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DIRECT ESTIMATES FOR DURRMEYER-BASKAKOV-STANCU TYPE OPERATORS USING HYPERGEOMETRIC REPRESENTATION

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Abstract. In the present article, we introduced and study hypergeometric representation of Durrmeyer-Baskakov-Stancu type operators. First, we estimate moments of these operators using hypergeometric series. Furthermore, we obtain an error estimation in simultaneous approximation for said operators.

1. Introduction

For $f \in C[0,\infty)$, the Durrmeyer-Baskakov operators were study by Sahai and Prashad [15] is defined as

$$\mathfrak{D}_n(f,x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \tag{1}$$

where $p_{n,k}(x) = \frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}$.

In [5] Gupta and Yadav introduced the Baskakov-Beta-Stancu operators and invetigated some approximation properties like asymptotic formula, moments of these operators using hypergeometric series and errors estimation in simultaneous approximation. The behavior of these operators is very similar to the operators recently introduced by Mishra et al. [8], [9].

It is observed that as an application of the special functions, we can write the different form of the operators $\mathfrak{D}_n(f,x)$ in terms of Hypergeometric series. For details on Hypergeometric series, we refer the readers to [3].
The hypergeometric function is defined as

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}x^k.$$

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The confluent hypergeometric function is a degenerate form of the hypergeometric function ${}_2F_1(a,b;c;x)$ which arises as a solution the confluent hypergeometric differential equation is defined as

$$ {}_1F_1(a;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{x^k}{k!}, $$

where the Pochhammer symbol $(n)_k$ is defined as

$$ (n)_k=n(n+1)(n+2)(n+3)....(n+k-1). $$

Motivated by the recent studies on certain operators by Gupta et al.[5] and Mishra et al.[9] using hypergeometric form, we can write the operators (1) as

$$ \begin{aligned} \mathfrak{D}_n(f,x) &= (n-1)\sum_{k=0}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^{\infty}\frac{(n)_k}{k!}\frac{t^k}{(1+t)^{n+k}}f(t)dt \\ &= (n-1)\int_0^{\infty}\frac{f(t)}{[(1+x)(1+t)]^n}\sum_{k=0}^{\infty}\frac{(n)_k^2}{(k!)^2}\frac{(xt)^k}{[(1+x)(1+t)]^k}dt. \end{aligned} $$

By hypergeometric series ${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$ and the Pochhammer symbol $(n)_k$ and using the equality $(1)_k=k!$, we can write

$$ \mathfrak{D}_n(f,x)=(n-1)\int_0^{\infty}\frac{f(t)}{[(1+x)(1+t)]^n}{}_2F_1\left(n,n;1;\frac{xt}{(1+x)(1+t)}\right)dt. $$

Now, applying Pfaff–Kummer transformation

$$ {}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right) $$

we have

$$ \mathfrak{D}_n(f,x)=(n-1)\int_0^{\infty}\frac{f(t)}{(1+x+t)^n}{}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)dt. \quad (2) $$

This is the another form of the operators (1) in terms of hypergeometric functions. In 1974, Khan [6] studied approximation of functions in various classes using different types of operators. Several other researchers have studied in this direction and obtained different approximation properties of many operators and we mention some of them as [1, 2, 7, 10, 11, 12, 13, 14]. Here, we introduce Durrmeyer-Baskakov-Stancu operators in terms of hypergeometric functions, for $0\le\alpha\le\beta$ as

$$ \mathfrak{D}_n^{(\alpha,\beta)}(f,x)=(n-1)\int_0^{\infty}f\left(\frac{nt+\alpha}{n+\beta}\right)\frac{1}{(1+x+t)^n}\;{}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)dt. \quad (3) $$

For $\alpha=\beta=0$ the operators (3) reduces to the operators (1).
We know that

$$ \sum_{k=0}^{\infty}p_{n,k}(x)=1,\ \int_0^{\infty}p_{n,k}(t)dt=\frac{1}{n-1}. $$

Let us consider

$$ C_\nu[0,\infty)=\{f\in C[0,\infty):f(t)=O(t)^\nu,\nu>0\}. $$

The operators $\mathfrak{D}_n^{(\alpha,\beta)}$ are well defined for $f \in C[0,\infty)$. In the present article we establish moments of Durrmeyer-Baskakov-Stancu operators using the technique of hypergeometric series. Next, we give an error estimation in simultaneous approximation for the operators (3)

2. Moment estimation and auxiliary results

In this section, we establish certain lemmas which will be useful for the proof of our main theorems.

Lemma 1 For $n > 0$ and $r > -1$, we have

$$\mathfrak{D}_n(t^r,x) = (n-1)\frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n)}(1+x)^r {}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right). \quad (4)$$

Moreover,

$$D_n(t^r,x) = \frac{(n-r-2)!(n+r-1)!}{(n-1)!(n-2)!}x^r + r^2\frac{(n+r-2)!(n-r-2)!}{(n-1)!(n-2)!}x^{r-1} + O(n^{-2}). \quad (5)$$

Proof. Taking $f(t) = t^r$, $t = (1+x)u$ and using Pfaff-Kummer transformation the right-hand side of (2), we get

$$\begin{aligned}
\mathfrak{D}_n(t^r,x) &= (n-1)\int_0^\infty \frac{(1+x)^{r+1}u^r}{[(1+x)(1+u)]^n}\sum_{k=0}^\infty \frac{(n)_k(1-n)_k}{(k!)^2}\frac{(-x(1+x)u)^k}{[(1+x)(1+u)]^k}du \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\int_0^\infty \frac{u^{r+k}}{(1+u)^{n+k}}du \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}B(r+k+1,n-r-1) \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\frac{\Gamma(r+k+1)\Gamma(n-r-1)}{\Gamma(n+k)}.
\end{aligned}$$

Using $\Gamma(n+k+1) = \Gamma(n+1)(n+1)_k$, we have

$$\begin{aligned}
\mathfrak{D}_n(t^r,x) &= (n-1)\sum_{k=0}^\infty \frac{(n)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n+1}\frac{\Gamma(r+1)(r+1)_k\Gamma(n-r-1)}{\Gamma(n)(n)_k} \\
&= (n-1)(1+x)^{r-n+1}\frac{\Gamma(r+1)\Gamma(n-r-1)}{\Gamma(n)}\sum_{k=0}^\infty \frac{(r+1)_k(1-n)_k}{(k!)^2}(-x)^k \\
&= (n-1)(1+x)^{r-n+1}\frac{\Gamma(r+1)\Gamma(n-r-1)}{\Gamma(n)}\ {}_2F_1(1-n,1+r;1;-x).
\end{aligned}$$

Using Pfaff–Kummer transformation transformation ${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$, we have

$$= (n-1)\frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n)}(1+x)^r {}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right).$$

The other consequence (5) follows from the above equation by writting the expansion of hypergeometric series. $\square$

Lemma 2 For $0 \leq \alpha \leq \beta$ we have

$$\begin{aligned}
\mathfrak{D}_n^{(\alpha,\beta)}(t^r,x) &= x^r \frac{n^r}{(n+\beta)^r}\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!} \\
&+ x^{r-1}\left\{r^2\frac{n^r}{(n+\beta)^r}\frac{(n+r-2)!(n-r-2)!}{(n-1)!(n-2)!} + r\alpha\frac{n^{r-1}}{(n+\beta)^r}\frac{(n+r-2)!(n-r+1)!}{(n-1)!(n-2)!}\right\} \\
&+ x^{r-2}\left\{r(r-1)^2\alpha\frac{n^{r-1}}{(n+\beta)^r}\frac{(n+r-3)!(n-r-1)!}{(n-1)!(n-2)!}\right. \\
&+ \left.\frac{r(r-1)}{2}\alpha^2\frac{n^{r-2}}{(n+\beta)^r}\frac{(n+r-3)!(n-r+2)!}{(n-1)!(n-2)!}\right\} + O(n^{-2}).
\end{aligned}$$

Proof. By using binomial theorem, the relation between operators (2) and (3) can be defined as

$$\begin{aligned}
\mathfrak{D}_n^{(\alpha,\beta)}(t^r,x) &= (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r dt \\
&= (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\sum_{j=0}^{\infty}\binom{r}{j}\frac{(nt)^j\alpha^{r-j}}{(n+\beta)^r}dt \\
&= \sum_{j=0}^{\infty}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}\mathfrak{D}_n(t^j,x)
\end{aligned}$$

Using (5), we get Lemma (2). □

Lemma 3[4] For $m \in \mathbb{N}\bigcup\{0\}$, if

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n}-x\right)^m,$$

then $U_{n,0}(x)=1, U_{n,1}(x)=0$ and we have the recurrence relation:

$$nU_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[m]$ is integral part of m.

Lemma 4 For $m \in \mathbb{N}\bigcup\{0\}$, if

$$\begin{aligned}
\mu_{n,m}(x) &= \mathfrak{D}_n^{(\alpha,\beta)}((t-x)^m,x) \\
&= (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt
\end{aligned}$$

then

$$\mu_{n,0}(x)=1,\ \ \mu_{n,1}(x) = \frac{(2n+2\beta-n\beta)x+(1+\alpha)n-2\alpha}{(n-2)(n+\beta)},$$

$$\begin{aligned}
\mu_{n,2}(x) &= \left(\frac{2n^3+(b^2-4\beta+6)n^2+(12\beta-5\beta^2)n+6\beta^2}{(n-2)(n-3)(n+\beta)^2}\right)x^2 \\
&+\left(\frac{2n^3+(6+4\alpha-2\beta-2\alpha\beta)n^2+(6\beta+10\alpha\beta-12\alpha)n-12\alpha\beta}{(n-2)(n-3)(n+\beta)^2}\right)x \\
&+\frac{(2+\alpha^2+2\alpha)n^2-(6\alpha+5\alpha^2)n+6\alpha^2}{(n-2)(n-3)(n+\beta)^2},
\end{aligned}$$

and for n>m we have recurrence relation:

$$\begin{aligned}(n+\beta)\mu_{n,m+1}(x) &= x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\ &+ [m+1+\alpha-nx-\beta x-x]\mu_{n,m}(x)-m\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m-1}(x)\end{aligned}$$

From the recurrence relation, it easily verified that for all $x\in[0,\infty)$, we have

$$\mu_{n,m}(x)=O(n^{-[(m+1)/2]}).$$

Proof. Taking derivative of $\mu_{n,m}(x)$

$$\begin{aligned}\mu'_{n,m}(x) &= (n-1)\sum_{k=0}^{\infty}p'_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\ &\quad -m(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^{m-1}dt\\ \mu'_{n,m}(x) &= -m\mu_{n,m-1}(x)+(n-1)\sum_{k=0}^{\infty}p'_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\end{aligned}$$

using $x(1+x)p'_{n,k}(x)=(k-(n+1)x)p_{n,k}(x)$, we get

$$\begin{aligned}x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)] &= (n-1)\sum_{k=0}^{\infty}(k-(n+1)x)p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\ &= (n-1)\sum_{k=0}^{\infty}(k-nx)p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt-x\mu_{n,m}(x)\\ &= (n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}[k-nt+n(t-x)]p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\ &\quad -x\mu_{n,m}(x)\\ &= I-x\,\mu_{n,m}(x). \qquad (6)\end{aligned}$$

We can write I as

$$\begin{aligned}I &= (n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}[k-nt+n(t-x)]p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\ &= \Big[(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}[k-nt]p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\ &\quad +(n-1)\Big(\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}n(t-x)p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\Big)\Big]\\ &= I_1+I_2,\ (say).\end{aligned}$$

To estimate I_2 using $t=\frac{n+\beta}{n}\Big[\Big(\frac{nt+\alpha}{n+\beta}-x\Big)-\Big(\frac{\alpha}{n+\beta}-x\Big)\Big]$, we have

$$
\begin{aligned}
I_2 &= (n-1)\Bigg(\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} n(t-x)p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\Bigg)\Bigg]\\
&= (n-1)\frac{n+\beta}{n}n\Bigg[\sum_{k=0}^{\infty} np_{n,k}(x)\int_0^{\infty} s_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^{m+1} dt\\
&\quad -(n-1)\Big(\frac{\alpha}{n+\beta}-x\Big)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\Bigg]\\
&= (n+\beta)\left[\mu_{n,m+1}(x)-\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m}\right].
\end{aligned}
$$

Next to estimate I_1 using the equality, $t\,p'_{n,k}(t)=[k-nt]p_{n,k}(t)$

$$
I_1 = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} tp'_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt,
$$

again putting $t=\frac{n+\beta}{n}\left[\left(\frac{nt+\alpha}{n+\beta}-x\right)-\left(\frac{\alpha}{n+\beta}-x\right)\right]$, we get

$$
\begin{aligned}
I_1 &= (n-1)\frac{n+\beta}{n}\Bigg[\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^{m+1} dt\\
&\quad -\Big(\frac{\alpha}{n+\beta}-x\Big)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\Bigg].
\end{aligned}
$$

Now integrating by parts and by simple computation, we get

$$
I_1 = \Bigg[-(m+1)\mu_{n,m}(x)+m\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m-1}(x)\Bigg].
$$

Put the values of I_1 and I_2 in I, we get

$$
I = \Bigg[-(m+1)\mu_{n,m}(x)+m\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m-1}(x)\Bigg]+(n+\beta)\left[\mu_{n,m+1}(x)-\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m}\right].
$$

Now, put value of I in (7), we get

$$
\begin{aligned}
x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)] &= -(m+1)\mu_{n,m}(x)+m\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m-1}(x)\\
&\quad +(n+\beta)\left(\mu_{n,m+1}(x)-\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m}\right)-x\mu_{n,m}(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(n+\beta)\mu_{n,m+1}(x) &= x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&\quad +[m+1+\alpha-nx-\beta x-x]\mu_{n,m}(x)-m\Big(\frac{\alpha}{n+\beta}-x\Big)\mu_{n,m-1}(x),
\end{aligned}
$$

which is the required result. □

Lemma 5[4] There exist the polynomials $q_{i,j,r}(x)$ on $[0,\infty)$, independent of n and k such that

$$x^r(1+x)^r\frac{d^r}{dx^r}p_{n,k}(x)=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(k-nx)^j q_{i,j,r}(x)p_{n,k}(x).$$

3. Main result

In this section, we give an estimate of the degree of approximation by $\mathfrak{D}^{(r)}_{n,\alpha,\beta}(f(t),x)$ for smooth functions.

Theorem 1 Let $f\in C_\nu[0,\infty)$ for some $\nu>0$ and $r\leq q\leq r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then for sufficiently large n

$$||\mathfrak{D}^{(r)}_{n,\alpha,\beta}(f,x)-f^{(r)}(x)||_{C[a,b]}\leq C_1n^{-1}\sum_{i=r}^{q}||f^i||_{C[a,b]}+C_2n^{1/2}\omega(f^{(q)},n^{1/2})+O(n^{-m}), \tag{7}$$

where C_1, C_2 are constants independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$ and $||.||_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. Using the Taylor's, expansion, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t))$$

where ξ lies between t and x, and $\chi(t)$ is the characteristic function on interval $(a-\eta,b+\eta)$.

For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}(t-\xi)^q.$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$\mathfrak{D}^{(r)}_{n,\alpha,\beta}(f,x)-f^{(r)}(x)=\left\{\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\mathfrak{D}^{(r)}_{n,\alpha,\beta}((t-x)^i,x)-f^{(r)}(x)\right\}+\mathfrak{D}^{(r)}_{n,\alpha,\beta}\Bigg(\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}$$
$$(t-x)^q\chi(t),x\Bigg)+\mathfrak{D}^{(r)}_{n,\alpha,\beta}(h(t,x)(1-\chi(t)),x)=S_1+S_2+S_3.$$

Using Lemma 2, we get

$$\begin{aligned}
S_1 &= \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\Bigg[x^j\frac{n^j}{(n+\beta)^j}\frac{(n+j-1)!(n-j-2)!}{(n-1)!(n-2)!}\\
&+ x^{j-1}\Bigg(j^2\frac{n^j}{(n+\beta)^j}\frac{(n+j-2)!(n-j-2)!}{(n-1)!(n-2)!}+j\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n+j-2)!(n-j+1)!}{(n-1)!(n-2)!}\Bigg)\\
&+ x^{j-2}\Bigg(j(j-1)^2\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n+j-3)!(n-j-1)!}{(n-1)!(n-2)!}\\
&+ \frac{j(j-1)}{2}\alpha^2\frac{n^{j-2}}{(n+\beta)^j}\frac{(n+j-3)!(n-j+2)!}{(n-1)!(n-2)!}\Bigg)+O(n^{-m})\Bigg]-f^{(r)}(x).
\end{aligned}$$

Hence

$$||S_1||_{C[a,b]}\le C_1n^{-1}\sum_{i=r}^{q}||f^i||_{C[a,b]}+O(n^{-m}),\ uniformly\ on\ [a,b].$$

Next, we estimate S_2 as

$$\begin{aligned}
|S_2| &\le (n-1)\sum_{k=0}^{\infty}|p_{n,k}^{(r)}(x)|\int_0^\infty p_{n,k}(t)\Bigg\{\Bigg|\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}\Bigg|\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^q\chi(t)\Bigg\}dt\\
&\le \frac{\omega(f^{(q)},\delta)}{q!}(n-1)\Bigg[\sum_{k=0}^{\infty}|p_{n,k}^{(r)}(x)|\int_0^\infty p_{n,k}(t)\left(1+\frac{\left|\frac{nt+\alpha}{n+\beta}-x\right|^q}{\delta}\right)\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^q dt\\
&\le \frac{\omega(f^{(q)},\delta)}{q!}(n-1)\Bigg[\sum_{k=0}^{\infty}|p_{n,k}^{(r)}(x)|\int_0^\infty p_{n,k}(t)\Bigg(\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^q+\delta^{-1}\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^{q+1}\Bigg)dt
\end{aligned}$$

Now, using Schwarz inequality for integration and then for summation, we get

$$\begin{aligned}
(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty p_{n,k}(t)\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^q &\le \sum_{k=0}^{\infty}p_{n,k}(x)|k-nx|^j\Bigg(\int_0^\infty p_{n,k}(t)dt\Bigg)^{\frac12}\\
&\quad\Bigg(\int_0^\infty p_{n,k}(t)\Bigg(\frac{nt+\alpha}{n+\beta}-x\Bigg)^{2q}dt\Bigg)^{\frac12}\\
&\le \Bigg(\sum_{k=0}^{\infty}p_{n,k}(x)(k-nx)^{2j}\Bigg)^{\frac12}\Bigg(\sum_{k=0}^{\infty}p_{n,k}(x)\\
&\quad\int_0^\infty p_{n,k}(t)\Bigg(\frac{nt+\alpha}{n+\beta}-x\Bigg)^{2q}dt\Bigg)^{\frac12}.
\end{aligned}$$

Hence

$$\begin{aligned}
(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty p_{n,k}(t)\Bigg|\frac{nt+\alpha}{n+\beta}-x\Bigg|^q dt &= O(n^{j/2})O(n^{-q/2})\\
&= O(n^{(j-q)/2}), \qquad (8)\\
&\quad uniformly\ on\ [a,b].
\end{aligned}$$

Therefore, by Lemma 2 and (8), we get

$$(n-1)\sum_{k=0}^{\infty}|p_{n,k}^{(r)}(x)|\int_0^{\infty}p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q dt \leq (n-1)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{n^i q_{i,j,r}(x)}{x^r(1+x)^r}p_{n,k}(x)|k-nx|^j\int_0^{\infty}p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^q dt$$

$$\leq K\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\left(\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^{\infty}p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q dt\right)$$

$$= K\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i O(n^{(j-q)/2}) = O(n^{(r-q)/2}),\ uniformly\ on\ [a,b], \quad (9)$$

where $K=\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\sup\limits_{x\in[a,b]}\frac{q_{i,j,r}(x)}{x^r(1+x)^r}$. Choosing $\delta=n^{-1/2}$ and making use of (8), we get for any $m>0$,

$$||S_2||_{C[a,b]} \leq \frac{\omega(f^{(q)},n^{-1/2})}{q!}[O(n^{(r-q)/2})+n^{1/2}O(n^{(r-q-1)/2})+O(n^{-q})] \leq C_2(n^{-(r-q)/2})\omega(f^{(q)},n^{-1/2}).$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$, we can choose δ such that $|t-x|\geq\delta$ for all $x\in[a,b]$. Thus by Lemma 2, we get

$$|S_3| \leq \sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{n^i q_{i,j,r}(x)}{x^r(1+x)^r}\sum_{k=0}^{\infty}p_{n,k}(x)|k-nx|^j\int_{|t-x|\geq\delta}p_{n,k}(t)|h(t,x)|dt$$

We can find a constant M_1 such that

$$|h(t,x)| \leq M_1\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\beta}\ for\ |t-x|\geq\delta,$$

where $\beta\geq(\nu,q)$. Hence applying Schwarz inequality and Lemma (2) and (2), it is easy to see that $S_3=O(n^{-r})$ for any $r>0$ uniformly on $[a,b]$. Combining the estimates of S_1, S_2 and S_3, the required result follows. □

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