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AN APPLICATION OF FRACTIONAL CALCULUS AND ITS IMPLICATIONS RELATING TO CERTAIN ANALYTIC FUNCTIONS AND COMPLEX EQUATIONS

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Abstract. The aim of this investigation is first to expose some comprehensive results constituted by fractional calculus and also related to both certain analytic functions and complex equations, and then to point some useful consequences of them out.

1. Introduction, definitions and notations

In the literature, as a term, fractional calculus (FC) is more than 300 years old. As we know, it is a generalization of the ordinary differentiation and integration to non-integer (arbitrary) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. Several mathematicians contributed to this subject over the years. Specially, people like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story of the fractional calculus also continued with contributions from Fourier, Abel, Leibniz, Grünwald, and Letnikov.

Nowadays, the fractional calculus attracts many scientists and engineers. There are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on. It is natural that many authors have been trying to determine the fractional derivatives and fractional integrals and also to solve fractional differential equations by using certain techniques.

Fractional differential equations (FDE), i.e., differential equations determined by FC, have also many applications in modeling of physical and chemical processes and in engineering. In its turn, mathematical aspects of studies on FDC were discussed by several authors.

The main purpose of this investigation is both to present a novel work relating to analytic and/or geometric function theory (AGFT) and FDC and to reveal some (comprehensive) results between certain complex valued functions and complex (differential) equations constituted by certain operators dealing with FC. In

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particular, special consequences of the main results are also pointed out in the second section of this paper.

For the main results, we begin to publicize a number of notations and certain well-known definitions.

Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{U}$ be the set of real numbers, the set of complex numbers, the set of positive integers and unit open disk: $\big\{z \in \mathbb{C} : |z| < 1\big\}$, respectively. Also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}^* := \mathbb{R} - \{0\}$.

For $0 \leq \mu < 1$ and an analytic function $\kappa := \kappa(z)$, the symbol $\mathcal{D}_z^{\mu}[\kappa]$ denotes an operator of FC, which is defined as follows (cf., e.g., [1], [2], [3], [7]):

Let $\kappa(z)$ be an analytic function in a simply-connected region of the $z-$plane containing the origin. Then, the fractional derivative of order μ is defined by

$$\mathcal{D}_z^{\mu}[\kappa] = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{\kappa(\xi)}{(z-\xi)^{\mu}}\, d\xi \qquad (0 \leq \mu < 1), \tag{1}$$

where the multiplicity of $(z-\xi)^{-\mu}$ above is removed by requiring $log(z-\xi)$ to be real when $z - \xi > 0$. Note that, here and throughout this paper, the function Γ is the well-known gamma function.

Under the hypotheses of the definition above, for the function $\kappa(z)$, the fractional derivative of order $m+\mu$ is also defined by

$$\mathcal{D}_z^{m+\mu}[\kappa] = \frac{d^m}{dz^m}\Big(\mathcal{D}_z^{\mu}[\kappa]\Big) \qquad \big(0 \leq \mu < 1; m \in \mathbb{N}_0\big). \tag{2}$$

By means of (1) and (2), for a function $\kappa(z) = z^{\sigma}$, it can be easily determined that

$$\mathcal{D}_z^{m+\mu}\big[z^{\sigma}\big] = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-m-\mu+1)} z^{\sigma-m-\mu} \tag{3}$$

for some $0 \leq \mu < 1$ and for all $m \in \mathbb{N}_0$ with $m < \sigma - \mu + 1$.

Let $\mathcal{A}$ denote the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \quad (a_{n+1} \in \mathbb{C};\ n \in \mathbb{N}), \tag{4}$$

which are analytic in the the domain $\mathbb{U}$. Also let $\mathcal{S}$ denote the family of functions belonging to $\mathcal{A}$ which are univalent in the open disk $\mathbb{U}$. As is known, the functions family $\mathcal{A}$ has an important roles for AGFT (see [4], [5]). In especial, some of the important and well-investigated families of the univalent function family $\mathcal{S}$ include the family $\mathcal{S}^*(\tau)$ of starlike functions of order τ and the family $\mathcal{K}(\tau)$ of convex functions of order τ $(0 \leq \tau < 1)$ in the domain $\mathbb{U}$. Indeed, we have

$$\mathcal{S}^*(\tau) := \left\{ f \in \mathcal{S} \ : \ \Re e\left(\frac{zf'(z)}{f(z)}\right) > \tau \ \ \big(0 \leq \tau < 1; z \in \mathbb{U}\big) \right\} \tag{5}$$

and

$$\mathcal{K}(\tau) := \left\{ f \in \mathcal{S} \ : \ \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \tau \ \ \big(0 \leq \tau < 1; z \in \mathbb{U}\big) \right\}. \tag{6}$$

It readily follows from the definitions (5) and (6) that $f(z) \in \mathcal{K}(\tau)$ if and only if $zf'(z) \in \mathcal{S}^*(\tau)$. For their details, one may refer to [4] and [5].

Next, by making use of the operator $\mathcal{D}_z^{\mu}[\cdot]$, for a function $f(z)$ belonging to the class $\mathcal{A}$, we can again define a linear operator $J_z^{\mu}[f]$ as in the form:

$$J_z^{\mu}[f] = \Gamma(2-\mu)\, z^{\mu}\, \mathcal{D}_z^{\mu}[f] = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)} a_k z^k, \tag{7}$$

where $\mu \in \mathbf{R} := \mathbb{R} - \{2, 3, 4, \cdots\}$.

For the scope of this investigation, we also define an analytic function $\mathcal{F} := \mathcal{F}(z)$ in the following form:

$$\mathcal{F}(z) = \big(1-\lambda\big) f(z) + \lambda z f'(z) \quad \Big(0 \leq \lambda \leq 1; f(z) \in \mathcal{S}\Big). \tag{8}$$

In order to prove the main result, it is a need to recall the well-known assertion given by [8].

Lemma 1. Let $q(z)$ be an analytic function in the disk $\mathbb{U}$ with $q(0) = 1$. If there exists a point z_0 in $\mathbb{U}$ such that

$$\Re e\Big(q(z)\Big) > 0 \quad \Big(|z| < |z_0|\Big) \quad , \quad \Re e\Big(q(z_0)\Big) = 0 \quad and \quad q(z_0) \neq 0, \tag{9}$$

then

$$q(z_0) = ia \quad and \quad \left.\frac{zq'(z)}{q(z)}\right|_{z=z_0} = i\rho\Big(a + \frac{1}{a}\Big) \quad \Big(a \in \mathbb{R}^*; \rho \geq \frac{1}{2}\Big). \tag{10}$$

2. The main result and its implications

We now state and prove our main result consisting of several comprehensive consequences dealing with certain analytic functions and complex equations given by the theorem below.

Theorem 1. Let $\phi(z)$ be an analytic function and satisfy any one of the inequalities given by

$$\Re e\Big(\phi(z)\Big) \neq 0 \quad \text{and} \quad \Big|\Im m\Big(\phi(z)\Big)\Big| < 1 \quad \big(z \in \mathbb{U}\big). \tag{11}$$

If the function $\mathcal{F}$, defined by (8), is a solution for the complex equation given by

$$\gamma\, \phi(z)\left(\frac{J_z^\beta\big[\mathcal{F}\big]}{J_z^\alpha\big[\mathcal{F}\big]}\right)^\omega + \omega\, z \left(\frac{J_z^\beta\big[\mathcal{F}\big]}{J_z^\alpha\big[\mathcal{F}\big]}\right)\left(\frac{J_z^\alpha\big[\mathcal{F}\big]}{J_z^\beta\big[\mathcal{F}\big]}\right)' - \phi(z) \;=\; 0\,, \tag{12}$$

then

$$\Re e\left[\;\left(\frac{J_z^\alpha\big[\mathcal{F}\big]}{J_z^\beta\big[\mathcal{F}\big]}\right)^\omega\;\right] > \gamma\,, \tag{13}$$

where $0 \leq \gamma < 1$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\omega \in \mathbb{R}^*$, $z \in \mathbb{U}$ and, here and throughout this paper, the value of the above complex power is also taken to be as its principal value.

Proof. Let the function $\mathcal{F}$ define as in the form (8). Then, in view of the operator in (7), it can be easily obtained that

$$\begin{aligned}
\left(\frac{J_z^\alpha\big[\mathcal{F}\big]}{J_z^\beta\big[\mathcal{F}\big]}\right)^\omega &= \left(\frac{\Gamma(2-\alpha)\, z^\alpha\, \mathcal{D}_z^\alpha[\mathcal{F}]}{\Gamma(2-\beta)\, z^\beta\, \mathcal{D}_z^\beta[\mathcal{F}]}\right)^\omega \\
&= \left(\frac{1 + \sum_{k=2}^\infty \frac{(k\lambda-\lambda+1)\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k-\alpha+1)} a_k z^{k-1}}{1 + \sum_{k=2}^\infty \frac{(k\lambda-\lambda+1)\Gamma(k+1)\Gamma(2-\beta)}{\Gamma(k-\beta+1)} a_k z^{k-1}}\right)^\omega \\
&= \big(1 + c_1 z + c_2 z^2 + c_3 z^3 \;+\cdots\;\big)^\omega \\
&= 1 + a_1 z + a_2 z^2 + a_3 z^3 \;+\cdots\;\;,
\end{aligned} \tag{14}$$

where $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\omega \in \mathbb{R}^*$ and $z \in \mathbb{U}$.

With the help of (14), define an implicit function $q(z)$ in the form:

$$\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)^{\omega}=\gamma+(1-\gamma)q(z), \tag{15}$$

where $0\leq\gamma<1$, $\omega\in\mathbb{R}^*$ and $z\in\mathbb{U}$. Obviously, $q(z)$ is an analytic function in $\mathbb{U}$ satisfying $p(0)=1$. By differentiating the both sides of (15) with respect to the complex varible z, it can be easily derived that

$$\frac{z\cdot\left\{\frac{1}{1-\gamma}\cdot\left[\left(\frac{J_z^{\alpha}[\mathcal{F}]}{J_z^{\beta}[\mathcal{F}]}\right)^{\omega}-\gamma\right]\right\}'}{\frac{1}{1-\gamma}\cdot\left[\left(\frac{J_z^{\alpha}[\mathcal{F}]}{J_z^{\beta}[\mathcal{F}]}\right)^{\omega}-\gamma\right]}=\frac{zq'(z)}{q(z)}$$

or, equivalently,

$$\frac{\omega\, z\left(\frac{J_z^{\alpha}[\mathcal{F}]}{J_z^{\beta}[\mathcal{F}]}\right)'\left(\frac{J_z^{\alpha}[\mathcal{F}]}{J_z^{\beta}[\mathcal{F}]}\right)^{\omega-1}}{\frac{J_z^{\alpha}[\mathcal{F}]}{J_z^{\beta}[\mathcal{F}]}-\gamma}=:\phi(z)\left(:=\frac{zq'(z)}{q(z)}\right).\quad (say) \tag{16}$$

After simple calculations, it is easily seen that the equation in (16) is equivalent to the equation given by (12).

We now assume that there exists a point $z_0\in\mathbb{U}$ satisfying the hypotheses in (9). From (10), we then have that

$$q(z_0)=ia\quad\text{and}\quad\left.\frac{zq'(z)}{q(z)}\right|_{z=z_0}=\ i\rho\Big(a+\frac{1}{a}\Big)\quad\Big(\rho\geq\frac{1}{2};a\in\mathbb{R}^*\Big).$$

If we use the hypotheses above in (16), we easily obtain that

$$\Re e\Big(\phi(z_0)\Big)=\Re e\left(\left.\frac{zq'(z)}{q(z)}\right|_{z=z_0}\right)\ =\ 0$$

and

$$\Big|\Im m\Big(\phi(z_0)\Big)\Big|=\left|\Im m\left(\left.\frac{zq'(z)}{q(z)}\right|_{z=z_0}\right)\right|\ =\ \rho\left|a+\frac{1}{a}\right|\ \geq\ 2\rho\geq1,$$

which are contradictions with the assumptions given by (11), respectively. Hence, the equality in (15) immediately yields that

$$\Re e\left[\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)^{\omega}\right]=\Re e\Big(\gamma+(1-\gamma)q(z)\Big)>\gamma,$$

where $0\leq\gamma<1$, $\alpha\in\mathbf{R}$, $\beta\in\mathbf{R}$, $\omega\in\mathbb{R}^*$ and $z\in\mathbb{U}$. This completes the desired proof.

As a great number of the implications of the main result, when one looks over the related theorem, it is easily observed that it includes several comprehensive results between complex functions relating to AGFT and several types of certain complex equations connecting with FDE, which were constituted by FC. As example, we want to center on only some of them and also their certain useful applications. The other possible consequences of the main result, which are here omitted, are presented to the attention of the researchers who have been working on the theory of differential equation and/or univalent function.

Firstly, upon setting $\omega:=-1$ in the Theorem 1, we immediately arrive at the following consequence of Theorem 1, which is the Corollary 1.

Corollary 1. Let $\phi(z)$ be an analytic function and satisfy any one of the following inequalities in (11). If the function $\mathcal{F}$, defined by (8), is a solution for the following complex equation:

$$\gamma\,\phi(z)\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)\,-\,z\left(\frac{J_z^{\beta}\big[\mathcal{F}\big]}{J_z^{\alpha}\big[\mathcal{F}\big]}\right)\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)'-\phi(z)\;=\;0\,,$$

then

$$\Re e\left(\frac{J_z^{\beta}\big[\mathcal{F}\big]}{J_z^{\alpha}\big[\mathcal{F}\big]}\right)>\gamma\quad\big(0\leq\gamma<1;\alpha\in\mathbf{R};\beta\in\mathbf{R};z\in\mathbb{U}\big).$$

Secondly, upon letting $\beta:=1+\alpha$ in the Corollary 1 above (or $\omega:=-1$ and $\beta:=1+\alpha$ $(\alpha\in\mathbf{R})$ in the Theorem 1), we thus receive the following consequence of the Theorem 1, which is Corollary 2.

Corollary 2. Let $\phi(z)$ be an analytic function and satisfy any one of the following inequalities in (11). If the function $\mathcal{F}$, defined by (8), is a solution for the following complex equation:

$$\gamma\,\phi(z)\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{1+\alpha}\big[\mathcal{F}\big]}\right)\,-\,z\left(\frac{J_z^{1+\alpha}\big[\mathcal{F}\big]}{J_z^{\alpha}\big[\mathcal{F}\big]}\right)\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{1+\alpha}\big[\mathcal{F}\big]}\right)'-\phi(z)\;=\;0\,,$$

then

$$\Re e\left(\frac{J_z^{1+\alpha}\big[\mathcal{F}\big]}{J_z^{\alpha}\big[\mathcal{F}\big]}\right)>\gamma\quad\big(0\leq\gamma<1;\alpha\in\mathbf{R};z\in\mathbb{U}\big).$$

The third consequence of the related theorem is contained in the following comprehensive proposition.

Proposition 1. Let $\phi(z)$ be an analytic function and satisfy any one of the inequalities given by (11), and also let $\mathcal{F}(z)$ be defined by (8). If $\mathcal{F}$ satisfies the following complex equation:

$$\Big[\,\alpha\,J_z^{2+\alpha}[\mathcal{F}]+\big(\phi(z)-1\big)\,J_z^{1+\alpha}[\mathcal{F}]\,\Big]\,J_z^{\alpha}[\mathcal{F}]$$
$$+\,(1-\alpha)\,\Big[\,J_z^{1+\alpha}[\mathcal{F}]\,\Big]^2\,-\,\gamma\,\phi(z)\,\Big[\,J_z^{\alpha}[\mathcal{F}]\,\Big]^2\;=\;0\,,$$

then

$$\Re e\left(\frac{J_z^{1+\alpha}[\mathcal{F}]}{J_z^{\alpha}[\mathcal{F}]}\right)>\gamma\quad\big(0\leq\gamma<1;\beta\in\mathbf{R};z\in\mathbb{U}\big).$$

Proof. If one takes the value of β as $\beta:=1+\alpha$ $\;(0\leq\alpha<1)$ in the Theorem 1 and also uses the following well-known identity:

$$z\,\Big(J_z^{\alpha}[\mathcal{F}]\Big)'\;=\;(1-\alpha)\,J_z^{1+\alpha}[\mathcal{F}]\;+\;\alpha\,J_z^{\alpha}[\mathcal{F}]\;,$$

it can be easily arrived at the proof of the Proposition 1. Its detail is here omitted.

By taking into consideration the values of the parameters α and λ as $\alpha:=0$ and $\lambda:=0$ in the Proposition 1 (or, equivalent choosing in the related theorem), respectively, the following corollary relating to starlikeness of order γ can be then revealed.

Corollary 3. Let $\psi(z)$ be an analytic function and satisfy any one of the inequalities given by (11). If the function $w:=f(z)\in\mathcal{S}$ satisfies the following nonlinear complex differential equation:

$$zw\big[zw'' + \big(1-\psi(z)\big)w'\big] - \big[zw'\big]^2 + \gamma\phi(z)w^2 = 0,$$

then $w \in \mathcal{S}^*(\gamma)$, where $0 \leq \gamma < 1$ and $z \in \mathbb{U}$.

By taking in consideration the values of the parameters α and λ as $\alpha := 0$ and $\lambda := 1$ in the Proposition 1 (or, equivalent choosing in the related theorem), respectively, the following corollary relating to convexity of order γ can be also found out.

Corollary 4. Let $\psi(z)$ be an analytic function and satisfy any one of the inequalities given by (11). If the function $w := f(z) \in \mathcal{S}$ satisfies the following nonlinear complex differential equation:

$$zw'\big[zw''' + \big(1-\psi(z)\big)w''\big] - \big[zw''\big]^2 - (1-\gamma)\phi(z)\big[w'\big]^2 = 0,$$

then $w \in \mathcal{K}(\gamma)$, where $0 \leq \gamma < 1$ and $z \in \mathbb{U}$.

By letting $\omega := 1$ and $\phi(z) := \psi(z)$ in the Theorem 1, the following result including the corrected hypothesis of the earlier results obtained by Irmak and Frasin ([6], Theorem 1) can be easily derived.

Remark 1. Let $\psi(z)$ be an analytic function and satisfy any one of the following inequalities in (11). If the function $\mathcal{F}$, defined by (8), is a solution for the following complex equation:

$$z\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)' - \psi(z)\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right) + \gamma\,\psi(z) \;=\; 0\;,$$

then

$$\Re e\left(\frac{J_z^{\alpha}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right) > \gamma \qquad \big(0 \leq \gamma < 1; \alpha \in \mathbf{R}; \beta \in \mathbf{R}; z \in \mathbb{U}\big).$$

By putting $\alpha := 1+\beta$ in the Theorem 1, the following corollary involving several consequences relating to AGFT and/or FDC can be lastly determined.

Corollary 5. Let $\phi(z)$ be an analytic function and satisfy any one of the following inequalities in (11). If the function $\mathcal{F}$, defined by (8), is a solution for the following complex equation:

$$\gamma\,\phi(z)\left(\frac{J_z^{\beta}\big[\mathcal{F}\big]}{J_z^{1+\beta}\big[\mathcal{F}\big]}\right)^{\omega} + \omega\,z\left(\frac{J_z^{\beta}\big[\mathcal{F}\big]}{J_z^{1+\beta}\big[\mathcal{F}\big]}\right)\left(\frac{J_z^{1+\beta}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)' - \phi(z) \;=\; 0\,,$$

then

$$\Re e\left[\;\left(\frac{J_z^{1+\beta}\big[\mathcal{F}\big]}{J_z^{\beta}\big[\mathcal{F}\big]}\right)^{\omega}\;\right] > \gamma \qquad \big(0 \leq \gamma < 1; \beta \in \mathbf{R}; \omega \in \mathbb{R}^*; z \in \mathbb{U}\big).$$

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