*Journal of Fractional Calculus and Applications*,
Vol. 6(1) Jan. 2015, pp. 120-130.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

# ON THE SOLUTION OF A GENERALIZED FRACTIONAL ORDER INTEGRAL EQUATION AND SOME APPLICATIONS

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Abstract. Here, we introduce a generalized fractional order integral equation and study the existence of solutions for this equation. Existence of maximal and minimal solutions for this integral equation will be proved. As an application we present some comparison theorems and inequalities.

## 1. Introduction

The field of fractional calculus is almost as old as calculus itself , but over the last decades the usefulness of this mathematical theory in applications as well as its merits in pure mathematics has become more and more evident. Recently a number of textbooks [16], [14], [17], [19] have been published in this field.

**Definition 1.** *The fractional-order integral of order $\beta$ of the function $f$ is defined on $[a, b]$ by (see* [7], [17], [16] *and* [19]*)*

$$I_a^\beta f(t) \; = \; \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\; ds, \; t > a \tag{1}$$

*and when $a = 0$, we have $I^\beta f(t) = I_0^\beta f(t), \; t > 0$.*

**Definition 2.** *The Riemann-Liouville fractional-order derivative of order $\beta \in (0, 1)$ of the function $f$ is given by (see* [7], [17], [16] *and* [19]*)*

$$_R D^\beta f(t) \; = \; \frac{d}{dt} I^{1-\beta} f(t).$$

An Erdélyi-Kober operator is a fractional integration operation introduced by Arthur Erdélyi (1940) and Hermann Kober (1940).

$$I_{t^m,a}^\alpha \; f(t) \; = \; \int_a^t \frac{(t^m \; - \; s^m)^{\alpha-1}}{\Gamma(\alpha)} \; m \; s^{m-1} \; f(s) \; ds.$$

The Erdélyi-Kober fractional integral is defined in many literature [1] and [3]-[5].
The aim of the short note [18] is to highlight that the generalized grey Brownian motion(ggBm) is an anomalous diffusion process driven by a fractional integral equation in the sense of ErdelyiKober, and for this reason here it is proposed to call such family of diffusive processes as Erdelyi Kober fractional diffusion. The ggBm is a parametric class of stochastic processes that provides models for both fast and slow anomalous diffusion.

2010 *Mathematics Subject Classification.* Primary 26A33; Secondary 45D05, 60G22, 33E30.
*Key words and phrases.* Fractional integration and derivative; Erdélyi-Kober fractional; Maximal and minimal solutions; Comparison theorem .
Submitted Sep. 10, 2014.

For the properties of Erdélyi-Kober operators see [1], [7] and [19] for example.

Now, we shall introduce the generalized fractional-order operators as (see [19]and [8])

$$I^{\alpha}_{a,\ \phi}\ f(t)\ =\ \int_a^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \phi^{'}(s)\ f(s)\ ds \tag{2}$$

and

$$D^{\alpha}_{a,\ \phi}\ f(t)\ =\ (\frac{1}{\phi'(t)}\frac{d}{dt})\int_a^t \frac{(\phi(t)\ -\ \phi(s))^{-\alpha-1}}{\Gamma(-\alpha)}\ \phi^{'}(s)\ f(s)\ ds,\ \phi^{'}(t)\ \neq\ 0,\ a<t<b \tag{3}$$

defined for any monotonic increasing function $\phi(t)\ \geq\ 0,$ having a continuous derivative. The integral $I^{\alpha}_{a,\ \phi}$ is usually called a fractional integral of a function $f(t)$ by a function $\phi(t)$ of order $\alpha\ >\ 0,$ when $a\ =\ 0$ we denote $I^{\alpha}_{a,\ \phi}$ by $I^{\alpha}_{\phi}$ .
If $\phi^{'}(t)\ \neq\ 0,\ a<t<b$ then the operator $I^{\alpha}_{\phi}$ is easily expressed via the usual Riemann-Liouville fractional integration (see [17]). So many properties of the operator $I^{\alpha}_{\phi},$ in particular the semigroup property

$$I^{\alpha}_{\phi}\ I^{\beta}_{\phi}\ f(t)\ =\ I^{\alpha\ +\ \beta}_{\phi}\ f(t)$$

follows directly from the corresponding properties of the Riemann-Liouville fractional integral.

When $\phi(t)\ =\ t$ we obtain the Riemann-Liouville fractional integral $I^{\alpha}_{a}$

$$I^{\alpha}_{a}\ f(t)\ =\ \int_a^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds.$$

When $\phi(t)\ =\ t^m,\ m>0$ we obtain the Erdelyi-Kober (see [1], [7], [8] and [19]) fractional order operator $I^{\alpha}_{t^m,a}$

Now, we shall denote by $L^1_{\phi}\ =\ L^1_{\phi}\ [a,b]$ the space of all real functions defined on $[a,b].$ such that $\phi'(t)\ f(t)\ \in\ L^1$ and $\int_a^b|\ \phi'(t)\ f(t)\ |\ dt\ \leq\ \infty.$ where $\phi$ is increasing function and absolutely continuous on $[a,b]$ and we introduce the norm

$$||\ f(t)\ ||_{L^1_{\phi}}\ =\ \int_a^b|\ \phi'(t)\ f(t)\ |\ dt\ t\in\ [a,b].$$

**Definition 3.** *the $\phi-$ fractional integral of order $\alpha\ \geq\ 0$ of the function $f(\ t\ )\ \in\ L^1_{\phi}$ is defined as*

$$I^{\alpha}_{a,\phi}\ f(t)\ =\ \int_a^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \phi'(s)\ f(s)\ ds.$$

$I^{\alpha}_{a,\phi}$ *may be known as the fractional integral of the function $f(t)$ with respect to $\phi(t).$*

We shall prove some properties of this integral operator.
Now, for the continuation in $L^1_{\phi}$ of the fractional integral to the usual ones we have the following lemmas.

**Lemma 1.** *If $f(t)\ \in\ L^1_{\phi},$ then*

$$\lim_{\alpha\ \to\ n}\ I^{\alpha}_{a,\phi}\ f(t)\ =\ I^{n}_{a,\phi}\ f(t)\ uniformly,\ n\ =\ 1,2,3,...$$

*where $I^1_{a,\phi}\ f(t)\ =\ \int_a^t\ \phi'(s)\ f(s)\ ds.$*

**proof:**
From the definition of $I^{\alpha}_{a,\phi},$ we have

$$|\ I^{\alpha}_{a,\phi}\ f(t)\ -\ I^{n}_{a,\phi}\ f(t)\ |\ \leq\ \int_a^t \left|\ \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\phi(t)-\phi(s))^{n-1}}{\Gamma(n)}\ \right|\ |\ \phi'(s)\ f(s)\ |\ ds.$$

Since $\phi$ is increasing and $\phi(t)-\phi(s)$ is positive, then let $\alpha = n-\frac{1}{p}$ we get

$$\begin{aligned}\frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} &= \frac{(\phi(t)-\phi(s))^{n-1-\frac{1}{p}}}{\Gamma(n-\frac{1}{p})}\\ &= \frac{(\phi(t)-\phi(s))^{n-1}}{\Gamma(n-\frac{1}{p})}\cdot\frac{1}{(\phi(t)-\phi(s))^{\frac{1}{p}}}\\ &\to \frac{(\phi(t)-\phi(s))^{n-1}}{\Gamma(n)}\ as\ p\ \to\ \infty.\end{aligned}$$

By similar way we can prove that

$$I^{\alpha}_{a,\phi}\ f(t)\ \to\ I^{1}_{a,\phi}\ f(t)\ as\ n\ \to\ 1.$$

**Lemma 2.** *$I^{\alpha}_{a,\phi}$ maps $L^1_{\phi}$ into itself continuously.*

**proof:**

Let $f(t)\ \in\ L^1_{\phi}$, we shall prove that $I^{\alpha}_{a,\phi}\ f(t)\ \in\ L^1_{\phi}$

$$|\ I^{\alpha}_{a,\phi}\ f(t)\ |\ \leq\ \int_a^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}\ |\ \phi'(s)\ f(s)\ |\ ds$$

$$\begin{aligned}||\ I^{\alpha}_{a,\phi}\ f(t)\ || &\leq \int_a^b\int_a^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}\ |\ \phi'(s)\ f(s)\ |\ ds\ dt\\ &\leq \int_a^b\int_s^b \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}\ |\ \phi'(s)\ f(s)\ |\ dt\ ds\\ &\leq \frac{(\phi(b))^{\alpha}}{\Gamma(\alpha+1)}\int_a^b |\phi'(s)\ f(s)\ |\ ds\\ &\leq \frac{(\phi(b))^{\alpha}}{\Gamma(\alpha+1)}\ ||\ f(t)\ ||_{L^1_{\phi}}\ \text{from definition of } L^1_{\phi}.\end{aligned}$$

**Lemma 3.** *Let $f(t)\ \in\ L_1$. If $f(t)$ is bounded and measurable on $[a,b]$, then*

$$I^{\beta}_{a,\phi}\ f(t)\ |_{t\ =\ a}\ =\ 0.$$

**Proof.**

Since $|\ f(t)\ |\ \leq\ M$, then

$$\begin{aligned}|\ I^{\beta}_{a,\phi}\ f(t)\ | &\leq \int_a^t \frac{(\phi(t)\ -\ \phi(s))^{\beta\ -\ 1}}{\Gamma(\beta)}\ |\ f(s)\ \phi'(s)|\ ds\\ &\leq M\int_a^t \frac{(\phi(t)\ -\ \phi(s))^{\beta\ -\ 1}}{\Gamma(\beta)}\ |\phi'(s)|\ ds\\ &= M\ \frac{(\phi(t)\ -\ \phi(a))^{\beta}}{\Gamma(\beta\ +\ 1)}\ \to\ 0\ as\ t\ \to\ a.\ \blacksquare\end{aligned}$$

**Example 1:**

If $f(t)=[\phi(t)-\phi(a)]^{\beta-1},\ \beta>0$. Then

$$I^{\alpha}_{a,\phi}f(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\phi(t)-\phi(a)]^{\alpha+\beta-1}$$

- $\star$ when $\phi(t)=t$, $f(t)=(t-a)^{\beta-1},\ \beta>0$. Then

$$I^{\alpha}_{a}f(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}$$

- $\star$ when $\phi(t)=t^m,\ m>0$, $f(t)=(t^m-a^m)^{\beta-1},\ \beta>0$. Then

$$I^{\alpha}_{a}f(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t^m-a^m)^{\alpha+\beta-1}$$

## 2. Existence theorem

It is well known that integral equations have many useful applications in describing numerous events and problems of real world, and the theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory. Many papers studied the fractional order integral equation

$$x(t) \ = \ a(t) \ + \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s,x(s)) \ ds, \ t \ \in \ I, \ \alpha > 0. \tag{4}$$

Here, we prove the existence of at least one continuous solution for the integral equation of fractional order

$$x(t) \ = \ a(t) \ + \ \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha-1}}{\Gamma(\alpha)} \ f(s,x(s)) \ \phi'(s) \ ds, \ t \ \in \ I=[0,1], \ \alpha > 0 \tag{5}$$

and the existence of a continuous solution of the nonlinear differential equation of fractional-order

$${}_RD^{\alpha}x(t) \ = \ f(t,x(t)), \ t \ \in \ I \ \text{ and } x(0) \ = \ 0, \ \alpha \in (0,1) \tag{6}$$

(where ${}_RD^{\alpha}$ is the Riemann-Liouville fractional order derivative ) will be given as an application. Also the results concerning the existence of continuous solution of the initial value problem

$$\frac{dx(t)}{dt} \ = \ f(t,x(t)), \ x(0) \ = \ x_0, \tag{7}$$

will be given as another application.
Finally, the existence of maximal and minimal solutions of (5) will be proved.
These results extend the results obtained by El-Sayed et al... [6].
Now, Equation (5) will be investigated under the assumptions:

- (i) $a: \ I \to \mathbb{R}$ is continuous and bounded with $k_1 = \sup_{t\in I} |a(t)|$.
- (ii) $f: \ I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathèodory condition (i.e. measurable in $t$ for all $x: I \to \mathbb{R}$ and continuous in $x$ for all $t \in I$ ).
- (iii) There exists a function $m \in L_1$ such that $|f(t,x)| \leq m(t)$ ($\forall$ $(t,x) \in I \times \mathbb{R}$ ) and $k_2 = \sup_{t\in I} I^{\beta} \ m(t)$ for any $\beta \leq \alpha$.
- (iv) $\phi: \ I \to I$ be any monotonic increasing function having a continuous derivative.

**Theorem 1.** *Let the assumptions (i)-(iv) be satisfied. Then the fractional integral equation (5) has at least one solution in the space $C(I)$.*

**Proof.**
Let $C = C(I)$ be the Banach space of all real functions defined and continuous on the interval $I$. Fix a number $r > 0$ and consider the ball $S_r$ in the space $C(I)$ defined as

$$S_r = \{x \in C(I) : |x(t)| \leq r \text{ for } t \in I\}.$$

Let $T$ be the operator defined on $S_r$ by the formula

$$(Tx)(t) \ = \ a(t) \ + \ \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha-1}}{\Gamma(\alpha)} \ f(s,x(s)) \ \phi'(s) \ ds, \ x \in S_r, \ t \in I.$$

Then, in view of our assumptions, for $x \in S_r$ and $t \in I$ we get

$$\begin{aligned} | \ Tx(t) \ | \quad &\leq \quad |a(t)| \ + \ \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha-1}}{\Gamma(\alpha)} \ |f(s,x(s))| \ \phi'(s) \ ds \\ &\leq \quad k_1 \ + \ I_{\phi}^{\alpha-\beta} \ I_{\phi}^{\beta} \ m(t) \\ &\leq \quad k_1 \ + \ k_2 \ \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \ \phi'(s) \ ds \\ &\leq \quad k_1 \ + \ \frac{k_2}{\Gamma(\alpha \ - \ \beta \ + \ 1)}. \end{aligned}$$

Hence, in view of assumption (ii) we have that $T$ transforms the ball $S_r$ into itself for

$$r = k_1 + \frac{k_2}{\Gamma(\alpha - \beta + 1)}.$$

Now, for $t_1$ and $t_2 \in I$ (without loss of generality assume that $t_1 < t_2$ ), we have

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &= |a(t_2) - a(t_1) \\ &+ I_\phi^\alpha f(t_2, x(t_2)) - I_\phi^\alpha f(t_1, x(t_1))| \\ &\leq |a(t_2) - a(t_1)| + \end{aligned}$$

$$|I_\phi^\alpha f(t_2, x(t_2)) - I_\phi^\alpha f(t_1, x(t_1))| \leq |a(t_2) - a(t_1)| + |\int_0^{t_1} \frac{(\phi(t_2) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds$$

$$+ \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds - \int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds|$$

$$\leq |\, a(t_2) - a(t_1)| + |\int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds + \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds$$

$$- \int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \phi'(s)\, ds| \leq |\, a(t_2) - a(t_1)| + \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s))\, \phi'(s)|\, ds.$$

Then we get

$$|\, (Tx)(t_2) - (Tx)(t_1)\, | \leq |\, a(t_2) - a(t_1)\, | + (k_2\, \frac{(\phi(t_2) - \phi(t_1))^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)})$$

i.e.,

$$\begin{aligned} |\, (Tx)(t_2) - (Tx)(t_1)\, | &\leq |\, a(t_2) - a(t_1)\, | + k_2\, \frac{(\phi(t_2) - \phi(t_1))^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} \\ &\leq |\, a(t_2) - a(t_1)\, | + \frac{k_2}{\Gamma(\alpha - \beta + 1)}\, (\phi(t_2) - \phi(t_1))^{\alpha-\beta} \to 0 \text{ as } t_2 \to t_1. \end{aligned}$$

This means that the functions of $TS_r$ are equi-continuous on $I$. Then by the Arzela-Ascoli Theorem [2] the closure of $TS_r$ is compact .
It is clear that the set $S_r$ is nonempty, bounded, closed and convex.
Assumptions (ii) and (iv) imply that $T: S_r \to C(I)$ is a continuous operator in $x$.
Since all conditions of the Schauder fixed-point theorem hold, then $T$ has a fixed point in $S_r$. ■

## 3. Spacial cases

**Corollary 1.** *Let the assumptions of Theorem 1 be satisfied (with $\phi(t) = t$), then the fractional-order integral equation*

$$x(t) = a(t) + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds$$

*has at least one solution $x \in C$.*

**Corollary 2.** *Let the assumptions of Theorem 1 be satisfied (with $\phi(t) = t^m,\ m > 0$), then the fractional-order integral equation*

$$x(t) = a(t) + \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, m\, s^{m-1}\, ds$$

*has at least one solution $x \in C$.*

Now letting $\alpha,\ \beta \to 1$, we obtain

**Corollary 3.** *Let the assumptions of Theorem 1 be satisfied (with $a(t) = x_0$ and letting $\alpha,\ \beta \to 1$ ), then the integral equation*

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

*has at least one solution $x \in C$ which is equivalent to the mild solution to the initial value problem (7).*

## 4. Fractional order functional differential equations

For the initial value problem of the nonlinear fractional-order differential equation (6) we have the following theorem.

**Theorem 2.** *Let the assumptions of Theorem 1 be satisfied (with* $a(t) = 0,\ \phi(t) = t$ *), then the Cauchy problem (6) has at least one solution* $x \in C$.

**Proof.**
Integrating (6) we obtain the integral equation

$$x(t) \ = \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s, x(s)) \ ds, \ t \ \in \ I \tag{8}$$

which by Theorem 1 has the desired solution.
Operating with ${}_RD^{\alpha}$ on (8) we obtain the initial value problem (6). So the equivalence between the initial value problem(6) and the integral equation (8) is proved and then the results follow from Theorem 1.■

## 5. Maximal and minimal solutions

**Definition 4.** *( see* [9]*) Let* $q(t)$ *be a solution of (5). Then* $q(t)$ *is said to be a maximal solution of (5) if every solution of (5) on* $I$ *satisfies the inequality* $x(t) \ \leq \ q(t),\ t \in I$ *. A minimal solution* $s(t)$ *can be defined in a similar way by reversing the above inequality i.e.* $x(t) \ \geq \ s(t),\ t \in I$.

we need the following lemma to prove the existence of maximal and minimal solutions of (5).

**Lemma 4.** *Let* $f(t, x)$ *satisfies the assumptions in Theorem 1 and let* $x(t),\ y(t)$ *be continuous functions on I satisfying*

$$\begin{aligned} x(t) \quad &\leq \quad a(t) \ + \ I_\phi^\alpha f(t, x(t)) \\ y(t) \quad &\geq \quad a(t) \ + \ I_\phi^\alpha f(t, y(t)) \end{aligned}$$

*where one of them is strict.*
*Suppose* $f(t, x)$ *is nondecreasing function in* $x$. *Then*

$$x(t) \ < \ y(t). \tag{9}$$

**proof**
Let the conclusion (9) be false; then there exists $t_1$ such that

$$x(t_1) \ = \ y(t_1), \ t_1 \ > \ 0$$

and

$$x(t) \ < \ y(t), \ 0 \ < \ t \ < \ t_1.$$

From the monotonicity of the function $f$ in $x$, we get

$$\begin{aligned} x(t_1) \quad &\leq \quad a(t_1) \ + \ I_\phi^\alpha \ f(t_1, x(t_1)) \\ &= \quad a(t_1) \ + \int_0^{t_1} \frac{(\phi(t_1) \ - \ \phi(s))^{\alpha \ - \ 1}}{\Gamma(\alpha)} \ f(s, x(s)) \ \phi'(s) \ ds \\ &< \quad a(t_1) \ + \int_0^{t_1} \frac{(\phi(t_1) \ - \ \phi(s))^{\alpha \ - \ 1}}{\Gamma(\alpha)} \ f(s, y(s)) \ \phi'(s) \ ds \\ &< \quad y(t_1). \end{aligned}$$

This contradicts the fact that $x(t_1) \ = \ y(t_1)$; then

$$x(t) \ < \ y(t).$$

As particular cases of Lemma 4 we obtain the following lemmas:

**Lemma 5.** *Let $f(t,x)$ satisfies the assumptions in Theorem 1 and let $x(t)$, $y(t)$ be continuous functions on I satisfying*

$$\begin{aligned} x(t) &\leq a(t) + I^{\alpha}_{t^m} f(t,x(t)) \\ y(t) &\geq a(t) + I^{\alpha}_{t^m} f(t,y(t)) \end{aligned}$$

*where one of them is strict.*
*Suppose $f(t,x)$ is nondecreasing function in $x$. Then*

$$x(t) < y(t).$$

**Lemma 6.** [6] *Let $f(t,x)$ satisfies the assumptions in Theorem 1 and let $x(t)$, $y(t)$ be continuous functions on I satisfying*

$$\begin{aligned} x(t) &\leq a(t) + I^{\alpha} f(t,x(t)) \\ y(t) &\geq a(t) + I^{\alpha} f(t,y(t)) \end{aligned}$$

*where one of them is strict.*
*Suppose $f(t,x)$ is nondecreasing function in $x$. Then*

$$x(t) < y(t).$$

**Theorem 3.** *Let the assumptions of Theorem 1 be satisfied. Furthermore, if $f(t,x)$ is nondecreasing functions in $x$, then there exist maximal and minimal solutions of (5).*

**Proof**

Firstly, we shall prove the existence of maximal solution of (5). Let $\epsilon > 0$ be given. Now consider the fractional-order functional integral equation

$$x_\epsilon(t) = a(t) + I^{\alpha}_{\phi} f_\epsilon(t, x_\epsilon(t)), \tag{10}$$

where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon.$$

Clearly the function $f_\epsilon(t, x_\epsilon)$ satisfies assumptions (ii), (iii) and

$$| f_\epsilon(t, x_\epsilon) | \leq m(t) + \epsilon = m'(t).$$

Therefore, equation (10) has a continuous solution $x_\epsilon(t)$ according to Theorem 1.
Let $\epsilon_1$ and $\epsilon_2$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_1}(t) = a(t) + I^{\alpha}_{\phi} f_{\epsilon_1}(t, x_{\epsilon_1}(t)),$$

$$\begin{aligned} x_{\epsilon_1}(t) &= a(t) + I^{\alpha}_{\phi} (f(t, x_{\epsilon_1}(t)) + \epsilon_1), \\ &> a(t) + I^{\alpha}_{\phi} (f(t, x_{\epsilon_1}(t)) + \epsilon_2), \end{aligned} \tag{11}$$

$$x_{\epsilon_2}(t) = a(t) + I^{\alpha}_{\phi} (f(t, x_{\epsilon_2}(t)) + \epsilon_2). \tag{12}$$

Applying Lemma 4, then (11) and (12) imply

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \ for \ t \in I.$$

As shown before in the proof of Theorem 1, the family of functions $x_\epsilon(t)$ defined by (10) is uniformly bounded and of equi-continuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence $\epsilon_n$ such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $I$. We denote this limit by $q(t)$. From the continuity of the functions $f_{\epsilon_n}$ in the second argument, we get

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + I^{\alpha}_{\phi} f(t, q(t))$$

which proves that $q(t)$ is a solution of (5).
Finally, we shall show that $q(t)$ is maximal solution of (5). To do this, let $x(t)$ be any solution of (1). Then

$$\begin{aligned} x_\epsilon(t) &= a(t) + I^{\alpha}_{\phi} f_\epsilon(t, x_\epsilon(t)) \\ &> a(t) + I^{\alpha}_{\phi} f(t, x_\epsilon(t)). \end{aligned}$$

and

$$x(t) \ = \ a(t) \ + \ I_{\phi}^{\alpha} \ f(t, x(t)).$$

Applying Lemma 4, we get

$$x_{\epsilon}(t) \ > \ x(t) \ for \ t \ \in \ I.$$

from the uniqueness of the maximal solution (see [9], [15]), it is clear that $x_{\epsilon}(t)$ tends to $q(t)$ uniformly in $t \ \in \ I \ as \ \epsilon \to 0$.
In a similar way we can prove that there exists a minimal solution of (5). ■

## 6. Comparison theorems

An important technique is concerned with comparing a function satisfying an integral inequality of fractional-order by the maximal and the minimal solutions of the corresponding fractional-order integral equation. Some of the results that are widely used are the following comparison theorems:

**Theorem 4.** *Let the assumptions of Theorem 3 be satisfied and*

$$x(t) \ \leq \ a(t) \ + \ I_{\phi}^{\alpha} f(t, x(t)), \ t \ \geq \ 0, \tag{13}$$

*where* $x(t)$ *is continuous function on* $I$. *Suppose that* $q(t)$ *is the maximal solution of the fractional-order integral equation*

$$u(t) \ = \ a(t) \ + \ I_{\phi}^{\alpha} \ f(t, u(t)) \tag{14}$$

*existing on* $I$. *Then*

$$x(t) \ \leq \ q(t), \ t \ \in I.$$

**Proof.** The proof can be done by direct calculations.

**Theorem 5.** *Let the assumptions of Theorem 3 be satisfied and reversing inequality (13). Then*

$$x(t) \ \geq \ s(t),$$

*where* $s(t)$ *is the minimal solution of (14) on* $I$ .

**Proof.** The proof can be done by direct calculations.

## 7. Approximate solutions

Let us define an approximate solution of (5).

**Definition 5.** *Let* $x(t)$ *be continuous on* $I$ *and satisfies*

$$| \ x(t) \ - \ a(t) \ - \ I_{\phi}^{\alpha} \ f(t, x(t)) \ | \ \leq \ \delta(t),$$

*where* $\delta$ *is continuous on* $I$. *Then* $x(t)$ *is said to be a* $\delta-$*approximate solution of (5).*

The following theorem shows the difference between an approximate solution and any other solution of (5).

**Theorem 6.** *Let* $f(t, x), \ g(t, x)$ *satisfy the assumptions of Theorem 1,* $g(t, x)$ *is monotonic nondecreasing in* $x$ *for each* $t$, *and*

$$| \ f(t, x) \ - \ f(t, y) \ | \ \leq \ g(t, | \ x \ - \ y \ |). \tag{15}$$

*If* $x(t, \delta)$ *is a* $\delta-$*approximate solution of (5) and* $y(t)$ *is any solution of (5). Then*

$$| \ x(t, \delta) \ - \ y(t)| \ \leq \ q(t),$$

*where* $q(t)$ *is the maximal solution of*

$$u(t) \ = \ \delta(t) \ + \ I_{\phi}^{\alpha} \ g(t, u(t)).$$

**Proof.**
Consider the function

$$n(t) \;=\; |\; x(t,\delta) \;-\; y(t)\;|,$$

where $x(t,\delta)$ is $\delta-$approximate solution of (5) and $y(t)$ is any solution of (5). Then, using the definition of $\delta-$approximate solution and (15), we get

$$\begin{aligned} n(t) &= |\; x(t,\delta) \;-\; y(t)\;| \\ &= |x(t,\delta) \;-\; a(t) \;-\; I_\phi^\alpha\; f(t,y(t))| \\ &\le |\; x(t,\delta) \;-\; a(t) \;-\; I_\phi^\alpha\; f(t,x(t,\delta))| \;+\; I_\phi^\alpha |\; f(t,x(t,\delta)) \;-\; f(t,y(t))\;| \\ &\le \delta(t) \;+\; I_\phi^\alpha\; g(t, |\; x(t,\delta) \;-\; y(t)\;|) \\ &= \delta(t) \;+\; I_\phi^\alpha\; g(t,n(t)). \end{aligned}$$

By a direct application of Comparison Theorem 4, we get

$$n(t) \;=\; |\; x(t,\delta) \;-\; y(t)\;| \;\le\; q(t),\; t\; \ge 0. \blacksquare$$

The next theorem offers an estimate of the growth of solutions of (5).

**Theorem 7.** *Let $f(t,x)$ and $g(t,x)$ satisfy the assumptions in Theorem 6. If*

$$|f(t,x)| \le g(t,|x|); \tag{16}$$

*then*

$$|x(t)| \le q(t),\; t \ge 0,$$

*where $x(t)$ is any solution of (5) and $q(t)$ is the maximal solution of*

$$u(t) \;=\; h(t) \;+\; I_\phi^\alpha\; g(t,u(t)) \tag{17}$$

*such that $|\; a(t)\;| \;\le\; h(t),\; t \;\in\; I$.*

**Proof.**
If $n(t) \;=\; |\; x(t)\;|$ , we have by (16) the fractional-order integral inequality

$$\begin{aligned} n(t) \;=\; |\; x(t)\;| &\le |\; a(t)\;| \;+\; I_\phi^\alpha \;|\; f(t,x(t))\;| \\ &\le h(t) \;+\; I_\phi^\alpha\; g(t,|\; x(t)\;|) \\ &= h(t) \;+\; I_\phi^\alpha\; g(t,n(t)), \end{aligned}$$

and consequently, Comparison Theorem 4 gives

$$n(t) \;=\; |\; x(t)\;| \;\le\; q(t),\; t \;\ge\; 0,$$

where $q(t)$ is the maximal solution of (17). $\blacksquare$

**Theorem 8.** *Assume that:*

(i) *$f_1,\; f_2,\; g$ satisfy the assumptions of Theorem 1, $g(t,u)$ is monotonic nondecreasing in $u$ for each $t$, and*

$$|f_1(t,x) \;-\; f_2(t,y)| \;\le\; g(t,\; |x \;-\; y\;|); \tag{18}$$

(ii) *$x(t),\; y(t)$ are any two solutions of*

$$\begin{aligned} x(t) &= a_1(t) \;+\; I_\phi^\alpha\; f_1(t,x(t)), \\ y(t) &= a_2(t) \;+\; I_\phi^\alpha\; f_2(t,y(t)), \end{aligned}$$

*respectively;*

(iii) *$q(t)$ is the maximal solution of*

$$u(t) \;=\; h(t) \;+\; I_\phi^\alpha\; g(t,u(t))$$

*such that*

$$|\; a_1(t) \;-\; a_2(t)\;| \;=\; h(t),\; t\; \in I,$$

*where $a_1,\; a_2,\; h$ are continuous on $I$.*

*Then*

$$| \ x(t) \ - \ y(t) \ | \ \leq \ q(t), \ t \ \in I.$$

**Proof.**

The proof is an easy modification of the proof of Theorem 6. For, setting $n(t) = |x(t) - y(t)|$ and using (18), we obtain

$$\begin{aligned} n(t) &= | \ x(t) \ - \ y(t) \ | \\ &\leq |a_1(t) \ - \ a_2(t) \ | \ + \ I_\phi^\alpha \ | \ f_1(t, x(t)) \ - \ f_2(t, y(t)) \ | \\ &\leq h(t) \ + \ I_\phi^\alpha \ g(t, | \ x(t) \ - \ y(t) \ |) \\ &= h(t) \ + \ I_\phi^\alpha \ g(t, n(t)). \end{aligned}$$

The result follows from Theorem 4. ■

**Corollary 4.** *Let the assumptions in Theorem 8 be satisfied with* $a_1 \ = \ a_2$ *, then*

$$| \ x(t) \ - \ y(t) \ | \ \leq \ q(t),$$

*where* $q(t)$ *is the maximal solution of*

$$u(t) \ = \ I_\phi^\alpha \ g(t, u(t)). \tag{19}$$

**Theorem 9.** *Let the assumptions in Theorem 8 be satisfied with* $a_1 \ = \ a_2$ *and* $f_1 \ = \ f_2$ *, then*

$$x(t) \ = \ y(t).$$

*If* $u(t) \ = \ 0$ *is the only solution of the fractional-order integral equation (19).*

**Proof.**

Let $x(t), \ y(t)$ be as in Theorem 8. Setting $n(t) \ = \ | \ x(t) \ - \ y(t) \ |$ and arguing as before, we get

$$n(t) \ \leq \ q(t), \ t \in I,$$

where $q(t)$ is the maximal solution of (19). Since $u(t) \ = \ 0$ is the only solution of (19), then

$$n(t) \ = \ | \ x(t) \ - \ y(t) \ | \ \leq \ 0 \ \Rightarrow \ x(t) \ = \ y(t). \ ■$$

**Remark 1.** *Clearly, when* $f_1 \ = \ f_2$ *condition (18) yields Perron's condition which implies an uniqueness theorem of Perron type.*

**Remark 2.** *When* $g(t, u) \ = \ K \ u$ *and* $f_1 \ = \ f_2$ *then condition (18) becomes Lipschitz condition.*

Now consider the following two fractional-order functional integral equations

$$x(t) \ = \ a_1(t) \ + \ I_\phi^\alpha \ f(t, x(t)), \ \alpha \ \in \ (0, 1), \tag{20}$$

$$y(t) \ = \ a_2(t) \ + \ I_\phi^\beta \ f(t, y(t)), \ \beta \ \leq \ \alpha, \tag{21}$$

where $a_1(t), \ a_2(t)$ are continuous functions on $I$.
The following theorem is another comparison theorem which is more general than Theorem 8.

**Theorem 10.** *Assume that:*

(i) $f, \ g$ *satisfy the assumptions of Theorem 1,* $g(t, u)$ *is monotonic nondecreasing in* $u$ *for each* $t$*, and*

$$| \ I_\phi^{\alpha-\beta} \ f(t, x) \ - \ f(t, y) \ | \ \leq \ g(t, | \ x \ - \ y \ |);$$

(ii) $x(t), \ y(t)$ *are any two solutions of (20) and (21) respectively;*

(iii) $q(t)$ *is the maximal solution of*

$$u(t) \ = \ h(t) \ + \ I_\phi^\beta \ g(t, u(t))$$

*such that* $| \ a_1(t) \ - \ a_2(t) \ | \ \leq \ h(t), \ t \ \in I$*, where* $h(t)$ *is continuous on* $I$*.*

*Then*

$$| \ x(t) \ - \ y(t) \ | \ \leq \ q(t), \ t \ \in I.$$

**Proof.**

Similarly as in Theorem 8, putting

$$\begin{aligned} n(t) &= \mid x(t) - y(t) \mid \\ &\leq \mid a_1(t) - a_2(t) + I_\phi^\beta \, |I_\phi^{\alpha - \beta} f(t, x(t)) - f(t, y(t))|, \ \beta \leq \alpha \\ &\leq h(t) + I_\phi^\beta \, g(t, \mid x(t) - y(t) \mid) \\ &= h(t) + I_\phi^\beta \, g(t, n(t)), \end{aligned}$$

it follows from Theorem 4 that

$$\mid x(t) - y(t) \mid \leq q(t), \ t \in I. \blacksquare$$

## Acknowledgement

The authors express their gratitude to the referee for his/her useful comments and suggestions that improve the paper and for the time taken to review this paper.

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