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FRACTIONAL DERIVATIVES FOR KOBER OPERATORS AND STATISTICAL DENSITIES IN THE REAL MATRIX-VARIATE CASES

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Abstract. Fractional integrals, fractional derivatives and fractional differential equations in the real scalar variable cases have found many applications and are very popular in the literature. Fractional integrals in the real and complex matrix-variate cases have been considered by this author recently. Some cases of fractional derivatives in matrix-variate case are discussed in the present article. Some matrix-variate differential operators are defined. These are only suitable to handle certain types of matrix-variate cases. Fractional derivatives in the Riemann-Liouville and in the Caputo senses are evaluated when the arbitrary function is compatible with right and left sided fractional integrals in the matrix-variate cases. Fractional derivatives involving Kober operators of the first and second kind in the matrix-variate case are also discussed here

1. Introduction

Fractional integrals and fractional derivatives in the real scalar variable case and their applications in stochastic processes and random walk problems may be seen from many papers, see for example [1],[2]. Solutions of fractional differential equations in the real variable case may be seen, for example, from [3],[13]. There are not many papers on fractional integrals in the matrix-variate case. Some discussions on functions of matrix argument may be seen from [4] - [10]. Fractional integrals in the matrix-variate case may be seen from [7], [8]. Some aspects of fractional derivatives in the matrix-variate case are discussed in [4]. In the present article we introduce fractional differential operators in the matrix-variate case and which are applicable when the arbitrary function of matrix argument has certain structures. As an illustration of the matrix differential operators, Kober operators of the first and second kinds are discussed in the Riemann-Liouville and Caputo senses.

The following standard notations will be used. All matrices appearing here are $p \times p$ symmetric and positive definite when real and Hermitian positive definite when in the complex domain unless otherwise stated. $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the

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trace and determinant of the square matrix $(\cdot)$ respectively. $|\det(\cdot)|$ denotes the absolute value of the determinant of $(\cdot)$. For example, if $\det(A) = a + ib, i = \sqrt{-1}$ and a and b are real scalars then

$$|\det(A)| = [(a+ib)(a-ib)]^{\frac{1}{2}} = [a^2+b^2]^{\frac{1}{2}} = [\det(AA^*)]^{\frac{1}{2}} \tag{1.1}$$

where A^* is the conjugate transpose of A. If $X = (x_{ij})$ is $m \times n$ and real then $\mathrm{d}X$ will denote the wedge product of all differentials

$$\mathrm{d}X = \prod_{i=1}^{m}\prod_{j=1}^{n} \wedge \mathrm{d}x_{ij},\ \mathrm{d}X = \prod_{i \geq j=1}^{p} \wedge \mathrm{d}x_{ij} \text{ for } X = X',\ p \times p \text{ and real,} \tag{1.2}$$

where X' denotes the transpose of X. If $\tilde{X} = X_1 + iX_2, i = \sqrt{-1}$ where X_1 and X_2 are real matrices then $\mathrm{d}\tilde{X} = \mathrm{d}X_1 \wedge \mathrm{d}X_2$. In order to distinguish between matrices in the real and complex cases, matrices in the complex domain will be denoted by a tilde as $\tilde{X}$. Real matrix variables X and real or complex constant matrices will be written without a tilde. The real matrix-variate gamma function will be denoted by $\Gamma_p(\alpha)$ where $\Gamma_p(\alpha)$ has the following expression and integral representation:

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2} \tag{1.3}$$

$$\Gamma_p(\alpha) = \int_{X>O} [\det(X)]^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X, \Re(\alpha) > \frac{p-1}{2} \tag{1.4}$$

where $X > O$ means X is positive definite and $\Re(\alpha)$ denotes the real part of α. The integration is done over all real positive definite matrices X. The matrix-variate gamma function in the complex domain will be denoted by a tilde as $\tilde{\Gamma}_p(\alpha)$. Then it has the following expression and integral representation:

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1)...\Gamma(\alpha-p+1),\ \Re(\alpha) > p-1 \tag{1.5}$$

$$\tilde{\Gamma}_p(\alpha) = \int_{\tilde{X}>O} |\det(\tilde{X})|^{\alpha-p}\mathrm{e}^{-\mathrm{tr}(\tilde{X})}\mathrm{d}\tilde{X}, \Re(\alpha) > p-1 \tag{1.6}$$

and the associated Jacobian will be given here as lemmas without proofs. For the proofs and for other Jacobians see [4].

Lemma 1.1. *Let X and Y be $m \times n$ matrices of distinct real elements and let $Y = AXB$ where A and B are nonsingular $m \times m$ and $n \times n$ constant matrices. Then*

$$Y = AXB \Rightarrow \mathrm{d}Y = [\det(A)]^n[\det(B)]^m\mathrm{d}X. \tag{1.7}$$

Let $\tilde{X}$ and $\tilde{Y}$ be $m \times n$ and in the complex domain. Then

$$\tilde{Y} = A\tilde{X}B \Rightarrow \mathrm{d}\tilde{Y} = |\det(A)|^{2n}|\det(B)|^{2m}\mathrm{d}\tilde{X}. \tag{1.8}$$

Note that $|\det(A)|^2 = [\det(AA^*)], |\det(B)|^2 = [\det(BB^*)]$ where A^* and B^* are the conjugate transposes of A and B respectively.

Lemma 1.2. *Let X be a symmetric $p \times p$ matrix with distinct real elements except for symmetry and let A be a nonsingular constant matrix. Let $\tilde{X}$ be Hermitian. Let $Y = AXA'$ and $\tilde{Y} = A\tilde{X}A^*$. Then*

$$\mathrm{d}Y = [\det(A)]^{p+1}\mathrm{d}X,\ \mathrm{d}\tilde{Y} = |\det(A)|^{2p}\mathrm{d}\tilde{X}. \tag{1.9}$$

Lemma 1.3. *Let X and $\tilde{X}$ be $p\times p$ nonsingular matrices with distinct elements, except for symmetry, and let $Y = X^{-1}$ and $\tilde{Y} = \tilde{X}^{-1}$ be the regular inverses of X and $\tilde{X}$ respectively. Then*

$$\mathrm{d}Y = \begin{cases} [\det(X)]^{-2p}\mathrm{d}X \text{ for a general } X \\ [\det(X)]^{-(p+1)}\mathrm{d}X \text{ for } X = X'; \end{cases} \tag{1.10}$$

$$\mathrm{d}\tilde{Y} = \begin{cases} |\det(\tilde{X})|^{-4p}\mathrm{d}\tilde{X} \text{ for a general } \tilde{X} \\ |\det(\tilde{X})|^{-2p}\mathrm{d}\tilde{X} \text{ for } \tilde{X} = \tilde{X}^* \text{ or } \tilde{X} = -\tilde{X}^*. \end{cases} \tag{1.11}$$

2. Some Fractional Differential Operators

Let $U = (u_{ij})$ be $p\times p$ matrix of distinct real variables. Let $\frac{\partial^*}{\partial U} = (\eta_{ij}\frac{\partial}{\partial u_{ij}})$ where $\eta_{ij} = \begin{cases} 1, & i = j \\ \frac{1}{2}, & i \neq j. \end{cases}$ Let $U = U'$ and $X = X'$ be $p\times p$ real symmetric matrices. Then

$$\frac{\partial^*}{\partial U}[\mathrm{e}^{-\mathrm{tr}(UX)}] = -\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \dots & \vdots \\ x_{p1} & x_{p2} & \dots & x_{pp} \end{bmatrix} \mathrm{e}^{-\mathrm{tr}(UX)} = -X\mathrm{e}^{-\mathrm{tr}(UX)},$$

for $x_{ij} = x_{ji}, u_{ij} = u_{ji}$ for all i and j. Let us consider the case $p = 2$. Then $\frac{\partial^*}{\partial u_{jj}}\mathrm{e}^{-\mathrm{tr}(UX)} = -x_{jj}\mathrm{e}^{-\mathrm{tr}(UX)}$ and $\frac{1}{2}\frac{\partial}{\partial u_{ij}}\mathrm{e}^{-\mathrm{tr}(UX)} = -x_{ij}\mathrm{e}^{-\mathrm{tr}(UX)} = \frac{\partial^*}{\partial u_{ij}}\mathrm{e}^{-\mathrm{tr}(UX)}, i \neq j$. Then

$$[\frac{\partial^*}{\partial u_{22}}\frac{\partial^*}{\partial u_{11}} - (\frac{\partial^*}{\partial u_{12}})^2]\mathrm{e}^{-\mathrm{tr}(UX)} = (-1)^2[x_{22}x_{11} - x_{12}^2]\mathrm{e}^{-\mathrm{tr}(UX)} = (-1)^2[\det(X)]\mathrm{e}^{-\mathrm{tr}(UX)}.$$

For the general p, consider the determinant of the operator $\frac{\partial^*}{\partial U}$, that is $[\det(\frac{\partial^*}{\partial U})]$ operating on $\mathrm{e}^{-\mathrm{tr}(UX)}$ then the result is $(-1)^p[\det(X)]\mathrm{e}^{-\mathrm{tr}(UX)}$. Then the operator $[(-1)^p\det(\frac{\partial^*}{\partial U})]^n = [(-1)^p\det(\frac{\partial^*}{\partial U})]...[(-1)^p\det(\frac{\partial^*}{\partial U})]$ operating on $\mathrm{e}^{-\mathrm{tr}(UX)}$ gives $[\det(X)]^n\mathrm{e}^{-\mathrm{tr}(UX)}$. This determinant operator will be denoted by D_{2U}. Then

$$D_{2U}^n\mathrm{e}^{-\mathrm{tr}(UX)} = [\det(X)]^n\mathrm{e}^{-\mathrm{tr}(UX)}. \tag{2.1}$$

Similarly, $\frac{\partial^*}{\partial U}$ operating on $\mathrm{e}^{\mathrm{tr}(UX)}$ gives $X\mathrm{e}^{\mathrm{tr}(UX)}$. Consider the operator $D_{1U}^n = [\det(\frac{\partial^*}{\partial U})]^n$. Then

$$D_{1U}^n\mathrm{e}^{\mathrm{tr}(UX)} = [\det(X)]^n\mathrm{e}^{\mathrm{tr}(UX)}. \tag{2.2}$$

With the help of these two operators we will establish a few basic results which will be stated as lemmas.

Lemma 2.1. *Let X be $p\times p$ real positive definite. Then*

$$D_{2U}^n[\det(U)]^{-\gamma} = [\det(U)]^{-(\gamma+n)}\frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}$$

for $\Re(\gamma) > \frac{p-1}{2}, n = 0, 1, 2, \dots$.

Proof: Consider the following integral, for $X = X' > O$ and $U = U' > O$:

$$\int_{X>O}[\det(X)]^{\gamma-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(UX)}\mathrm{d}X = \int_{X>O}[\det(X)]^{\gamma-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(U^{\frac{1}{2}}XU^{\frac{1}{2}})}\mathrm{d}X$$

where $U^{\frac{1}{2}}$ is the unique positive definite square root of the positive definite matrix U. Let $V = U^{\frac{1}{2}}XU^{\frac{1}{2}} \Rightarrow \mathrm{d}V = [\det(U)]^{\frac{p+1}{2}}\mathrm{d}X$ by using Lemma 1.2. Then

$$\int_{X>O}[\det(X)]^{\gamma-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(U^{\frac{1}{2}}XU^{\frac{1}{2}})}\mathrm{d}X = [\det(U)]^{-\gamma}\int_{V>O}[\det(V)]^{\gamma-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(V)}\mathrm{d}V$$
$$= [\det(U)]^{-\gamma}\Gamma_p(\gamma),\ \Re(\gamma) > \frac{p-1}{2}$$

by using (1.4). Hence we have the following identity:

$$[\det(U)]^{-\gamma} = \frac{1}{\Gamma_p(\gamma)}\int_{V>O}[\det(V)]^{\gamma-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(UV)}\mathrm{d}V,\ \Re(\gamma) > \frac{p-1}{2}. \tag{2.3}$$

Now, operate on both sides with D^n_{2U}. That is,

$$\begin{aligned}D^n_{2U}[\det(U)]^{-\gamma} &= \frac{1}{\Gamma_p(\gamma)}\int_{V>O}[\det(V)]^{\gamma-\frac{p+1}{2}}[D^n_{2U}\mathrm{e}^{-\mathrm{tr}(UV)}]\mathrm{d}V\\ &= \frac{1}{\Gamma_p(\gamma)}\int_{V>O}[\det(V)]^{\gamma+n-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(UV)}\mathrm{d}V\\ &= \frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}[\det(U)]^{-(\gamma+n)}, \Re(\gamma) > \frac{p-1}{2}, n = 0,1,2,.. \end{aligned} \tag{2.4}$$

This establishes the result.

Now, let us look at a basic result of D^n_{1U} operating on $\mathrm{e}^{\mathrm{tr}(UX)}$, which will be stated as a lemma.

Lemma 2.2. *Let X and U be $p\times p$ real positive definite matrices. Let D^n_{1U} be the operator defined in (2.2). Then*

$$D^n_{1U}\frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)} = \frac{[\det(U)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}, \Re(\gamma) > n + \frac{p-1}{2}. \tag{2.5}$$

Proof: Observe that (2.3) can be taken as the Laplace transform of the function $\frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}$ with Laplace parameter matrix U. If $U = U' = (\eta_{ij}u_{ij})$ with $\eta_{ij} = \begin{cases}1, i=j\\ \frac{1}{2}, i\neq j\end{cases}$ then it is the multivariable Laplace transform of all elements in V, taking each element once. If $U = (u_{ij}), U = U'$ then it is the Laplace transform of all elements in V, taking the diagonal elements once and the off-diagonal elements twice. Hence as an inverse Laplace transform we can write, for $\Re(\gamma) > \frac{p-1}{2}$,

$$\frac{[\det(X)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)} = \begin{cases}\frac{1}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(U)>U_o}[\det(U)]^{-\gamma}\mathrm{e}^{\mathrm{tr}(UX)}\mathrm{d}U, U = (\eta_{ij}u_{ij})\\ \frac{2^{\frac{p(p-1)}{2}}}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(U)>U_o}[\det(U)]^{-\gamma}\mathrm{e}^{\mathrm{tr}(UX)}\mathrm{d}U, U = (u_{ij}).\end{cases} \tag{2.6}$$

Then, operating on both sides with the operator D^n_{1X} we have

$$
\begin{aligned}
D_{1X}^{n}\frac{[\det(X)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)} &= \frac{2^{\frac{p(p-1)}{2}}}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(U)>U_o}[\det(U)]^{-\gamma}(D_{1X}^{n}\mathrm{e}^{\mathrm{tr}(UX)}\mathrm{d}U \\
&= \frac{2^{\frac{p(p-1)}{2}}}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(U)>U_o}[\det(U)]^{-\gamma+n}\mathrm{e}^{\mathrm{tr}(UX)}\mathrm{d}U.
\end{aligned}
$$

Interpreting the right side as an inverse Laplace transform the right side corresponds to $\frac{[\det(X)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}$ for $\Re(\gamma-n)>\frac{p-1}{2}$ or $\Re(\gamma)>n+\frac{p-1}{2}$. Hence the result.

With the help of Lemmas 2.1 and 2.2 we can look at some fractional derivatives when the arbitrary function $f(X)$ is of the form $[\det(V)]^{-\gamma}$ or of the form $[\det(V)]^{\gamma-\frac{p+1}{2}}$ or of the form $\mathrm{e}^{\pm\mathrm{tr}(V)}$. Let $D_{2,U}^{-\alpha}f$ and $D_{2,U}^{\alpha}f$ denote the fractional integral and fractional derivative of order α of the second kind or right-sided situation respectively. Similarly, let $D_{1,U}^{-\alpha}f$ and $D_{1,U}^{\alpha}f$ be the fractional integral and fractional derivative of the first kind (left-sided) and of order α respectively. The following symbolic representations will be used to write fractional derivatives from fractional integrals:

$$D_{2,U}^{\alpha}f = D_{2U}^{n}[D_{2,U}^{-(n-\alpha)}f] =$$

the fractional derivative of order α of the second kind, in the Riemann-Liouville sense for $n>\Re(\alpha)+\frac{p-1}{2}$;

$$D_{2,U}^{\alpha}f = D_{2,U}^{-(n-\alpha)}(D_{2U}^{n}f) =$$

the fractional derivative of order α, of the second kind, in the Caputo sense for $n>\Re(\alpha)+\frac{p-1}{2}$;

$$D_{1,U}^{\alpha}f = D_{1U}^{n}[D_{1,U}^{-(n-\alpha)}f] =$$

the fractional derivative of order α, of the first kind, in the Riemann-Liouville sense for $n>\Re(\alpha)+\frac{p-1}{2}$;

$$D_{1,U}^{\alpha}f = D_{1,U}^{-(n-\alpha)}[D_{1U}^{n}f] =$$

the fractional derivative of order α, of the first kind, in the Caputo sense. The operator of the second kind is also called right-sided operator and the operator of the first kind is also called the left-sided operator.

3. Fractional Derivatives in Some Special Cases

We will examine a few cases of the arbitrary function with reference to first and second kinds of fractional derivatives of order α.

Case 3.1: $f(V)=\mathrm{e}^{-\mathrm{tr}(V)}$, right-sided fractional derivative in the Riemann-Liouville sense

For $n>\Re(\alpha)+\frac{p-1}{2}$,

$$\begin{aligned}
D^{\alpha}_{2,U}f &= D^n_{2U}[D^{-(n-\alpha)}_{2,U}f]\\
&= D^n_{2U}\frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(V)}\mathrm{d}V\\
&= D^n_{2U}\frac{\mathrm{e}^{-\mathrm{tr}(U)}}{\Gamma_p(n-\alpha)}\int_{W>O}[\det(W)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(W)}\mathrm{d}W, W=V-U\\
&= D^n_{2U}\mathrm{e}^{-\mathrm{tr}(U)}=\mathrm{e}^{-\mathrm{tr}(U)} \qquad (3.1)
\end{aligned}$$

where $D^n_{2U}\mathrm{e}^{-\mathrm{tr}(U)}=[(-1)^{np}\det((-I)^n)]=1$. In this case, the right-sided fractional derivative in the Caputo sense is the following for $n>\Re(\alpha)+\frac{p-1}{2}$:

$$D^{\alpha}_{2,U}f=D^{-(n-\alpha)}_{2,U}[D^n_{2U}f]=\frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(V)}\mathrm{d}V.$$

But $D^n_{2U}\mathrm{e}^{-\mathrm{tr}(V)}=\mathrm{e}^{-\mathrm{tr}(V)}$ and

$$\frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(V)}\mathrm{d}V=\mathrm{e}^{-\mathrm{tr}(U)}\frac{\Gamma_p(n-\alpha)}{\Gamma_p(n-\alpha)}=\mathrm{e}^{-\mathrm{tr}(U)}. \qquad (3.2)$$

In this case, both the Riemann-Liouville and the Caputo derivatives are the same.

Note 3.1. In this case it is easy to note that the semigroup property holds for both the Riemann-Liouville and the Caputo derivatives. That is,

$$D^{\alpha}_{2,U}D^{\beta}_{2,U}\mathrm{e}^{-\mathrm{tr}(U)}=D^{\beta}_{2,U}D^{\alpha}_{2,U}\mathrm{e}^{-\mathrm{tr}(U)}=D^{\alpha+\beta}_{2,U}\mathrm{e}^{-\mathrm{tr}(U)}.$$

Case 3.2a: $f(V)=[\det(V)]^{-\gamma},\Re(\gamma)>0$, right-sided fractional derivative of order α in the Riemann-Liouville sense

This is the following:

$$\begin{aligned}
D^{\alpha}_{2,U}f &= D^n_{2U}[D^{-(n-\alpha)}_{2,U}f]\\
&= D^n_{2U}\frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}[\det(V)]^{-\gamma}\mathrm{d}V\\
&= D^n_{2U}[\frac{1}{\Gamma_p(n-\alpha)}\int_{W>O}[\det(W)]^{n-\alpha-\frac{p+1}{2}}[\det(U+W)]^{-\gamma}\mathrm{d}W]\\
&= D^n_{2U}[\det(U)]^{-\gamma+n-\alpha}\frac{\Gamma_p(n-\alpha)\Gamma_p(\gamma-n+\alpha)}{\Gamma_p(n-\alpha)\Gamma_p(\gamma)}, T=U^{-\frac{1}{2}}VU^{-\frac{1}{2}}\\
&= D^n_{2U}[\det(U)]^{-(\gamma-n+\alpha)}\frac{\Gamma_p(\gamma-n+\alpha)}{\Gamma_p(\gamma)}. \qquad (3.3)
\end{aligned}$$

But from Lemma 2.1

$$D^n_{2U}[\det(U)]^{-(\gamma-n+\alpha)}=\frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma-n+\alpha)}[\det(U)]^{-(\gamma+\alpha)} \qquad (3.4)$$

for $\Re(\gamma)>\frac{p-1}{2},\Re(\gamma+\alpha)>\frac{p-1}{2}$. Substituting (3.4) in (3.3) we have for $\Re(\gamma)>\frac{p-1}{2}$,

$$D^{\alpha}_{2,U}[\det(U)]^{-\gamma}=[\det(U)]^{-(\gamma+\alpha)}\frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)}.$$

Note 3.2. It is not difficult to see that the semigroup property holds here. Note that, for $n > \Re(\alpha) + \frac{p-1}{2}$ and $n > \Re(\beta) + \frac{p-1}{2}$,

$$\begin{aligned}
D_{2,U}^{\beta}\{D_{2,U}^{\alpha}[\det(U)]^{-\gamma}\} &= D_{2,U}^{\beta}\{\frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)}[\det(U)]^{-(\gamma+\alpha)}\} \\
&= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} D_{2U}^{m}\{\frac{1}{\Gamma_p(m-\beta)}\int_{V>U}[\det(V-U)]^{m-\beta-\frac{p+1}{2}}[\det(V)]^{-(\gamma+\alpha)}\mathrm{d}V\} \\
&= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} D_{2U}^{m}\{[\det(U)]^{-(\gamma+\alpha+\beta-m)}\frac{\Gamma_p(\gamma+\alpha+\beta-m)}{\Gamma_p(\gamma+\alpha)}\} \\
&= \frac{\Gamma_p(\gamma+\alpha+\beta-m)}{\Gamma_p(\gamma)} D_{2U}^{m}[\det(U)]^{-(\gamma+\alpha+\beta-m)} \\
&= \frac{\Gamma_p(\gamma+\alpha+\beta)}{\Gamma_p(\gamma)}[\det(U)]^{-(\gamma+\alpha+\beta)} = D_{2,U}^{\alpha+\beta}[\det(U)]^{-\gamma}. \qquad (3.5)
\end{aligned}$$

Thus, semigroup property is proved for the Riemann-Liouville type derivative of order α and of the second kind.

Case 3.2b: $f(V) = [\det(V)]^{-\gamma}, \Re(\gamma) > 0$, right-sided fractional derivative of order α in the Caputo sense

Here

$$D_{2,U}^{\alpha} f = D_{2,U}^{-(n-\alpha)}[D_{2U}^{n} f] = \frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}} D_{2V}^{n}[\det(V)]^{-\gamma}\mathrm{d}V$$

for $n > \Re(\alpha) + \frac{p-1}{2}$. But

$$D_{2V}^{n}[\det(V)]^{-\gamma} = \frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}[\det(V)]^{-(\gamma+n)}.$$

Therefore,

$$\begin{aligned}
D_{2,U}^{\alpha} f &= \frac{1}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}\frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}[\det(V)]^{-(\gamma+n)}\mathrm{d}V \\
&= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)}[\det(U)]^{-(\gamma+\alpha)} \qquad (3.6)
\end{aligned}$$

for $\Re(\gamma) > \frac{p-1}{2}, \Re(\gamma+\alpha) > \frac{p-1}{2}$. This is the same result in the Riemann-Liouville case also. Hence for both cases here we have the same expression for the fractional derivative of order α and also the semigroup property holds good in both the cases.

Note 3.3: If $f(V) = [\det(U)]^{\gamma}, \Re(\gamma) > 0$ then it is easy to see that the conditions in the above procedure are violated. In fact the right-sided or second kind fractional integral diverges for this situation whereas the left-sided integrals will be available in this case.

4. First Kind Fractional Derivative for Some Special Cases

Here we consider two special cases of the arbitrary function.

Case 4.1a: $f(V) = \mathrm{e}^{\mathrm{tr}(V)}$, fractional integral of order α in the Riemann-Liouville sense

$$\begin{aligned}
D^{\alpha}_{1,U}f &= D^{n}_{1U}[D^{-(n-\alpha)}_{1,U}f] \\
&= D^{n}_{1U}\frac{1}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(U-V)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{\mathrm{tr}(V)}\mathrm{d}V \\
&= D^{n}_{1U}\frac{\mathrm{e}^{\mathrm{tr}(U)}}{\Gamma_p(n-\alpha)}\int_{W>O}[\det(W)]^{n-\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(W)}\mathrm{d}W, U-V=W \\
&= D^{n}_{1U}\mathrm{e}^{\mathrm{tr}(U)}\frac{\Gamma_p(n-\alpha)}{\Gamma_p(n-\alpha)} = \mathrm{e}^{\mathrm{tr}(U)}.
\end{aligned}$$

It is trivial to see that the semigroup property holds. Since $D^{n}_{1U}\mathrm{e}^{\mathrm{tr}(U)} = \mathrm{e}^{\mathrm{tr}(U)}$ the derivative in the Caputo sense also gives the same result.

Note 4.1. If $f(V) = \mathrm{e}^{-\mathrm{tr}(V)}$ then the above procedure does not hold. But we can take out U from $[\det(U-V)]^{n-\alpha-\frac{p+1}{2}}$, make a transformation $W = U^{-\frac{1}{2}}VU^{-\frac{1}{2}}$. Then expand $\mathrm{e}^{-\mathrm{tr}(UW)}$ for $O < W < I$ and integrate out to obtain a confluent hypergeometric series of matrix argument, see [4] for details.

Case 4.2a: $f(V) = \frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}$, left-sided fractional derivative in the Riemann-Liouville sense

In this case, for $n > \Re(\alpha) + \frac{p-1}{2}$,

$$\begin{aligned}
D^{\alpha}_{1,U}f &= D^{n}_{1U}[D^{-(n-\alpha)}_{1,U}f] \\
&= D^{n}_{1U}\frac{1}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(U-V)]^{n-\alpha-\frac{p+1}{2}}\frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}\mathrm{d}V \\
&= D^{n}_{1U}\frac{[\det(U)]^{\gamma+n-\alpha-\frac{p+1}{2}}}{\Gamma_p(\gamma+n-\alpha)}, T = U^{-\frac{1}{2}}VU^{-\frac{1}{2}} \\
&= \frac{[\det(U)]^{\gamma-\alpha-\frac{p+1}{2}}}{\Gamma_p(\gamma-\alpha)}
\end{aligned}$$

by Lemma 2.2, for $\Re(\gamma-\alpha) > \frac{p-1}{2}, n > \Re(\alpha) + \frac{p-1}{2}$. Let us see whether it satisfies the semigroup property.

$$\begin{aligned}
D^{\beta}_{1,U}[D^{\alpha}_{1,U}f] &= D^{\beta}_{1,U}\{\frac{[\det(U)]^{\gamma-\alpha-\frac{p+1}{2}}}{\Gamma_p(\gamma-\alpha)}\}, \Re(\gamma-\alpha) > \frac{p-1}{2} \\
&= D^{m}_{1U}[\det(U)]^{\gamma-\alpha-\beta+m-\frac{p+1}{2}}\frac{\Gamma_p(m-\beta)\Gamma_p(\gamma-\alpha)}{\Gamma_p(m-\beta)\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\alpha-\beta+m)} \\
&= D^{m}_{1U}\frac{[\det(U)]^{\gamma-\alpha-\beta+m-\frac{p+1}{2}}}{\Gamma_p(\gamma-\alpha-\beta+m)} = \frac{[\det(U)]^{\gamma-\alpha-\beta-\frac{p+1}{2}}}{\Gamma_p(\gamma-\alpha-\beta)} \\
&= D^{\alpha+\beta}_{1,U}f = D^{\alpha}_{1,U}[D^{\beta}_{1,U}f]
\end{aligned}$$

for $\Re(\gamma-\alpha-\beta) > \frac{p-1}{2}$. Hence the semigroup property is satisfied.

Case 4.2b: $f(V) = \frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}, \Re(\gamma) > \frac{p-1}{2}$, left-sided fractional derivative in the Caputo sense

In this case we have the following, for $n > \Re(\alpha) + \frac{p-1}{2}$,:

$$
\begin{aligned}
D^{\alpha}_{1,U} f &= D^{-(n-\alpha)}_{1,U}[D^{n}_{1U} f] = D^{-(n-\alpha)}_{1,U}\{\frac{[\det(U)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}\} \\
&= \frac{1}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(U-V)]^{n-\alpha-\frac{p+1}{2}}\frac{[\det(V)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}\mathrm{d}V \\
&= \frac{[\det(U)]^{\gamma-\alpha-\frac{p+1}{2}}}{\Gamma_p(n-\alpha)\Gamma_p(\gamma-n)}\frac{\Gamma_p(n-\alpha)\Gamma_p(\gamma-n)}{\Gamma_p(\gamma-\alpha)}, T = U^{-\frac{1}{2}}VU^{-\frac{1}{2}} \\
&= \frac{[\det(U)]^{\gamma-\alpha-\frac{p+1}{2}}}{\Gamma_p(\gamma-\alpha)}, \Re(\gamma-\alpha) > \frac{p-1}{2}.
\end{aligned}
$$

It is the same result as in the Riemann-Liouville case also. It is evident that this Caputo derivative also satisfies the semigroup property.

Note 4.2: If $f(V) = [\det(V)]^{-\gamma}, \Re(\gamma) > 0$ then it is easy to see that the above conditions on the parameters are violated. Hence the left-sided fractional derivatives are not available for this situation.

5. Fractional Derivatives of Kober Operators

The Kober integral operator of the second kind and of order α, in the real matrix-variate case is given by the following:

$$
K^{-\alpha}_{2,U} f = \frac{[\det(U)]^{\rho}}{\Gamma_p(\alpha)}\int_{V>U}[\det(V)]^{-\rho-\alpha}[\det(V-U)]^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \tag{5.1}
$$

for $\Re(\alpha) > \frac{p-1}{2}$. Then, the Kober fractional derivative of order α and of the second kind in the Riemann-Liouville sense is given by the following, see [8]:

$$
\begin{aligned}
K^{\alpha}_{2,U} f &= D^{n}_{2U} K^{-(n-\alpha)}_{2,U} f \\
&= D^{n}_{2U}\frac{[\det(U)]^{\rho}}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V)]^{-\rho+\alpha-n}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V
\end{aligned} \tag{5.2}
$$

for $n > \Re(\alpha) + \frac{p-1}{2}$. The Kober fractional derivative of order α and of the second kind, in the Caputo sense is given by the following:

$$
\begin{aligned}
K^{\alpha}_{2,U} &= K^{-(n-\alpha)}_{2,U}[D^{n}_{2U} f] \\
&= \frac{[\det(U)]^{\rho}}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V)]^{-\rho-(n-\alpha)}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}[D^{n}_{2V} f(V)]\mathrm{d}V
\end{aligned} \tag{5.3}
$$

for $n > \Re(\alpha) + \frac{p-1}{2}$. Now, we will evaluate (5.2) and (5.3) for various cases for $f(V)$.

Case 5.1a: $f(V) = [\det(V)]^{-\gamma}, \Re(\gamma) > 0$, evaluation of (5.2)

$$
\begin{aligned}
K_{2,U}^{\alpha}f &= D_{2U}^{n}K_{2,U}^{-(n-\alpha)}f \\
&= D_{2U}^{n}\{\frac{[\det(U)]^{\rho}}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V)]^{-\rho+\alpha-n-\gamma}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}\mathrm{d}V\} \\
&= D_{2U}^{n}\{\frac{[\det(U)]^{-\gamma}}{\Gamma_p(n-\alpha)}\int_{W>O}[\det(W)]^{n-\alpha-\frac{p+1}{2}}[\det(I+W)]^{-\rho+\alpha-n-\gamma}\mathrm{d}W\} \\
&= D_{2U}^{n}[\det(U)]^{-\gamma}\frac{\Gamma_p(\rho+\gamma)}{\Gamma_p(\rho+\gamma+n-\alpha)}, Y=V-U, W=U^{-\frac{1}{2}}YU^{-\frac{1}{2}} \\
&= \frac{\Gamma_p(\rho+\gamma)}{\Gamma_p(\rho+\gamma+n-\alpha)}\frac{1}{\Gamma_p(\gamma)}\int_{S>O}[\det(S)]^{\gamma-\frac{p+1}{2}}D_{2U}^{n}\mathrm{e}^{-\mathrm{tr}(US)}\mathrm{d}S \\
&= \frac{\Gamma_p(\rho+\gamma)}{\Gamma_p(\rho+\gamma+n-\alpha)}\frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}[\det(U)]^{-(\gamma+n)}. \qquad (5.4)
\end{aligned}
$$

This is the αth order Kober fractional derivative of the second kind in the Riemann-Liouville sense.

Case 5.1b: The Caputo derivative in Case 5.1a

Consider

$$
\begin{aligned}
K_{2,U}^{\alpha}f &= K_{2,U}^{-(n-\alpha)}[D_{2V}^{n}f] \\
&= \frac{[\det(U)]^{\rho}}{\Gamma_p(n-\alpha)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}[\det(V)]^{-\rho-n+\alpha} \\
&\times \{D_{2V}^{n}[\det(V)]^{-\gamma}\}\mathrm{d}V \\
&= \frac{[\det(U)]^{\rho}}{\Gamma_p(n-\alpha)}\frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}\int_{V>U}[\det(V-U)]^{n-\alpha-\frac{p+1}{2}}[\det(V)]^{-\rho-2n-\gamma+\alpha}\mathrm{d}V \\
&= \frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}\frac{\Gamma_p(\rho+n+\gamma)}{\Gamma_p(\gamma+\rho+2n-\alpha)}[\det(U)]^{-(\gamma+n)}
\end{aligned}
$$

for $\Re(\gamma)>\frac{p-1}{2}, n>\Re(\alpha)+\frac{p-1}{2}$. Note that the Caputo derivative is different from the Riemann-Liouville derivative in this case.

Case 5.2a: $f(V)=\mathrm{e}^{-\mathrm{tr}(V)}$

This will go to a Whittaker function of matrix argument, both in the Riemann-Liouvile and in the Caputo senses. For the final integrals, see [4]. Final integrals will be of the form

$$\int_{S>O}[\det(S)]^{\alpha_1-\frac{p+1}{2}}[\det(I+S)]^{-(\alpha_1+\beta_1)}\mathrm{e}^{-\mathrm{tr}(S)}\mathrm{d}S$$

for some $\Re(\alpha_1)>\frac{p-1}{2}, \Re(\beta_1)>\frac{p-1}{2}$. Hence this case will not be discussed here.

6. Kober Fractional Derivatives of the First Kind, of order α

We will consider fractional derivative of order α in the Riemann-Liouville and in the Caputo senses for some special cases of $f(V)$. The Kober integral operator of the first kind is given by the following, see [8]:

$$K_{1,U}^{-\alpha}f = \frac{[\det(U)]^{-\rho-\alpha}}{\Gamma_p(\alpha)}\int_{V<U}[\det(V)]^{\rho}[\det(U-V)]^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V. \tag{6.1}$$

The αth order fractional derivative of the first kind in the Riemann-Liouville sense is then given by the following:

$$\begin{aligned} K_{1,U}^{\alpha}f &= D_{1U}^{n}\{K_{1,U}^{-(n-\alpha)}f\} \\ &= D_{1U}^{n}\frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(V)]^{\rho}[\det(U-V)]^{n-\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \end{aligned} \tag{6.2}$$

for $n > \Re(\alpha) + \frac{p-1}{2}$, and that in the Caputo sense is given by the following:

$$K_{1,U}^{\alpha}f = \frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(V)]^{\rho}[\det(U-V)]^{n-\alpha-\frac{p-1}{2}}\{D_{1V}^{n}f(V)\}\mathrm{d}V. \tag{6.3}$$

Let us examine these two types of derivatives for some special cases of $f(V)$.

Case 6.1: $f(V) = \mathrm{e}^{\pm\mathrm{tr}(V)}$

In this case the integrals to be evaluated, corresponding to (6.2) and (6.3) will be of the form

$$\int_{O<W<I}[\det(W)]^{\gamma_1-\frac{p+1}{2}}[\det(I-W)]^{\gamma_2-\frac{p+1}{2}}\mathrm{e}^{\pm\mathrm{tr}(UW)}\mathrm{d}W$$

for $\Re(\gamma_i) > \frac{p-1}{2}, i = 1, 2$ and the integral will go to confluent hypergeometric function of matrix argument, see [4], and hence it will not be discussed here.

Case 6.2: $f(V) = \frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}, \Re(\gamma) > \frac{p-1}{2}$

In this case (6.2) will reduce to the following for $n > \Re(\alpha) + \frac{p-1}{2}, \Re(\gamma) > \frac{p-1}{2}$:

$$\begin{aligned} K_{1,U}^{\alpha}f &= D_{1U}^{n}\{K_{1,U}^{-(n-\alpha)}f\} \\ &= D_{1U}^{n}\{\frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)}\int_{V<U}[\det(V)]^{\rho}[\det(U-V)]^{n-\alpha-\frac{p+1}{2}} \\ &\times \frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}\mathrm{d}V\} \\ &= D_{1U}^{n}\{\frac{[\det(U)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)}\frac{1}{\Gamma_p(n-\alpha)}\int_{O<W<I}[\det(W)]^{\rho+\gamma-\frac{p+1}{2}} \\ &\times [\det(I-W)]^{n-\alpha-\frac{p+1}{2}}\mathrm{d}W\} \\ &= \frac{[\det(U)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}\frac{\Gamma_p(\rho+\gamma)}{\Gamma_p(n-\alpha+\rho+\gamma)} \end{aligned} \tag{6.4}$$

for $\Re(\gamma) > n + \frac{p-1}{2}, \Re(\rho+\gamma) > \frac{p-1}{2}, n > \Re(\alpha) - \Re(\gamma)$. This is the Kober fractional derivative of order α of the first kind in the Riemann-Liouville sense. Now, consider (6.3).

$$D_{1V}^n f(V) = D_{1V}^n \frac{[\det(V)]^{\gamma-\frac{p+1}{2}}}{\Gamma_p(\gamma)} = \frac{[\det(V)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)}, \Re(\gamma) > n + \frac{p-1}{2}.$$

Then

$$\begin{aligned} K_{1,U}^{\alpha} f &= \frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)} \int_{V<U} \frac{[\det(V)]^{\rho+\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)} [\det(U-V)]^{n-\alpha-\frac{p+1}{2}} \mathrm{d}V \\ &= \frac{[\det(U)]^{\gamma-n-\frac{p+1}{2}}}{\Gamma_p(\gamma-n)} \frac{\Gamma_p(\rho+\gamma-n)}{\Gamma_p(\rho+\gamma-\alpha)} \end{aligned} \tag{6.5}$$

for $n > \Re(\alpha) + \frac{p-1}{2}, n < \Re(\gamma) - \frac{p-1}{2}, \Re(\rho+\gamma) > \Re(\alpha) + \frac{p-1}{2}$. Note that the expressions in (6.4) and (6.5) are different.

7. Fractional Derivatives of Matrix-variate Statistical Densities

In [8] it is shown that matrix-variate statistical densities are directly connected to Kober fractional integral operators. Let X_1 and X_2 be $p \times p$ statistically independently distributed real matrix-variate random variables. Let $U_2 = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$ and $U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}$. Then U_2 and U_1 are called product and ratio of matrices X_1 and X_2 where $X_2^{\frac{1}{2}}$ denotes the real positive definite square root of the real positive definite matrix X_2. If the densities of U_2 and U_1 are denoted by $g_2(U_2)$ and $g_1(U_1)$ respectively then it is shown in [8] that

$$g_2(U_2) = \frac{\Gamma_p(\alpha+\gamma+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{p+1}{2})} K_{2,U}^{-\alpha} \text{ and } g_1(U_1) = \frac{\Gamma_p(\alpha+\gamma)}{\Gamma_p(\gamma)} K_{1,U}^{-\alpha} \tag{7.1}$$

where $K_{2,U}^{-\alpha}$ is the Kober fractional integral operator of order α and of the second kind, given in (5.1), and $K_{1,U}^{-\alpha}$ is the Kober fractional integral operator of order α and of the first kind, given in (6.1). Hence, fractional derivatives of order α of the second kind, in the Riemann-Liouville and in the Caputo senses, of the density $g_2(U_2)$, are available from (5.2) and (5.3) by multiplying with the constant $\frac{\Gamma_p(\alpha+\gamma+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{p+1}{2})}$. For the density $g_1(U_1)$ the fractional derivative of order α and of the first kind, in the Riemann-Liouville and in the Caputo senses, are available from (6.2) and (6.3) by multiplying with the constant $\frac{\Gamma_p(\alpha+\gamma)}{\Gamma_p(\gamma)}$. In both these cases $f(V)$ is assumed to be a statistical density of the real positive definite $p \times p$ matrix V. For $f(V)$ we have considered two special cases in Section 6. One was $[\det(V)]^{-\rho}$. Note that the results will go through for the function $[\det(I+V)]^{-\rho}$ also. This can be made into a statistical density by multiplying with a constant. Note that

$$\begin{aligned} f(V) &= \frac{\Gamma_p(\rho)}{\Gamma_p(\frac{p+1}{2})\Gamma_p(\rho-\frac{p+1}{2})} [\det(I+V)]^{-\rho} \\ &= \frac{\Gamma_p(\rho)}{\Gamma_p(\frac{p+1}{2})\Gamma_p(\rho-\frac{p+1}{2})} [\det(V)]^{\frac{p+1}{2}-\frac{p+1}{2}} [\det(I+V)]^{-\rho}, V > O \end{aligned} \tag{7.2}$$

which is a type-2 matrix-variate beta density with parameters $(\frac{p+1}{2}, \rho - \frac{p+1}{2})$. Hence the fractional derivative of $g_2(U_2)$ is available from those of the Kober fractional integral operator of order α, namely $K_{2,U}^{-\alpha}$. Thus, we can define the fractional derivative of the density, $D_{2,U_2}^{\alpha} g_2(U_2)$, by using the density in (7.2).

The second function that we have considered for $f(V)$ was of the form

$$f(V) = \frac{[\det(V)]^{\rho - \frac{p+1}{2}}}{\Gamma_p(\rho)} = \frac{1}{\Gamma_p(\rho)} [\det(V)]^{\rho - \frac{p+1}{2}} [\det(I-U)]^{\frac{p+1}{2} - \frac{p+1}{2}}. \qquad (7.3)$$

Hence $\frac{\Gamma_p(\rho + \frac{P+1}{2})}{\Gamma_p(\frac{p+1}{2})} f(V)$ is a statistical density, which is a type-1 matrix-variate beta density with parameters $(\rho, \frac{p+1}{2})$. Hence the fractional derivative of order α for the density $g_1(U_1)$ in (7.1) is available from the corresponding derivative of $K_{1,U}^{-\alpha}$ with $f(V)$ in (7.3) multiplied by the constant $\frac{\Gamma_p(\rho + \frac{p+1}{2})}{\Gamma_p(\frac{p+1}{2})}$. Thus, we can define the fractional derivative of the density $g_1(U_1)$, that is, $D_{1,U_1}^{\alpha} g_1(U_1)$, with the help of $K_{1,U}^{-\alpha}$ of (7.1) and the density in (7.3).

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