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A GENERALIZED ALGORITHM BASED ON LEGENDRE POLYNOMIALS FOR NUMERICAL SOLUTIONS OF COUPLED SYSTEM OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study shifted Legendre polynomials and provide a simple algorithm for the approximate solution of coupled system of fractional differential equations. We generalize some operational matrices. Based on these matrices a coupled system is analytically converted to easily solvable algebraic equations. Two types of orthogonal systems are used, the Legendre polynomials and Haar wavelets. The results of both the systems is compared. The method is computer oriented and provide highly accurate solution. To demonstrate the efficiency of the method, several examples are solved and the results are displayed graphically. For some problems the results are also compared with some other results available in the literature.

1. Introduction

Coupled systems of differential equations are of basic importance in modeling various phenomena like Cascades and Compartment Analysis, Pond Pollution, Home Heating, Chemostats and Microorganism Culturing, Nutrient Flow in an Aquarium, Biomass Transfer, Forecasting Prices, Electrical Network, Earthquake Effects on Buildings see for example [2, 5, 15, 18, 23, 25] and many more. After the discovery of fractional calculus it is investigated by many authors that the fractional derivatives can best approximate the situation under consideration as compared to ordinary derivatives. But due to computational complexities of fractional derivatives the non availability of exact analytical solution is the great problem for the researcher in the field of fractional calculus. Therefore establishment of numerical schemes is of great importance in the current field of fractional calculus.

The operational matrix technique is a simple technique and is used widely for solving a wide class of fractional differential equations with different kinds of conditions see for example [16, 17, 20, 21, 22, 24] and the references quoted there. These operational matrices are based on various orthogonal polynomials and wavelets. Orthogonal polynomials are frequently applied by many mathematicians. M. M.

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Khader [7, 10, 26] used Legendre and Chebyshev polynomials to establish an efficient method for the approximate solution of some important class of fractional differential equations. In [26] the author studied fractional order logistic equations with two different delays and used Chebyshev polynomials to approximate the solution of the problem. In [10] M. M. Khader used Legendre polynomials and establish an efficient algorithm to approximate the solution of high order fractional differential equations. In [10] the same author applied Legendre polynomial and efficiently solve fractional order advection dispersion equations. In [8] the operational matrices are used to approximate the solution of nonlinear fractional differential equations. In [9] the orthogonal polynomials are used to establish a method for the solution of delay differential equations.

Unfortunately the coupled system of fractional differential equations got less attention by solving with the operational matrix techniques. In our previous paper [11, 12] we successfully developed a scheme for a small class of coupled system of fractional order partial differential equations with initial condition.

In this paper, we generalized the operational matrix techniques to solve a wide class of coupled system of fractional order differential equations with initial condition. Here the most simplest one that is Legendre polynomials and Haar wavelets are used, operational matrices are modified and a simple but highly efficient technique is developed to solve the corresponding system. The results obtained with Legendre polynomials and Haar wavelets are compared.
The article is organized as follows : in section 2, we provide some preliminaries of fractional calculus, orthogonal polynomials and Haar wavelets, in section 3 we present some operational matrices of integration and differentiation for Legendre polynomials and Haar wavelets ,in section 4 the operational matrices are used to generalize the numerical schemes for a generalized class of coupled systems of fractional order differential equations, in section 5 we solve some models and provide the numerical results of the schemes. At last in section 6 a short conclusion is made.

2. Preliminaries

In this section, we summarizes some necessary concepts, definitions and basic results from fractional calculus and orthogonal polynomials which are useful for development in this paper.

Definition 2.1. [13, 23] *According to Riemann-Liouville, the fractional order integral of order $\alpha \in \mathbf{R}_+$ of a function $\phi \in (L^1[a,b], \mathbf{R})$ on interval $[a,b] \subset \mathbf{R}$, is defined by*

$$\mathcal{I}_{a+}^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\phi(s)ds, \tag{1}$$

provided that the integral on right hand side exists.

Definition 2.2. *For a given function $\phi(x) \in C^n[a,b]$, the Caputo fractional order derivative of order α is defined as*

$$D^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\phi^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt,\ n-1 \leq \alpha < n\ ,\ n \in N, \tag{2}$$

provided that the right side is pointwise defined on (a,∞), where $n = [\alpha]+1$.

From (1),(2) it is easily deducted that

$$D^{\alpha}x^{k}=\frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}x^{k-\alpha},\ I^{\alpha}x^{k}=\frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}x^{k+\alpha}\ \text{and}\ D^{\alpha}C=0,\ \text{for a constant } C. \tag{3}$$

2.1. **The shifted Legendre polynomials.** The Legendre polynomials are defined by the following recurrence relation

$$L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z),\ i=1,2....,\ \text{where } L_0(z)=0,\ L_1(z)=z.$$

These polynomials are defined on $[-1,1]$. For our purpose we use the transformation $z=(2x-1)/\eta$ which transforms the interval $[-1,1]$ to $[0,\eta]$. The analytical expression for the shifted Legendre polynomials on $[0,\eta]$ is given by

$$P_i^{\eta}(x)=\sum_{k=o}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(\eta^k)(k!)^2},\ i=0,1,..., \tag{4}$$

where $P_i^{\eta}(0)=(-1)^i$, $P_i^{\eta}(\eta)=1$. The orthogonality condition is

$$\int_0^{\eta}P_i^{\eta}(x)P_j^{\eta}(x)dx=\begin{cases}\frac{\eta}{2i+1}, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases} \tag{5}$$

Which implies that any $f(x)\in C[0,\eta]$ can be approximated by Legendre polynomials as follows

$$f(x)\approx\sum_{a=0}^{m}C_aP_a^{\eta}(x),\ \text{where } C_a=\langle f(x),P_a^{\eta}(x)\rangle=(2a+1)\int_0^{\eta}f(x)P_a^{\eta}(x)dx. \tag{6}$$

In vector notation, we write

$$f(x)=K_M^T\Psi_M(x). \tag{7}$$

Where $M=m+1$, K_M is the coefficient vector and $\Psi_M(x)$ is a function vector which contains the Legendre polynomials. M represents the order of these vectors. For sufficiently smooth function $f(x)$ on $[0,\eta]$, the error of the approximation is given by

$$\|f(x)-\sum_{i=0}^{m}c_iP_i^{\eta}(x)\|_2\leq(C_1\frac{1}{M^{M+1}}), \tag{8}$$

where

$$C_1=\frac{1}{4}\max_{x\in[0,\eta]}|\frac{d^{M+1}}{dx^{M+1}}f(x)|. \tag{9}$$

By the arguments given in [14, 4] we can easily prove the above equations. The spectral accuracy and decay of the expansion coefficient can be guaranteed by the following lemma.

Lemma 2.1. *Let $g(x)\in\prod_M(x)$ where $\prod_M(x)$ is the space span by first M Legendre polynomials and*

$$g(x)=\sum_{k=0}^{m}c_kP_k^{\eta}(x),$$

then

$$|\ c_k\ |\simeq\frac{C}{(\lambda_k)^m}\parallel g_{(m)}\parallel, \tag{10}$$

and

$$\| g(x) - \sum_{k=0}^{m} c_k P_k^{\eta}(x) \|^2 = \sum_{k=m}^{\infty} \gamma_k c_k^2. \tag{11}$$

Where $\lambda_k = k(k+1)$ *and* $c_k = \frac{1}{R_{\eta,j}^{(\alpha,\beta)}} \int_0^{\eta} y(x) P_k^{\eta}(x) dx$. *C is a constant and m can be chosen in a way such that* $y_{(2m)} \in \prod_M(x)$. *Also we have the equality*

$$g_{(m)} = L g_{(m-1)}(x) = L^m g(x).$$

Where L is the Sturm-Liouville operator, and $g_{(0)} = g(x)$.

Proof. By following the steps in [6] we can easily proof this lemma. □

From above Lemma, we conclude that if the function $g(x)? \in C?^{\infty}[0,1]$, we recover spectral decay of the expansion coefficients that is, $|c_k|$ decays faster than any algebraic order of λ_k. This result is valid and independent of specific boundary conditions on $g(x)$.

2.2. **Haar Wavelets.** Haar wavelets are frequently used in many problems. For our purpose we use the notation used in [19]. Let $\widetilde{I} = \widetilde{I}_{00} = [0,\eta]$ and $\widetilde{I}_{j,k} = [2^{-j}k\eta, 2^{-j}(k+1)\eta]$, then the Haar scaling and Wavelet function on $[0,\eta]$ are defined as follows:

$$\phi(x) = \frac{1}{\sqrt{\eta}} \chi_{\widetilde{I}}, \qquad \psi_{j,k} = \frac{2^{j/2}}{\sqrt{\eta}} (\psi_{\widetilde{I}_{j,k}^{l}}(x) - \psi_{\widetilde{I}_{j,k}^{r}}(x)), J \geq 0, j \leq J, k \leq 2^j - 1. \tag{12}$$

Where $\psi_{0,0} = \psi_1(t) = \frac{1}{\sqrt{\eta}}(\chi_{[0,\frac{\eta}{2}]}(x) - \chi_{[\frac{\eta}{2},\eta]}(x))$ is the mother wavelet function for the Haar system $\{\psi_{j,k} = 2^{j/2}\psi_1(2^j x - k)\}$. An arbitrary function $y \in L^2[0,\eta]$ can be expanded into the Haar wavelet series as follows:

$$\begin{aligned} \widetilde{y}(x) &= \langle y, \phi \rangle \phi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle y, \psi_{j,k} \rangle \psi_{j,k}(x) \\ &= \widetilde{c}\phi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} c_{j,k} \psi_{j,k}(x) = C_M^T \hat{\Psi}_M(x). \end{aligned} \tag{13}$$

Where $M = 2^J$ for some fixed $J \in N$. The Haar function vector $\hat{\Psi}_M(x)$ is given as

$$\hat{\Psi}_M(x) = [\phi(x), \psi_{0,0}(x), \psi_{1,0}(x), \psi_{1,1}(x), \psi_{2,0}(x), \cdots, \psi_{J-1,0}(x), \psi_{J-1,1}(x), \cdots, \psi_{J-1,2^J-1}(x)]. \tag{14}$$

It is also known that the Haar wavelet function vector can be represented in terms of Block pulse function such as

$$\hat{\Psi}_M(x) = \Psi_{M \times M} B_M(t), \tag{15}$$

where $\Psi_{M \times M}$ is the Haar matrix defined as

$$\Psi_{M \times M} = [\hat{\Psi}_M(\frac{\eta}{2M}) \hat{\Psi}_M(\frac{3\eta}{2M}) \cdots \hat{\Psi}_M(\frac{(2M-1)\eta}{2M})]. \tag{16}$$

For the proof and detail study we refer the reader to [19].

2.2.1. *Convergence of Haar approximation.* Babolian and Shahsavaran [1] derived the relation for the convergence of Haar wavelets approximation in the following form.

Lemma 2.2. *Let $y(x)$ be a differentiable function and assume that $y(x)$ have bounded first derivative on $[0,\eta]$, that is, there exist $K > 0$ such that $y'(x) \leq K$ then*

$$\|y(x) - \widetilde{y}(x)\|^2 \leq \frac{K^2}{3}\frac{1}{(2M)^2}. \tag{17}$$

For the proof of this relation we refer the reader to [1].

3. Operational Matrices of integration and differentiation of fractional order

The operational matrices of integration based on Legendre polynomials is also discussed by the famous mathematician A. Saadatmandi in [24], however these matrices will be efficient when we are interested in $[0,1]$, here we want to seek the solution on any finite domain $[0,\eta]$ so a slight modification in these result will make us comfortable with any finite domain. The following lemmas are important to establish our result.

Lemma 3.1. *Let $\Psi_M(x)$ be the function vector as defined in* (7) *then the integration of order α of $\Psi_M(x)$ is generalized as*

$$I^{\alpha}(\Psi_M(x)) \simeq H^{\eta,\alpha}_{M\times M}\Psi_M(x), \tag{18}$$

where $H^{\eta,\alpha}_{M\times M}$ is the operational matrix of integration of order α and is defined as

$$H^{\eta,\alpha}_{M\times M} = \begin{pmatrix} \sum_{k=0}^{0}\Theta_{0,0,k,\eta} & \sum_{k=0}^{0}\Theta_{0,1,k,\eta} & \cdots & \sum_{k=0}^{0}\Theta_{0,j,k,\eta} & \cdots & \sum_{k=0}^{0}\Theta_{0,m,j,\eta} \\ \sum_{k=0}^{1}\Theta_{1,0,k,\eta} & \sum_{k=0}^{1}\Theta_{1,1,k,\eta} & \cdots & \sum_{k=0}^{1}\Theta_{1,j,k,\eta} & \cdots & \sum_{k=0}^{1}\Theta_{1,m,j,\eta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{i}\Theta_{i,0,k,\eta} & \sum_{k=0}^{i}\Theta_{i,1,k,\eta} & \cdots & \sum_{k=0}^{i}\Theta_{i,j,k,\eta} & \cdots & \sum_{k=0}^{i}\Theta_{i,m,k,\eta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{m}\Theta_{m,0,k,\eta} & \sum_{k=0}^{m}\Theta_{m,1,k,\eta} & \cdots & \sum_{k=0}^{m}\Theta_{m,j,k,\eta} & \cdots & \sum_{k=0}^{m}\Theta_{m,m,k,\eta} \end{pmatrix}. \tag{19}$$

where

$$\Theta_{i,j,k,\eta} = \frac{(2j+1)}{\eta}\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!(\eta^{\alpha+1})}{(i-k)!k!\Gamma(k+\alpha+1)(j-l)!(l!)^2(k+l+\alpha+1)}. \tag{20}$$

Proof. Using (3) along with (4) we have

$$\begin{aligned} I^{\alpha}P_i^{\eta}(x) &= \sum_{k=o}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{I^{\alpha}x^k}{(\eta^k)(k!)^2} \\ &= \sum_{k=o}^{i}\frac{(-1)^{i+k}(i+k)!}{(k!)(i-k)!(\eta^k)\Gamma(k+\alpha+1)}x^{k+\alpha}. \end{aligned} \tag{21}$$

Now approximating $x^{k+\alpha}$ by $m+1$ terms of Legendre polynomials we get

$$x^{k+\alpha} \simeq \sum_{j=0}^{m} b_{k,j}P_j^{\eta}(x), \tag{22}$$

where

$$\begin{aligned} b_{k,j} =& \frac{(2j+1)}{\eta}\int_0^\eta x^{k+\alpha}P_j^\eta(x) \\ =& \frac{(2j+1)}{\eta}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(\eta^l)(j-l)(l!)^2}\int_0^\eta x^{k+l+\alpha} \\ =& \frac{(2j+1)}{\eta}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!(\eta)^{k+l+\alpha+1}}{(\eta^l)(j-l)(l!)^2(k+l+\alpha+1)}. \end{aligned} \tag{23}$$

Using (21), and (23) we get

$$I^\alpha P_i^\eta(x) = \sum_{k=o}^{i}\sum_{j=0}^{m}\frac{(-1)^{i+k}(i+k)!}{(k!)(i-k)!(\eta^k)\Gamma(k+\alpha+1)}b_{k,j}P_j^\eta(x).$$

$$I^\alpha P_i^\eta(x) = \sum_{k=o}^{i}\sum_{j=0}^{m}\frac{(-1)^{i+k}(i+k)!}{(k!)(i-k)!(\eta^k)\Gamma(k+\alpha+1)}(\frac{(2j+1)}{\eta}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)(l!)^2(\eta^l)(k+l+\alpha+1)})P_j^\eta(x),$$

or on rearranging we get

$$I^\alpha P_i^\eta(x) = \sum_{j=0}^{m}\sum_{k=o}^{i}\frac{(2j+1)}{\eta}\sum_{l=0}^{i}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!(\eta^{k+l+\alpha+1})}{(k!)(i-k)!(j-l)(l!)^2(\eta^{k+l})\Gamma(k+\alpha+1)(k+l+\alpha+1)}P_j^\eta(x).$$

Or

$$I^\alpha P_i^\eta(x) = \sum_{j=0}^{m}\sum_{k=0}^{i}\frac{(2j+1)}{\eta}\sum_{l=0}^{i}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!(\eta^{\alpha+1})}{(k!)(i-k)!(j-l)(l!)^2\Gamma(k+\alpha+1)(k+l+\alpha+1)}P_j^\eta(x).$$

Setting

$$\frac{(2j+1)}{\eta}\sum_{l=0}^{i}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!(\eta^{\alpha+1})}{(k!)(i-k)!(j-l)(l!)^2\Gamma(k+\alpha+1)(k+l+\alpha+1)} = \Theta_{i,j,k.\eta} \tag{24}$$

$$I^\alpha P_i^\eta(x) = \sum_{j=0}^{m}\sum_{k=0}^{i}\Theta_{i,j,k.\eta}P_j^\eta(x).$$

Evaluating for different i we get the desired result. □

Lemma 3.2. *Let $\Psi_M(x)$ be the function vector as defined in (7) then the derivative of order β of $\Psi_M(x)$ is generalized as*

$$D^\beta(\Psi_M(x)) \simeq A_{M\times M}^{\eta,\beta}\Psi_M(x), \tag{25}$$

where $A_{M\times M}^{\eta,\beta}$ is the operational matrix of derivative of order β and is defined as

$$A_{M\times M}^{\eta,\beta} = \begin{pmatrix} \sum_{k=0}^{0}\Theta_{0,0,k,\eta} & \sum_{k=0}^{0}\Theta_{0,1,k,\eta} & \cdots & \sum_{k=0}^{0}\Theta_{0,j,k,\eta} & \cdots & \sum_{k=0}^{0}\Theta_{0,m,j,\eta} \\ \sum_{k=0}^{1}\Theta_{1,0,k,\eta} & \sum_{k=0}^{1}\Theta_{1,1,k,\eta} & \cdots & \sum_{k=0}^{1}\Theta_{1,j,k,\eta} & \cdots & \sum_{k=0}^{1}\Theta_{1,m,j,\eta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{i}\Theta_{i,0,k,\eta} & \sum_{k=0}^{i}\Theta_{i,1,k,\eta} & \cdots & \sum_{k=0}^{i}\Theta_{i,j,k,\eta} & \cdots & \sum_{k=0}^{i}\Theta_{i,m,k,\eta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{m}\Theta_{m,0,k,\eta} & \sum_{k=0}^{m}\Theta_{m,1,k,\eta} & \cdots & \sum_{k=0}^{m}\Theta_{m,j,k,\eta} & \cdots & \sum_{k=0}^{m}\Theta_{m,m,k,\eta} \end{pmatrix}, \tag{26}$$

where

$$\Theta_{i,j,k,\eta} = \frac{(2j+1)}{\eta} \sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!(l+j)!(\eta^{\beta+1})}{(i-k)!k!\Gamma(k-\beta+1)(j-l)!(l!)^2(k+l-\beta+1)}. \tag{27}$$

Proof. Using (3) along with (4) we have

$$D^{\beta} P_i^{\eta}(x) = \sum_{k=o}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{D^{\beta} x^k}{(\eta^k)(k!)^2}, \tag{28}$$

$$= \sum_{k=\lceil\beta\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)(i-k)!(\eta^k)\Gamma(k-\beta+1)} x^{k-\beta}. \tag{29}$$

Now approximating $x^{k-\beta}$ by $m+1$ terms of Legendre polynomials we get

$$x^{k-\beta} \simeq \sum_{j=0}^{m} b_{k,j} P_j^{\eta}(x), \tag{30}$$

where

$$b_{k,j} = \frac{(2j+1)}{\eta} \int_0^{\eta} x^{k-\beta} P_j^{\eta}(x) dx.$$

$$= \frac{(2j+1)}{\eta} \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(\eta^l)(j-l)(l!)^2} \int_0^{\eta} x^{k-\beta+1} dx.$$

$$= \frac{(2j+1)}{\eta} \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!(\eta)^{k+l-\beta+1}}{(\eta^l)(j-l)(l!)^2(k+l-\beta+1)}, \tag{31}$$

employing (28) and (31) we get

$$D^{\beta} P_i^{\eta}(x) = \sum_{k=\lceil\beta\rceil}^{i} \sum_{j=0}^{m} \frac{(-1)^{i+k}(i+k)!}{(k!)(i-k)!(\eta^k)\Gamma(k-\beta+1)} b_{k,j} P_j^{\eta}(x).$$

Or

$$D^{\beta} P_i^{\eta}(x) = \sum_{j=0}^{m} \sum_{k=\lceil\beta\rceil}^{i} \frac{(2j+1)}{\eta} \sum_{l=0}^{i} \frac{(-1)^{i+k+j+l}(i+k)!(j+l)!(\eta^{\beta+1})}{(k!)(i-k)!(j-l)(l!)^2\Gamma(k-\beta+1)(k+l-\beta+1)} P_j^{\eta}(x).$$

Setting

$$\frac{(2j+1)}{\eta} \sum_{l=0}^{i} \frac{(-1)^{i+k+j+l}(i+k)!(j+l)!(\eta^{\beta+1})}{(k!)(i-k)!(j-l)(l!)^2\Gamma(k-\beta+1)(k+l-\beta+1)} = \Theta_{i,j,k,\eta}. \tag{32}$$

$$D^{\beta} P_i^{\eta}(x) = \sum_{j=0}^{m} \sum_{k=\lceil\beta\rceil}^{i} \Theta_{i,j,k.\eta} P_j^{\eta}(x),$$

or evaluating for different i we get the desired result. □

Remark 1. *If $f(X^n(x),, X''(x), X'(x), X(x))$ is any linear combination of the corresponding component and $X(x) = K_M \Psi(x)$ then*

$$f(X^n(x),, X''(x), X'(x), X(x)) = K_M Q^f_{M\times M} \Psi(x). \tag{33}$$

Where

$$Q^f_{M\times M} = f(A^{\eta,n}_{M\times M},, A^{\eta,2}_{M\times M}, A^{\eta,1}_{M\times M}, I_{M\times M}). \tag{34}$$

The proof of this remark is straight forward due to the linearity of f.

The following results are known and discussed in many papers.

Lemma 3.3. *Let $\hat{\Psi}_M(x)$ be the function vector of Haar Wavelets as defined in* (15)*, then*

$$I^\alpha \hat{\Psi}_M(x) = B^{\eta,\alpha}_{M\times M} \hat{\Psi}_M(x).$$

Where $B^{\eta,\alpha}_{M\times M}$ is the operational matrix of integration and is defined as

$$B^{\eta,\alpha}_{M\times M} = \Psi_{M\times M} F^{\eta,\alpha} \Psi^{-1}_{M\times M}.$$

The matrix $F^{\eta,\alpha}$ is defined as

$$F^{\eta,\alpha} = (\frac{\eta}{M})^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \vdots & \zeta_{M-1} \\ 0 & 1 & \zeta_1 & \vdots & \zeta_{M-2} \\ 0 & 0 & 1 & \vdots & \zeta_{M-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{35}$$

where $\zeta_j = (j+1)^{(\alpha+1)} - 2j^{(\alpha+1)} + (j-1)^{(\alpha+1)}$, $j = 1, 2\cdots m-i+1$, $i = 1, 2\cdots M+1$.

Proof. For the proof of this lemma we refer the reader to [19]. □

4. Solving fractional order system of differential equations

In this section we derive a simple scheme for the solution of coupled system of fractional differential equations using Legendre polynomials and Haar wavelets.

4.1. **Scheme based on Legendre polynomials.** Consider the following class of coupled systems.

$$\begin{aligned} D^\alpha_x Y(x) &= f_1(X^{n'}(x),, X''(x), X'(x), X(x)) + f_2(Y^n(x),, Y''(x), Y'(x), Y(x)) + f_3(x) \\ D^\beta_x X(x) &= g_1(X^{n'}(x),, X''(x), X'(x), X(t)) + g_2(Y^n(x),, Y''(x), Y'(x), Y(x)) + g_3(x) \end{aligned} \tag{36}$$

with initial conditions

$$Y^{(n)}(0) = a_n, \qquad X^{(n')}(0) = b_{n'}, n = 0, 1, 2...\lfloor\alpha\rfloor, n' = 0, 1, 2...\lfloor\beta\rfloor \tag{37}$$

where $x \in [0, \eta]$, $n < \alpha \leq n+1$, $n' < \beta \leq n'+1$ and $f_1, f_2, f_3, g_1\ g_2, and\ g_3$ are the corresponding linear functions. We seek the solution of the above problem in terms of Legendre polynomials such that

$$D^\alpha_x Y(x) = K_M \Psi_M(x) \qquad D^\beta_x X(x) = L_M \Psi_M(x). \tag{38}$$

We apply I^α and I^β on the corresponding equation (38) and using the initial conditions (37) to get

$$Y(x) - \sum_{s=0}^{n} a_s x^s = K_M H^{\eta,\alpha}_{M\times M} \Psi_M(x), \qquad X(x) - \sum_{s=0}^{n'} b_s x^s = L_M H^{\eta,\beta}_{M\times M} \Psi_M(x). \tag{39}$$

We can also write them as

$$Y(x) = (K_M H^{\eta,\alpha}_{M\times M} + F_1)\Psi_M(x), \qquad X(x) = (L_M H^{\eta,\beta}_{M\times M} + F_2)\Psi_M(x), \tag{40}$$

where $\sum_{s=0}^{n} a_s x^s(0) \simeq F_1\Psi_M(x)$ and $\sum_{s=0}^{n} b_s x^s \simeq F_2\Psi_M(x)$. For simplicity of notation we can write

$$(K_M H^{\eta,\alpha}_{M\times M} + F_1) = R^1_M, \qquad (L_M H^{\eta,\beta}_{M\times M} + F_2) = R^2_M. \tag{41}$$

We can get $Y(x) = R^1_M\Psi_M(x)$ and $X(x) = R^2_M\Psi_M(x)$. Now using (33) we get

$$f_1(X^n(x),, X''(x), X'(x), X(x)) = R^2_M Q^{f_1}_{M\times M}\Psi_M(x), \tag{42}$$

$$f_2(Y^n(x),, Y''(x), Y'(x), Y(x)) = R^1_M Q^{f_2}_{M\times M}\Psi_M(x), \tag{43}$$

$$g_1(X^n(x),, X''(x), X'(x), X(x)) = R^2_M Q^{g_1}_{M\times M}\Psi_M(x), \tag{44}$$

$$g_2(Y^n(x),, Y''(x), Y'(x), Y(x)) = R^1_M Q^{g_2}_{M\times M}\Psi_M(x), \tag{45}$$

where the matrices $Q\ 's$ are defined as in (34). Using this simplified notation along with (38) in the system we get

$$\begin{pmatrix} K^T_M\Psi_M(x) \\ L^T_M\Psi_M(x) \end{pmatrix} = \begin{pmatrix} R^1_M Q^{f_2}_{M\times M}\Psi_M(x) + R^2_M Q^{f_1}_{M\times M}\Psi_M(x) + F_3\Psi_M(x) \\ R^2_M Q^{g_1}_{M\times M}\Psi_M(x) + R^1_M Q^{g_2}_{M\times M}\Psi_M(x) + F_4\Psi_M(x) \end{pmatrix}$$

Where $F_3\Psi_M(x) = f_3(x)$ and $F_3\Psi_M(x) = g_3(x)$. On further simplification we get

$$\begin{pmatrix} K^T_M\Psi_M(x) \\ L^T_M\Psi_M(x) \end{pmatrix} = \begin{pmatrix} R^1_M Q^{f_2}_{M\times M}\Psi_M(x) \\ R^2_M Q^{g_1}_{M\times M}\Psi_M(x) \end{pmatrix} + \begin{pmatrix} R^2_M Q^{f_1}_{M\times M}\Psi_M(x) \\ R^1_M Q^{g_2}_{M\times M}\Psi_M(x) \end{pmatrix} + \begin{pmatrix} F_3\Psi_M(x) \\ F_4\Psi_M(x) \end{pmatrix}. \tag{46}$$

Taking the transpose of the above matrix equation we get

$$\begin{aligned} \begin{pmatrix} K^T_M\Psi_M(x) & L^T_M\Psi_M(x) \end{pmatrix} = & \begin{pmatrix} R^1_M Q^{f_2}_{M\times M}\Psi_M(x) & R^2_M Q^{g_1}_{M\times M}\Psi_M(x) \end{pmatrix} \\ & + \begin{pmatrix} R^2_M Q^{f_1}_{M\times M}\Psi_M(x) & R^1_M Q^{g_2}_{M\times M}\Psi_M(x) \end{pmatrix} \\ & + \begin{pmatrix} F_3\Psi_M(x) & F_4\Psi_M(x) \end{pmatrix}. \end{aligned} \tag{47}$$

On further simplification we can easily get

$$\begin{aligned} \begin{pmatrix} K^T_M & L^T_M \end{pmatrix} \begin{pmatrix} \Psi_M(x) & O_M \\ O_M & \Psi_M(x) \end{pmatrix} = & \begin{pmatrix} R^1_M & R^2_M \end{pmatrix} \begin{pmatrix} Q^{f_2}_{M\times M} & O_{M\times M} \\ O_{M\times M} & Q^{g1}_{M\times M} \end{pmatrix} \begin{pmatrix} \Psi_M(x) & O_{M\times M} \\ O_{M\times M} & \Psi_M(x) \end{pmatrix} \\ & + \begin{pmatrix} R^1_M & R^2_M \end{pmatrix} \begin{pmatrix} O_{M\times M} & Q^{g_2}_{M\times M} \\ Q^{f_1}_{M\times M} & O_{M\times M} \end{pmatrix} \begin{pmatrix} \Psi_M(x) & O_M \\ O_M & \Psi_M(x) \end{pmatrix} \\ & + \begin{pmatrix} F_3 & F_4 \end{pmatrix} \begin{pmatrix} \Psi_M(x) & O_M \\ O_M & \Psi_M(x) \end{pmatrix}, \end{aligned} \tag{48}$$

where $O_{M\times M}$ is square matrix of order M with all entries equal to zero, and O_M is a zero vector of order M. Let

$$A_O = \begin{pmatrix} \Psi_M(x) & O_M \\ O_M & \Psi_M(x) \end{pmatrix}.$$

Then we have

$$\begin{aligned}\begin{pmatrix} K_M^T & L_M^T \end{pmatrix} A_o = & \begin{pmatrix} R_M^1 & R_M^2 \end{pmatrix} \begin{pmatrix} Q_{M\times M}^{f_2} & O_{M\times M} \\ O_{M\times M} & Q_{M\times M}^{g1} \end{pmatrix} A_o \\ & + \begin{pmatrix} R_M^1 & R_M^2 \end{pmatrix} \begin{pmatrix} O_{M\times M} & Q_{M\times M}^{g_2} \\ Q_{M\times M}^{f_1} & O_{M\times M} \end{pmatrix} A_o \\ & + \begin{pmatrix} F_3 & F_4 \end{pmatrix} A_o, \end{aligned} \tag{49}$$

or

$$\begin{aligned}\begin{pmatrix} K_M^T & L_M^T \end{pmatrix} - \begin{pmatrix} R_M^1 & R_M^2 \end{pmatrix} & \begin{pmatrix} Q_{M\times M}^{f_2} & Q_{M\times M}^{g_2} \\ Q_{M\times M}^{f_1} & Q_{M\times M}^{g_1} \end{pmatrix} \\ & - \begin{pmatrix} F_3 & F_4 \end{pmatrix} = 0. \end{aligned} \tag{50}$$

Using (41), we get

$$\begin{aligned}\begin{pmatrix} K_M^T & L_M^T \end{pmatrix} - \begin{pmatrix} K_M & L_M \end{pmatrix} & \begin{pmatrix} H_{M\times M}^{\eta,\alpha} Q_{M\times M}^{f_2} & H_{M\times M}^{\eta,\beta} Q_{M\times M}^{g_2} \\ H_{M\times M}^{\eta,\alpha} Q_{M\times M}^{f_1} & H_{M\times M}^{\eta,\beta} Q_{M\times M}^{g_1} \end{pmatrix} \\ - \begin{pmatrix} F_1 & F_2 \end{pmatrix} & \begin{pmatrix} Q_{M\times M}^{f_2} & Q_{M\times M}^{g_2} \\ Q_{M\times M}^{f_1} & Q_{M\times M}^{g_1} \end{pmatrix} - \begin{pmatrix} F_3 & F_4 \end{pmatrix} = 0. \end{aligned} \tag{51}$$

Now we can see that (51) is a generalized lypanov type matrix equation and can be easily solved for the unknown K_m and L_M. Using K_M ,L_M along with (40) we can get approximate solutions of the problem.

4.2. **Scheme based on Haar wavelets.** It is clear that all the functions in the Haar function vector (14) are piecewise defined. Due to this reason we believe that the operational matrix for fractional order derivative is impossible (or at least difficult). The operational matrix of integration is the only mean to solve the problem. The idea of the scheme is similar as developed in the previous subsection. Consider the problem (36) and assume the solutions in terms of Haar wavelets such that

$$D_x^{\alpha} Y(x) = K_M^T \hat{\Psi}_M(x) \qquad\qquad D_x^{\beta} X(x) = L_M^T \hat{\Psi}_M(x). \tag{52}$$

Applying integration of order $\alpha - n$ and $\beta - n'$ on the corresponding equations, and using lemma 3.3 we get

$$D_x^{n} Y(x) = K_M^T B_{M\times M}^{\eta,(\alpha-n)} \hat{\Psi}_M(x) + a_n \qquad\qquad D_x^{n'} X(x) = L_M^T B_{M\times M}^{\eta,(\beta-n')} \hat{\Psi}_M(x) + b_{n'} \tag{53}$$

On repeating the integration process, using the operational matrices and making use of initial conditions we get

$$\begin{aligned} D_x^{n-i} Y(x) &= K_M^T B_{M\times M}^{\eta,(\alpha-n+i)} \hat{\Psi}_M(x) + \sum_{l=0}^{i} a_{n-l} x^{i-l}, \qquad i = 1, 2, \cdots n \\ D_x^{n'-j} X(x) &= L_M^T B_{M\times M}^{\eta,(\beta-n'+j)} \hat{\Psi}_M(x) + \sum_{l=0}^{j} b_{n'-l} x^{j-l} \qquad j = 1, 2, \cdots n'. \end{aligned} \tag{54}$$

Now as f_1 is linear therefore in view of (53) and (54) we may write

$$\begin{aligned} f_1(X^{n'},,X'',X',X) &= L_M^T f_1(B_{M\times M}^{\eta,(\beta-n')},\cdots B_{M\times M}^{\eta,(\beta-1)},B_{M\times M}^{\eta,(\beta)})\hat{\Psi}_M(x) \\ &+ f_1(b_{n'},b_{n'}x+b_{n'-1},\cdots\sum_{l=0}^{n'-1}b_{n'-l}x^{n'-1-l},\sum_{l=0}^{n'}b_{n'-l}x^{(n'-l)}). \end{aligned} \tag{55}$$

Equation (55) can be written in simplified form as

$$f_1(X^{n'},,X'',X',X) = L_M^T Q^{f_1}\hat{\Psi}_M(x) + \hat{F_1}_M^T\hat{\Psi}_M(x), \tag{56}$$

where $\hat{F_1}_M^T\hat{\Psi}_M(x)$ is the Haar wavelets approximation of the 2nd term of (55) which is known function. By the similar arguments we can write

$$f_2(Y^{n},,Y'',Y',Y) = K_M^T Q^{f_2}\hat{\Psi}_M(x) + \hat{F_2}_M^T\hat{\Psi}_M(x), \tag{57}$$

$$g_1(X^{n'},,X'',X',X) = L_M^T Q^{g_1}\hat{\Psi}_M(x) + \hat{G_1}_M^T\hat{\Psi}_M(x), \tag{58}$$

$$g_2(Y^{n},,Y'',Y',Y) = K_M^T Q^{g_2}\hat{\Psi}_M(x) + \hat{G_2}_M^T\hat{\Psi}_M(x). \tag{59}$$

Using the estimates (52), (56), (57), (58) and (59) in the problem (36) we get

$$\begin{pmatrix} K_M^T\Psi_M(x) \\ L_M^T\Psi_M(x) \end{pmatrix} = \begin{pmatrix} K_M^T Q_{M\times M}^{f_2}\Psi_M(x) + L_M^T Q_{M\times M}^{f_1}\Psi_M(x) + \overbrace{F_M^T}\Psi_M(x) \\ L_M^T Q_{M\times M}^{g_1}\Psi_M(x) + K_M^T Q_{M\times M}^{g_2}\Psi_M(x) + \overbrace{G_M^T}\Psi_M(x) \end{pmatrix}.$$

Where $\overbrace{F_M^T} = \hat{F_1}_M^T + \hat{F_2}_M^T + \hat{F_3}_M^T$, and $\hat{F_3}_M^T$ is the Haar wavelets coefficients vector of the source term $f_3(x)$. $\overbrace{G_M^T}$ is analogously defined. Repeating the same procedure ((47) to (51)) we get the resulting system of algebraic equations.

$$\begin{aligned} &\begin{pmatrix} K_M^T & L_M^T \end{pmatrix} - \begin{pmatrix} K_M^T & L_M^T \end{pmatrix}\begin{pmatrix} Q_{M\times M}^{f_2} & Q_{M\times M}^{g_2} \\ Q_{M\times M}^{f_1} & Q_{M\times M}^{g_1} \end{pmatrix} \\ &\qquad - \begin{pmatrix} \overbrace{F_M^T} & \overbrace{G_M^T} \end{pmatrix} = 0. \end{aligned} \tag{60}$$

Equation (60) can be solved for the unknowns K and L and using them in (54) (setting $i = n$,$j = n'$) will lead us to the approximate solution of the problem.

5. Examples

We check the efficiency of the proposed techniques with some example whose exact solution is known and as expected we get high accuracy of the approximate solution. In the figures $\overbrace{X(x)_L}$ represents the approximate solution obtained using Legendre polynomials and $\overbrace{X(x)_H}$ represents solution obtained using Haar wavelets.

Example 5.1. *Consider the fractional order two tank mixing problem.*

$$D_t^{\alpha}Y(t) = -0.02(Y(t)) + 0.02(X(t)) \tag{61}$$

$$D_t^{\alpha}X(t) = 0.02(Y(t)) - 0.02(X(t)) \tag{62}$$

with initial condition $Y(0) = 150$ and $X(0) = 0$ the exact solution for $\alpha = 1$ is

$$Y(t) = 75 + 150e^{-0.04t}$$

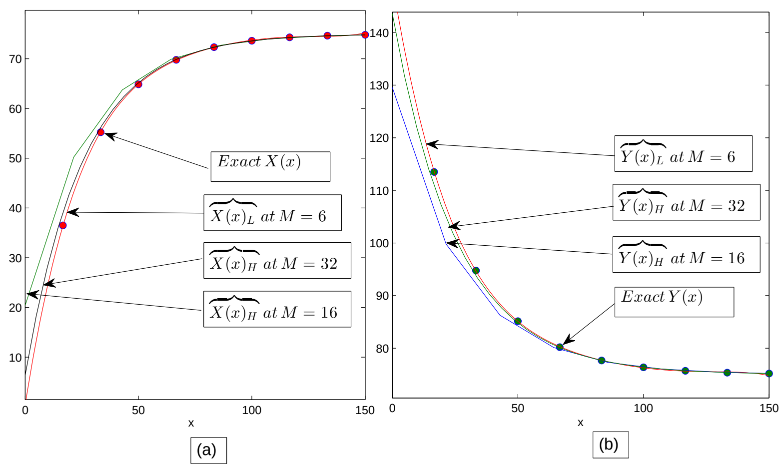

FIGURE 1. Comparison of exact and approximate solution of Example 1 by setting $\alpha = 1, \eta = 150$. (a) Comparison of $X(x)$ with approximate solution of Legendre and Haar wavelets.(b) Comparison of $Y(x)$ with approximate solution of Legendre and Haar wavelets.

and

$$X(t) = 75 - 150e^{-0.04t}.$$

We simulate the problem with the new techniques developed in the paper. We observe that the solutions obtained with the Legendre polynomials are more accurate as compare to the Haar wavelets. Fig. (1) *shows the comparison of the approximate solutions with the exact solutions. We see that Legendre polynomials solutions at* $M = 6$ *is more accurate as compare to Haar wavelet solution at* $M = 32$. *It is known that the solution of fractional differential equations approaches to the solution at integer order as the order of derivative approaches from fractional to integer. We approximate the solution at different value of* α *and observe the solution obtained with Legendre polynomials and Haar wavelets approaches to the solution at* $\alpha = 1$ *as* $\alpha \to 1$. *Fig.* (2) *shows the approximate solutions at different value of* α. *We observe that both the solutions obtained with Legendre and Haar wavelets agree at fractional value of* α. *We also study the convergence of approximate solution for both the schemes. We see that Fig.* (3) *Legendre polynomials converge more rapidly to the exact solution as compare to the Haar wavelets (Fig.* (4)*). The absolute error at* $M = 8$ *is much more less than* 10^{-4} *for Legendre polynomials, however that of Haar wavelets is less than* 10^{-3} *at scale level* $M = 64$.

Example 5.2. *Consider the following couple system fractional differential equations* [3]

$$D_x^{\alpha} Y(x) = Y(x) + X(x) \tag{63}$$

$$D_x^{\beta} X(x) = Y(x) - X(x) \tag{64}$$

with initial condition $X(0) = 0$ *and* $Y(0) = 1$. *The exact solution at* $\alpha = 1$, $\beta = 1$ *is* $X(x) = e^x sin(x)$ *and* $Y(x) = e^x cos(x)$. *In* [3] *Shaher Momani solved this problem*

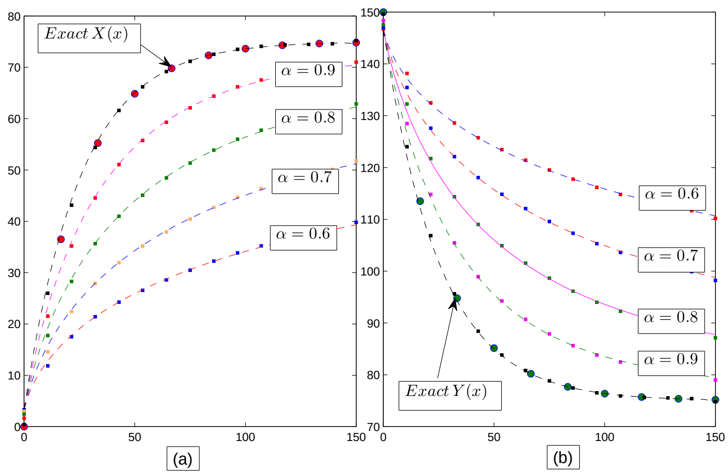

FIGURE 2. (a) Comparison of approximate X(x) of Haar wavelets (square dots), Legendre polynomials (dashed lines) at different value of α with the exact solution (red circles).(b) Comparison of approximate Y(x) of Haar wavelets (square dots), Legendre polynomials (dashed lines) at different value of α with the exact solution (green circles).

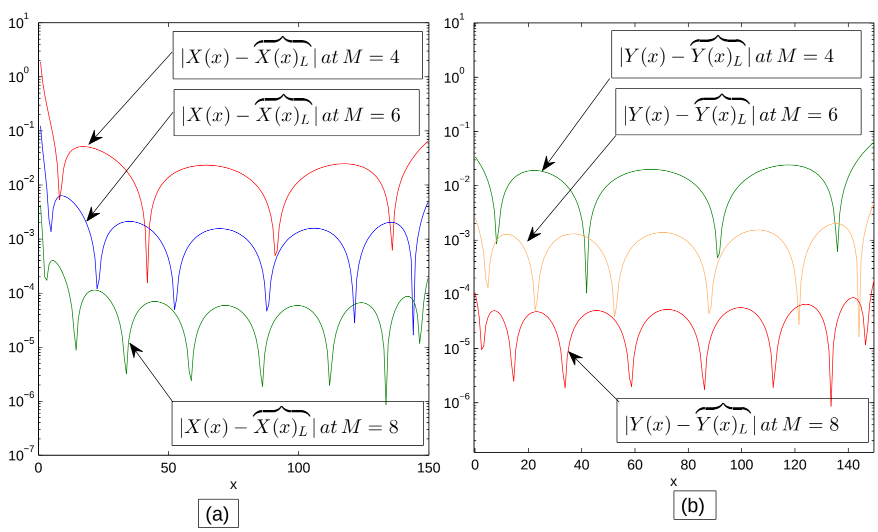

FIGURE 3. (a) Absolute error in $X(x)$ at different value of M using Legendre polynomials.(b) Absolute error in $Y(x)$ at different value of M using Legendre polynomials.

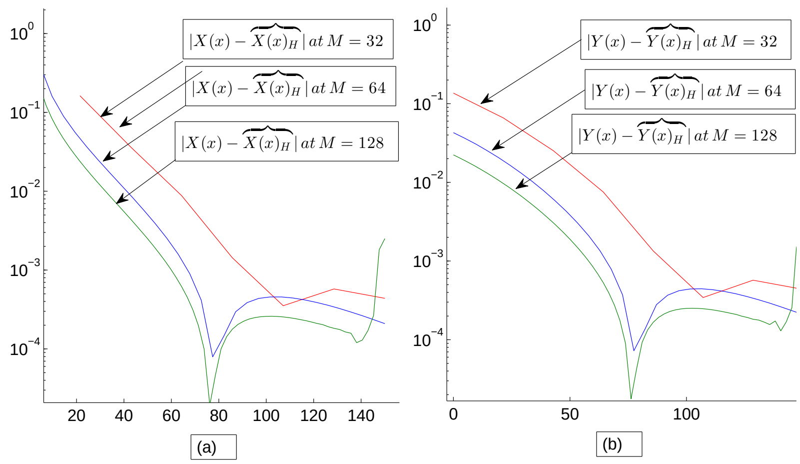

FIGURE 4. (a) Absolute error in $X(x)$ at different value of M using Haar wavelets. (b) Absolute error in $Y(x)$ at different value of M using Haar wavelets.

using differential transform method. We fix $\alpha = \beta = 1$ and approximate the solution using Legendre polynomials. We compare it with the exact solutions and the solutions obtained in [3] *(Fig.* (5)*). It is clear that the approximate solutions with the new method is in good agreement with the exact solution. We compare the absolute error of the presented method at $M = 8$ with the absolute error of differential transform method at $k = 10$ in Fig.* (6) *and observe that the new technique provide more accurate solution. In* [3] *this problem is also solved at $\alpha = 0.9$ and $\beta = 0.7$. We compare our solution at these values with the approximate solution reported in* [3] *and observe that our solution is in good agreement with that in* [3]*, see Fig.* (7) .

Example 5.3. *Consider the mathematical model of fractionally damped coupled system of two masses. The governing equation is*

$$m_1\ D_x^{\alpha} Y(x) = -(c_1+c_2)\frac{dY(x)}{dx} - (k_1+k_2)Y(x) + c_2\frac{dX(x)}{dx} + k_2X(x) + F_1(x), \quad (65)$$

$$m_2\ D_x^{\alpha} X(x) = c_2\frac{dY(x)}{dx} + k_2Y(x) - c_2\frac{dX(x)}{dx} - k_2X(x) + F_2(x). \quad (66)$$

Where $1 \leq \alpha \leq 2$,c_1, c_2, c_3 are the damping parameter k_1, k_2, k_3 are the spring constant. Where

$$F_1(x) = \frac{21\ \sin(\pi\, x)}{200} - \frac{9\ \cos(\pi\, x)}{100} + \frac{119\,\pi\ \cos(\pi\, x)}{100} + \frac{171\,\pi\ \sin(\pi\, x)}{200} - \frac{7\,\pi^2\ \sin(\pi\, x)}{10}$$

and

$$F_2(x) = \frac{153\ \cos(\pi\, x)}{200} - \frac{7\ \sin(\pi\, x)}{100} - \frac{133\,\pi\ \cos(\pi\, x)}{200} - \frac{63\,\pi\ \sin(\pi\, x)}{50} - \frac{9\,\pi^2\ \cos(\pi\, x)}{10}.$$

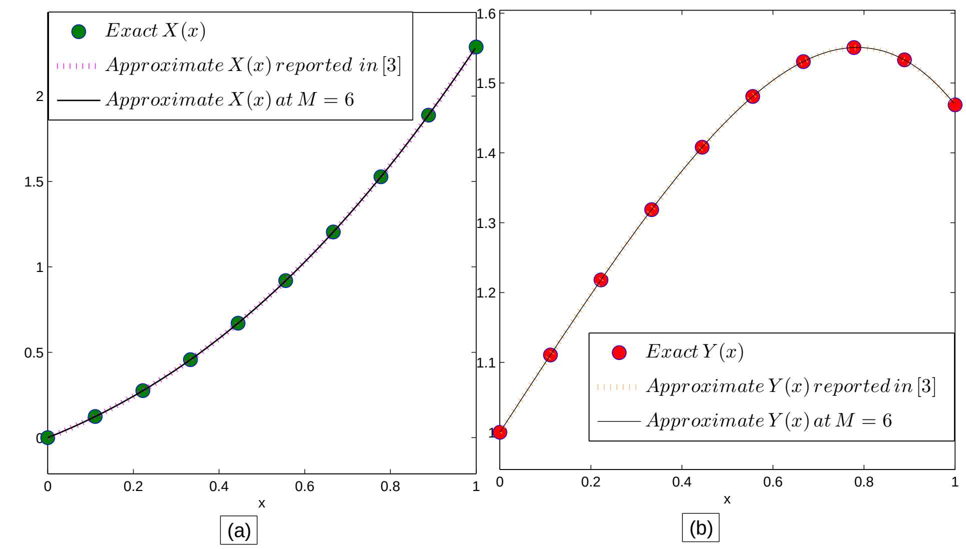

FIGURE 5. Comparison of approximate solution of example 2 with the exact solution at scale level $M = 6, \alpha = 1, \beta = 1$.

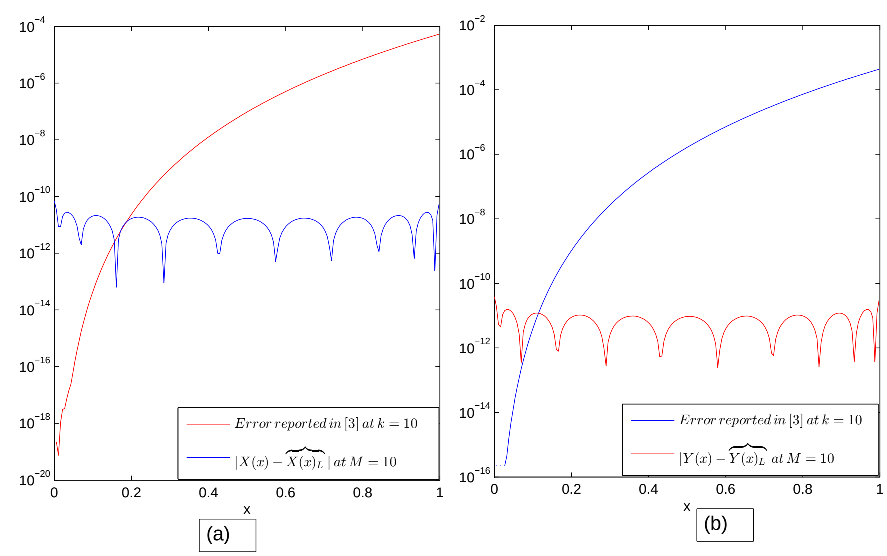

FIGURE 6. Comparison of absolute error of example 2 with the absolute error reported in [3].

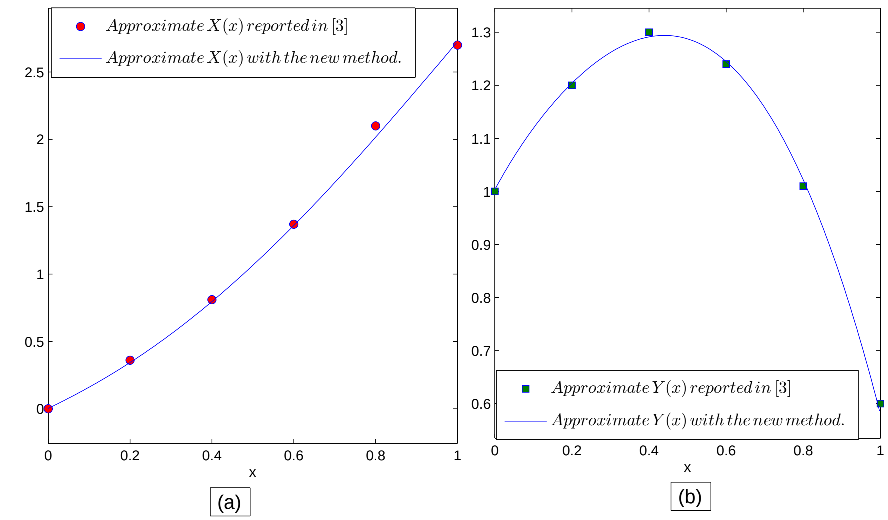

FIGURE 7. Comparison of approximate solution of example 2 with the solutions reported in [3], fixing $\alpha = 0.7$, $\beta = 0.9$.

For $\alpha = 2$,$m_1 = 1, m_2 = 1, c_1 = 0.75, c_2 = 0.95, c_3 = 0.45, k_1 = 0.05, k_2 = 0.1$ *and* $k_3 = 0.75$ *along with the initial conditions*

$$X(0) = 9/10 \text{ and } X'(0) = 0,\ Y(0) = 0 \text{ and } Y'(0) = 7\pi/10$$

the exact solution to the problem is

$$Y(x) = \frac{7\,\sin(\pi\, x)}{10},$$

and

$$X(x) = \frac{9\,\cos(\pi\, x)}{10}.$$

We solve the problem with our new technique and as expected we get highly accurate solution. As evident from Fig. (8) *as we increase the scale level the approximate result become more and more accurate. We also approximate the solution of the problem with Haar wavelets and compare it with the solution obtained with Legendre polynomials. We see that the solution of Haar wavelets are less accurate as compare with Legendre polynomials, see Fig.* (9)*. We approximate the solution at different value of* α *and the same conclusion is made see Fig.* (10) *and Fig.* (11)*. We approximate the absolute error at different value of* M *and observe that the amount of absolute error is much more less than* 10^{-3} *see Fig.* (12) *and Fig.* (13).

6. Conclusion

From the above results and observations we conclude that the Legendre polynomials provide a very good approximation to the problem as compare to Haar wavelets. The results we obtained are in good agreement with the results obtained with differential transform method. It is also expected that the scheme may provide

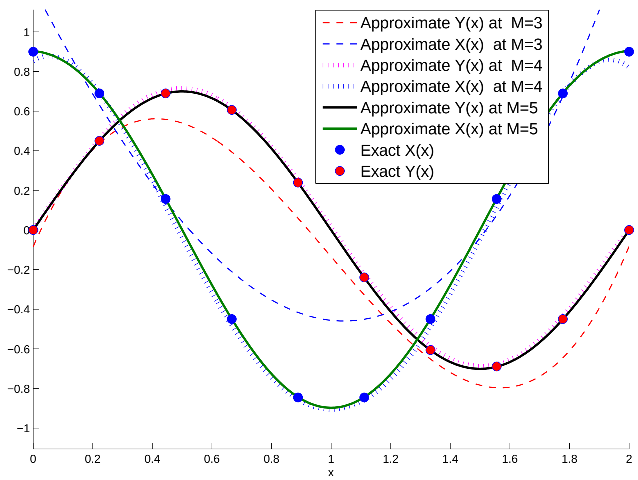

FIGURE 8. Comparison of exact solution of example 3 with the approximate solutions of Legendre polynomials at different scale level.

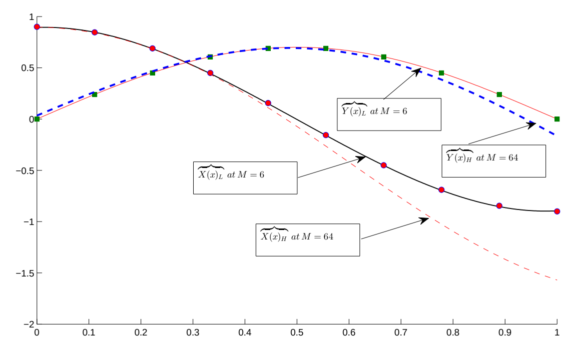

FIGURE 9. Comparison of exact solutions with the approximate solutions using Haar Wavelets and Legendre polynomials.

a more accurate solution if other orthogonal polynomials like Bernstein or Jacobi polynomials are used. If it is necessary to approximate the solution on half line then Laguerre polynomials can be efficiently applied in the scheme. Our future work is related to solve the same problem under different types of boundary conditions.

7. ACKNOWLEDGEMENT

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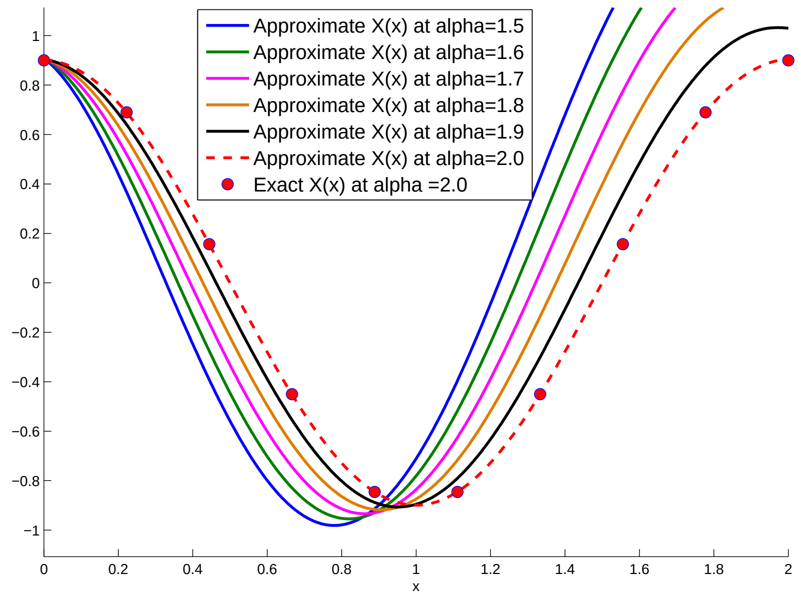

FIGURE 10. Approximate X(x) of example 3 at different values of α.

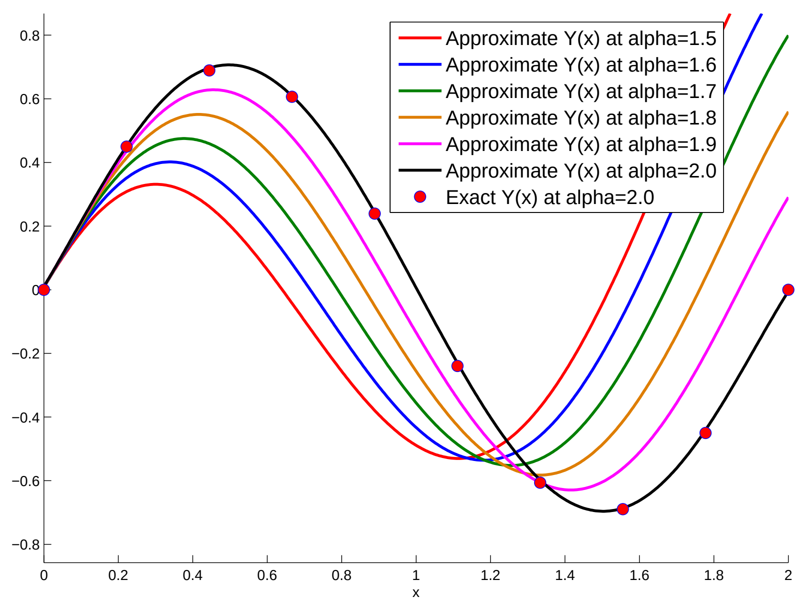

FIGURE 11. Approximate Y(x) of example 3 at different values of α.

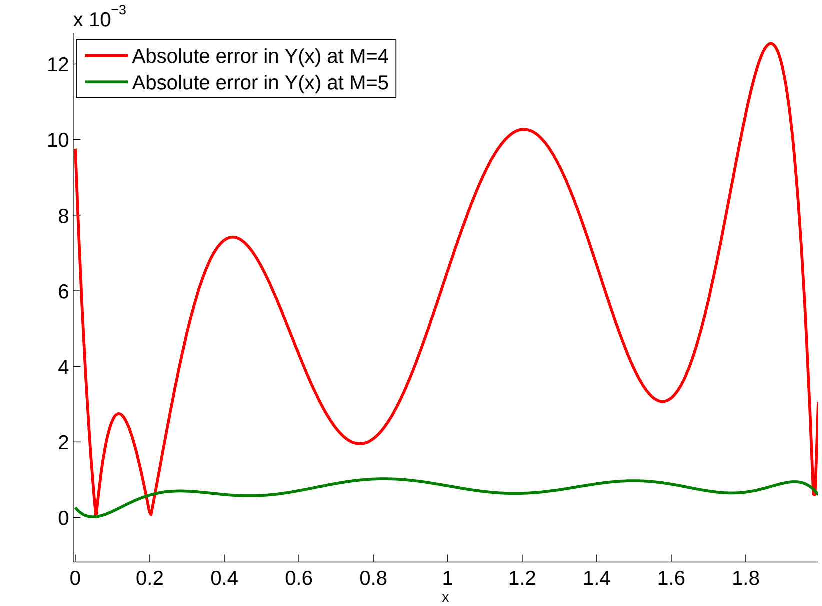

FIGURE 12. Absolute error in $Y(x)$ of Example 3 at different scale level (M=4,5).

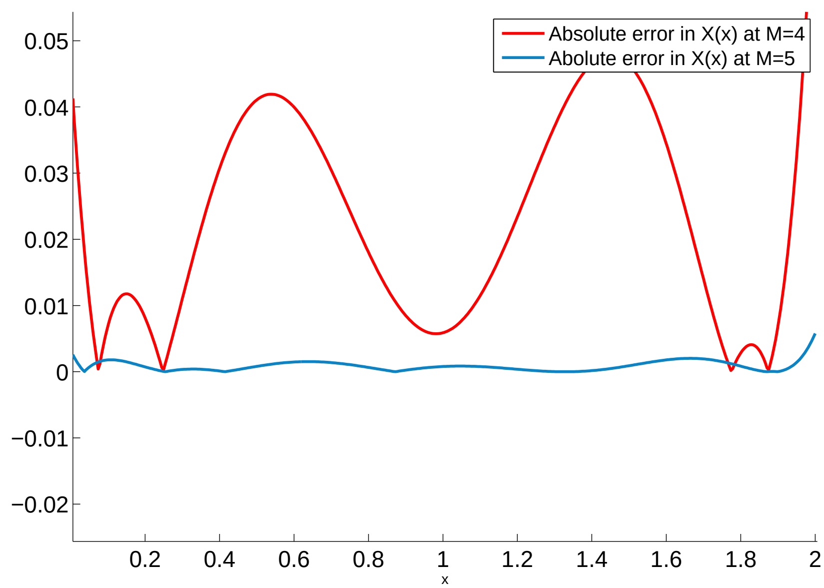

FIGURE 13. Absolute error in $X(x)$ of Example 3 at different scale level (M=4,5).

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