ is h-convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}&\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|\\&\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}t^{\alpha+1}h(t)dt\right.\right.\\&\left.+|f^{''}(a)|^{q}\int_{0}^{1}t^{\alpha+1}h(1-t)dt\right)^{\frac{1}{q}}\\&\left.+\left(\left|f^{''}(b)\right|^{q}\int_{0}^{1}(1-t)^{\alpha+1}h(t)dt+\left|f^{''}\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}t^{\alpha+1}h(t)dt\right)^{\frac{1}{q}}\right] \quad (19)\end{aligned}$$

Proof. From Lemma 1 and the power mean inequality, we have that the following inequality holds:

$$\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\left[\int_0^1 t^{\alpha+1}\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.$$
$$\left.+\int_0^1(1-t)^{\alpha+1}\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$
$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\left[\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha+1}\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^{\alpha+1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha+1}\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

By the h-convexity of $|f''|^q$, we have

$$\int_0^1 t^{\alpha+1}\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt$$
$$\leq\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^{\alpha+1}h(t)dt+|f''(a)|^q\int_0^1 t^{\alpha+1}h(1-t)dt$$

and

$$\int_0^1(1-t)^{\alpha+1}\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt$$
$$\leq|f''(b)|^q\int_0^1(1-t)^{\alpha+1}h(t)dt+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1(1-t)^{\alpha+1}h(1-t)dt$$
$$=|f''(b)|^q\int_0^1(1-t)^{\alpha+1}h(t)dt+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^{\alpha+1}h(t)dt$$

Using the fact that

$$\int_0^1 t^{\alpha+1}dt=\int_0^1(1-t)^{\alpha+1}dt=\frac{1}{\alpha+2}$$

and the last two inequalities in we obtain (19). This completes the proof.

Corrolary 5. If in Theorem 3 we take $h(t)=t$ then the inequality (19) reduces to the following inequality for the convex function:

$$\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{\alpha+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q-\frac{1}{(\alpha+1)(\alpha+2)}|f''(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{\alpha+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q-\frac{1}{(\alpha+1)(\alpha+2)}|f''(b)|^q\right)^{\frac{1}{q}}\right]. \quad (20)$$

Corrolary 6. If in Theorem 3 we take $h(t) = t^s$ then the inequality (19) reduces to the following inequality for the s-convex function:

$$\left|(\alpha+1)\Gamma(\alpha+1)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-2\left(\frac{b-a}{2}\right)^{\alpha}(\alpha+1)f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{\alpha+s+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$\left.+\frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+3)}|f''(a)|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{\alpha+s+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right]. \quad (21)$$

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