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Midpoint Time Allocation and Value Estimation of Frequency Component for Time-Limited Signal Using Continuous Fractional Fourier Transform

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ABSTRACT

Analyzing the frequency components of the time-varying signals to extract their frequency values and time allocation is required in some applications, i.e.; filter design in signal processing, optics, acoustics and other applications. There are potential tools that have been used for this purpose like Wigner transform, Wavelet transforms, short-time Fourier transform, and fractional Wavelet transform. In this paper, a novel method is proposed for the time-limited signal to estimate both the frequency component value and its midpoint (mean) time using the continuous fractional Fourier transform. The proposed method is based on the rotation property of the continuous fractional Fourier transform for function (signal) in the time-frequency plane. Two main applications are discussed. The first is the frequency estimation and its time location for a signal that consists of low frequency time-limited signal in addition to two separable high-frequency components; the other is the time and frequency value estimation of low frequency time-limited signal and two shared additive high-frequency components. Simulation verifies the validity of the proposed method, besides the importance of the continuous fractional Fourier transform as a valuable tool in engineering applications like signal processing, and optics. Also, there are many actual engineering components based on Fourier Analysis such as Electrical relay. Its practical applications are expected to grow significantly.

1. Introduction

Locating the frequency component of a signal at a certain time duration is required in some engineering applications, like accurate determination of the fault location of transmission line. Fourier transform (FT) is one of the powerful tools used in signal processing and analysis, but it is used to represent the signal in the frequency domain where there is no information about the occurrence of the frequency component at a certain time as the Fourier coefficients define the average spectral content over the entire duration of the signal. The short time Fourier transform (STFT) [1] has overcome the limitation of the FT as it divides a longer time signal into shorter segments of equal length by using a windowing function, and then computing Fourier transform separately of each of them. One of the pitfalls of the STFT is that it has a fixed resolution. The width of the windowing function relates to how the signal is represented, it determines whether there is good frequency resolution (frequency components close together can be separated) or good time resolution (the time at which frequencies change). A wide window gives better frequency resolution but poor time resolution. A narrower window gives better time resolution but poor frequency resolution. So, the STFT is not applicable in the case of real signals having low frequencies of long duration and high frequencies of short duration. Such signals could be better described by a transform which has a high time resolution for short-lived high-frequency phenomena and has high frequency resolution for long-lasting low-frequency phenomena. In these types of situations, Wavelet transform (WT) can provide a better description of the signal instead of the STFT. Wavelets have special ability to analyze signal in both time and frequency domain simultaneously and can easily detect the local properties of a signal, but WT has difficulties to be implemented practically in some engineering applications.

Also, sparse fractional Fourier transform (SFRFT) used for time-frequency analysis, but this technique limited to be applied only on sparse signal [2]

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The continuous fractional Fourier transform (FRFT) is a recent powerful time-frequency analysis tool [3-6]. FRFT corresponds to a rotation in the time-frequency plane over an angle $\alpha = \frac{a\pi}{2}$; where $a \in \mathcal{R}$; it can be considered as a generalization of FT which corresponds to the case of $a = 1$.

FRFT is the representation of a signal in the fractional Fourier domain (FRFD) [7]. But it was claimed that FRFT provides overall FRFD frequency component with no indication about the occurrence of the FRFD spectral component at a certain time duration [8]. Since the FRFT uses a global kernel like FT, other transforms were introduced to overcome that claim, these types of transforms combining the time and FRFD-frequency information, which is termed the time-FRFD frequency representation (TFFR) like the short-time FRFT [9,10], and the fractional WT (FRWT) [11-16].

The purpose of this paper is to define a new technique for using the FRFT to get both the value and the midpoint time of frequency components for time-limited function (signal). In Section II, preliminaries about the definition and basic properties of FRFT are introduced. In Section III, the new technique for estimating the midpoint time of frequency components for time-limited function (signal) by using the FRFT is proposed. In Section IV, simulation results are shown. Finally, in Section V, a conclusion is drawn.

2. Preliminaries

A. *The continuous Fractional Fourier transform*

The a^{th} order continuous FRFT of a function (signal) $x(t)$ has been defined as [17, 18]

$$\mathcal{F}^{\alpha}\{x(t)\}(t_\alpha) = X(u) = \int_{-\infty}^{\infty} x(t)\ K_\alpha(t,u)dt \#(1a)$$

Where:

$$K_\alpha(t,u) = \begin{cases} \sqrt{\frac{1-i\,cot(\alpha)}{2\pi}}\ exp\left[i\pi\left((t^2+u^2)cot(\alpha) - 2tu\,csc(\alpha)\right)\right], & if\ \alpha \neq n\pi \\ \delta(u-t), & if\ \alpha = 2n\pi \\ \delta(u-t), & if\ \alpha = (2n+1)\pi \end{cases} \#(1b)$$

and

$$\alpha = \frac{a\pi}{2}\ . \#(1c)$$

The FRFT generalizes the continuous time Fourier transform (CTFT) to an arbitrary fractional order a and reduces to the CTFT for $a = 1$ [19].

B. *Time-Frequency plane and FRFT*

For analyzing signals in both time and frequency it is important to consider the relation of the FRFT with the Time Frequency plane.

$X_\alpha(u)$ is a representation of the function (signal) $x(t)$ along $u - axis$ in the fractional Fourier domain, making an angle α with the time axis in the time-frequency plane as shown in Fig. 1.

For $\alpha = 0$; $X_\alpha(u)$ is the original function. Here $\mathcal{F}^\alpha$ is the identity operator.

For $\alpha = \frac{\pi}{2}$; $X_\alpha(u)$ is the function represented on the frequency axis. Here $\mathcal{F}^\alpha$ is the Fourier transform operator.

For $\alpha = \pi$; $X_\alpha(u)$ is the function represented on the negative time axis. Here $\mathcal{F}^\alpha$ has the same effect as flipping the original function.

For $\alpha = \frac{3\pi}{2}$; $X_\alpha(u)$ is the function represented on the negative frequency axis. Here $\mathcal{F}^\alpha$ has the same effect as flipping the Fourier transform of the function.

In general

$$X_\alpha(u) = X_{\alpha+2n\pi}(u)\ \text{, n is an integer.} \#(2)$$

For $0 < \alpha < \frac{\pi}{2}$; $X_\alpha(u)$ is the function represented along an axis lying in the first quadrant of time-frequency plane.

From the above one can deduce that; FRFT represents a rotation of the signal in the time-frequency plan through an angle α measured from $t - axis$, and related to other time-frequency representation like Wigner distribution and STFT.

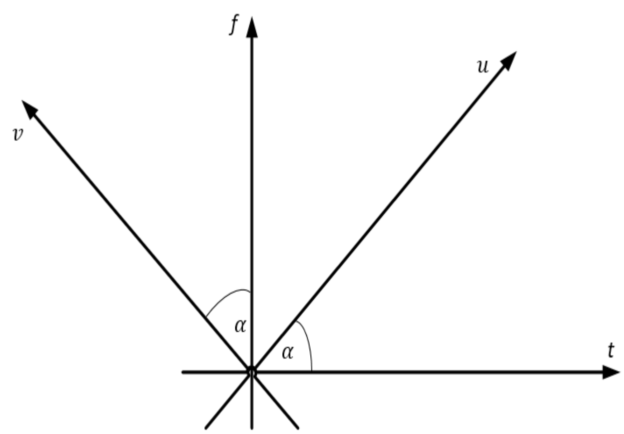

Fig. 1: The time-frequency plane

3. Proposed Technique

In this Section, a new technique for finding the frequency component values and the midpoint time of each frequency component of Time-Limited Signal will be illustrated.

First, the description of the problem will be clarified, then full derivation will be presented, at the end; the summary of the technique will be mentioned.

A. *Problem Description*

Consider a time-limited function (signal) $x(t)$; $t_1 \leq t \leq t_2$ that includes several frequency components; one of these frequency components ($f = f_1$), with a time duration $t_i \leq t \leq t_f$; where f_1 lies between , t_i and t_f.

The objective is to use the FRFT of $x(t)$ to find f_1 and the midpoint time (mean time) t_m, where $t_m = \frac{t_i + t_f}{2}$.

B. *The Proposed method*

The FRFT with angle α of the function (signal) $x(t)$; $t_1 \leq t \leq t_2$, is equivalent to rotation of axes by angle α, then; the new axes will be $u-axis$, and $v-axis$, as in Fig. (1).

Where:

$u\ (v)$: The independent variable of FRFT Domain at angle $\alpha\left(\alpha+\frac{\pi}{2}\right)$.

The relation between $t-axis$, $f-axis$ and $u-axis$, $v-axis$ is given by [20]

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix} \#(3a)$$

Where $\begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix}$ represents the rotation matrix.

This can be written as

$$u = t\cos(\alpha) + f\sin(\alpha) \#(3b)$$

$$v = -t\sin(\alpha) + f\cos(\alpha) \#(3c)$$

for $\alpha = \frac{\pi}{4}$

$$\sqrt{2}\, u = t\ + f\#(4a)$$

$$\sqrt{2}\, v = -t\ + f\#(4b)$$

Lemma (1) will clarify the relation between $\left|X_{\frac{\pi}{4}}(u)\right|$ and $\left|X_{\frac{3\pi}{4}}(v)\right|$; and lemma (2) will derive the expression for both f_1 and t_m for the frequency component$(f=f_1)$.

Lemma (1)*:* For function (signal) $x(t)$; $t_1 \leq t \leq t_2$, then $\left|X_{\frac{\pi}{4}}(u)\right| = \left|X_{\frac{3\pi}{4}}(-u)\right|$.

Proof

$$X_{\frac{\pi}{4}}(u) = \sqrt{1-i}\int_{t_1}^{t_2} x(t)\, e^{i\pi[(t^2+u^2)-2\sqrt{2}tu]}\, dt \#(5)$$

$$X_{\frac{3\pi}{4}}(v) = \sqrt{1+i}\int_{t_1}^{t_2} x(t)\, e^{-i\pi[(t^2+v^2)+2\sqrt{2}tv]}\, dt \#(6)$$

Equations (5) and (6) can be expressed as

$$X_{\frac{\pi}{4}}(u) = 2^{\frac{1}{4}}\, e^{-i\pi\left(u^2+\frac{1}{8}\right)} \int_{t_1}^{t_2} x(t)\, e^{i\pi\left(t-\sqrt{2}u\right)^2}\, dt \#(7)$$

$$X_{\frac{3\pi}{4}}(v) = 2^{\frac{1}{4}}\, e^{i\pi\left(v^2+\frac{1}{8}\right)} \int_{t_1}^{t_2} x(t)\, e^{-i\pi\left(t+\sqrt{2}v\right)^2}\, dt \#(8)$$

For real $x(t)$

$$\left|X_{\frac{3\pi}{4}}(v)\right| = \left|\overline{X_{\frac{3\pi}{4}}(v)}\right| = 2^{\frac{1}{4}}\left|\overline{\int_{t_1}^{t_2} x(t)\, e^{-\iota\pi\left(t+\sqrt{2}v\right)^2}\, dt}\right|$$

$$= 2^{\frac{1}{4}}\left|\int_{t_1}^{t_2} x(t)\, e^{i\pi\left(t+\sqrt{2}v\right)^2}\, dt\right| \#(9)$$

$$\left|X_{\frac{3\pi}{4}}(-u)\right| = 2^{\frac{1}{4}}\left|\int_{t_1}^{t_2} x(t)\, e^{i\pi\left(t-\sqrt{2}u\right)^2}\, dt\right| \#(10)$$

$$\left|X_{\frac{\pi}{4}}(u)\right| = 2^{\frac{1}{4}}\left|\int_{t_1}^{t_2} x(t)\, e^{i\pi\left(t-\sqrt{2}u\right)^2}\, dt\right| \#(11)$$

$$\left|X_{\frac{\pi}{4}}(u)\right| = \left|X_{\frac{3\pi}{4}}(-u)\right| \#(12)$$

Lemma (1) clarified that both $\left|X_{\frac{\pi}{4}}(u)\right|$ and the flipped version of $\left|X_{\frac{3\pi}{4}}(v)\right|$ will have the same shape.

Lemma (2): For the frequency component $(f=f_k)$ of the time-limited function (signal) $x(t)$; $t_i \leq t \leq t_f$, then $f_k = \frac{(u_m+v_m)}{\sqrt{2}}$, and $t_m = \frac{(u_m-v_m)}{\sqrt{2}}$.

Where:

$u_m(v_m)$: The midpoint value in the peak area for the FRFT domain corresponding to $(f=f_k)$ at $\alpha = \frac{\pi}{4}\left(\alpha = \frac{3\pi}{4}\right)$.

Proof

Apply Eqs. (4a) and (4b) for the frequency component $\left(f=f_k;\ t_i \leq t \leq t_f\right)$ of the time-limited function (signal) $x(t)$; $t_i \leq t \leq t_f$, and at $\alpha = \frac{\pi}{4}$.

$$u_i = \frac{(t_i+f_k)}{\sqrt{2}} \#(13a)$$

$$u_f = \frac{(t_f+f_k)}{\sqrt{2}} \#(13b)$$

and

$$v_i = \frac{(-t_f\ + f_k)}{\sqrt{2}} \#(13c)$$

$$v_f = \frac{(-t_i\ + f_k)}{\sqrt{2}} \#(13d)$$

Where:

$u_i(u_f)$: The FRFT Domain at $\alpha = \frac{\pi}{4}\ corresponding\ to\ t_i\ (t_f)$.

$v_i(v_f)$: The FRFT Domain at $\alpha = \frac{3\pi}{4}\ corresponding\ to\ t_f\ (t_i)$.

$$u_m = \frac{u_i+u_f}{2} \#(14a)$$

$$v_m = \frac{v_i+v_f}{2} \#(14b)$$

Upon substituting $(13a)$ and $(13b)$ in $(14a)$, and $(13c)$ and $(13d)$ in $(14b)$ one obtains

$$u_m = \frac{\left[\frac{(t_i + f_k)}{\sqrt{2}}\right] + \left[\frac{(t_f + f_k)}{\sqrt{2}}\right]}{2} \#(15a)$$

$$v_m = \frac{\left[\frac{(-t_f + f_k)}{\sqrt{2}}\right] + \left[\frac{(-t_i + f_k)}{\sqrt{2}}\right]}{2} \#(15b)$$

Subtracting $(15b)$ from $(15a)$, one gets

$$u_m - v_m = \frac{t_i + t_f}{\sqrt{2}} \#(16)$$

$$\frac{u_m - v_m}{\sqrt{2}} = \frac{t_i + t_f}{2} = t_m \#(17)$$

Adding $(15b)$ from $(15a)$, one gets

$$u_m + v_m = \frac{2f_k}{\sqrt{2}} \#(18)$$

$$\frac{u_m + v_m}{\sqrt{2}} = f_k \#(19)$$

u_i , u_f , v_i , and v_f corresponding to frequency component $(f = f_k)$ can't be calculated in a rigorous form but u_m and v_m can be estimated by using the similarity of both $\left|X_{\frac{\pi}{4}}(u)\right|$ and $\left|X_{\frac{3\pi}{4}}(v)\right|$ that was proved in lemma (1).

C. *Summary of suggested Procedure*

Find the FRFT of the function (signal) $x(t)$; $t_1 \le t \le t_2$ at $\alpha = \frac{\pi}{4}$ and at $\alpha = \frac{3\pi}{4}$

Locate $u_m(v_m)$ midpoint in the peak area for $\left|X_{\frac{\pi}{4}}(u)\right| \left(\left|X_{\frac{3\pi}{4}}(u)\right| \right)$.

Use (19) to get f and (17) to get t_m for frequency component f per peak point.

4. Simulation Results

In preceding sections, it has been shown that the FRFT can be used to get the values of the frequency components and locate the midpoint time of each frequency component for time-limited function (signal).

The effectiveness of the FRFT in that application is demonstrated by considering two simulation applications.

A. *Application (1)*

Consider low frequency time-limited signal $x_1(t)$ and two separable additive high frequency components $x_2(t)$, and $x_3(t)$, such that the function (signal) $x(t) = x_1(t) + x_2(t) + x_3(t)$; $(t_i = 5 \text{ s}) \le t \le (t_f = 10 \text{ s})$;

Where

$$x_1(t) = sin(2f_1\pi t) \,; 5\,s \le t \le 10\,s;\; f_1 = 50\,Hz \#(20a)$$

$$x_2(t) = \cos(2f_2\pi t) \,; 6.6666 \text{ s} \le t \le 7.6664 \text{ s};\; f_2 = 1\,kHz \#(20b)$$

$$x_3(t) = \cos(2f_3\pi t) \,; 8.3334 \text{ s} \le t \le 9.3332 \text{ s};\; f_3 = 2\,kHz \#(20c)$$

As mentioned in $(20b)$ and $(20c)$; the frequency component $(f_2 = 1\,kHz)$ starts at $t_1 + \frac{1}{3}(t_2 - t_1)$ with duration 0.9998 , and the frequency component $(f_3 = 2\,kHz)$ starts at $t_1 + \frac{2}{3}(t_2 - t_1)$ with duration of 0.9998; all f_1, f_2, f_3 and their time locations are assumed to be unknown and the use of FRFT should deduce them.

The actual midpoints time have the following values

$$t_{m1} = \frac{5 + 10}{2} = 7.5\,s \#(21a)$$

$$t_{m2} = \frac{6.6666 + 7.6664}{2} = 7.1665\,s \#(21b)$$

$$t_{m3} = \frac{8.3334 + 9.3332}{2} = 8.8333\,s \#(21c)$$

The plot in Fig. 2 shows $x(t)$ by using sampling rate $f_s = 2.5 * 2000 = 5\,kHz$.

The Recursive Adaptive Lobatto Quadrature (RALQ) method is used to evaluate (10) and (11) to get both $\left|X_{\frac{\pi}{4}}(u)\right|$ and $\left|X_{\frac{3\pi}{4}}(v)\right|$, as mentioned in Fig. 3.

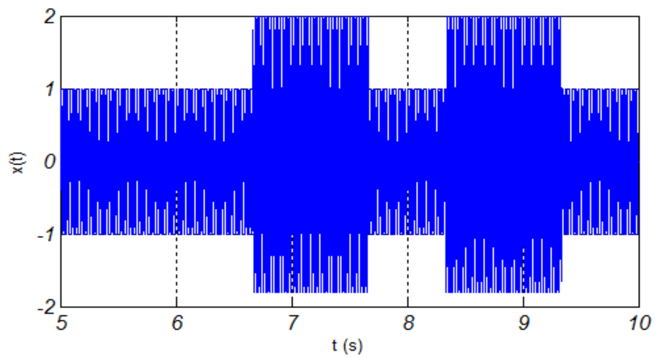

Fig. 2. The composite 3-frequency components pilot signal $x(t)$ for application (1)

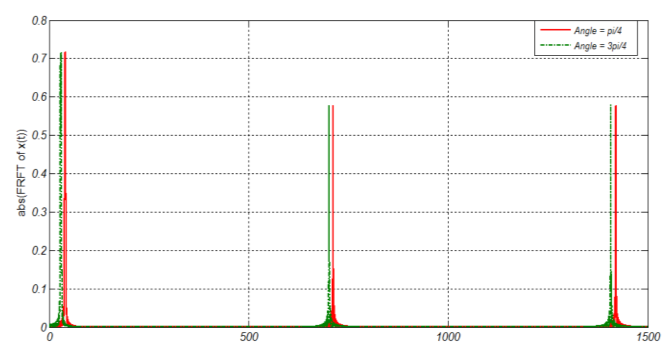

Fig. 3. $\left|X_{\frac{\pi}{4}}(u)\right|$ and $\left|X_{\frac{3\pi}{4}}(v)\right|$ for the three frequency components for application (1)

As illustrated in Fig. 3; we have clearly three peak points per angle $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ corresponding to three values of u_m and v_m which can be used to deduce the frequency component and its corresponding mid time as mentioned below:

- ***For the first peak point***

By using MATLAB software, the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$ is indicated as in Fig. 4; and the value of $u_m = 40.66$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; the value of $v_m = 30.05$.

According to (19)

$$f_1 = \frac{40.65 + 30.05}{\sqrt{2}} = 49.99\ Hz. \#(22a)$$

According to (17)

$$\tilde{t}_{m1} = \frac{40.65 - 30.05}{\sqrt{2}} = 7.502\ s. \#(22b)$$

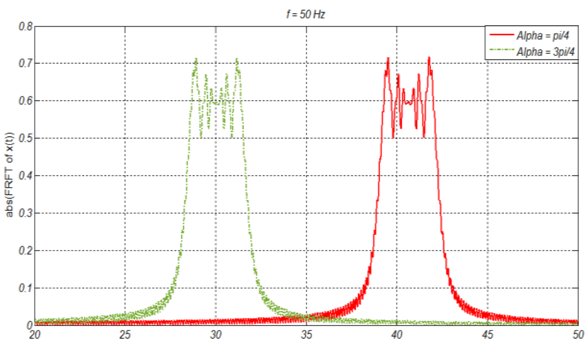

Fig. 4. $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the first frequency component for application (1)**

- ***For the second peak point***

By using MATLAB software, the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$ is indicated as in Fig. 5; and the value of $\left|X_{\frac{\pi}{4}}(u)\right|$; $u_m = 712.17$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; the value of $v_m = 702.04$,

then

$$f_2 = \frac{712.17 + 702.04}{\sqrt{2}} \cong 999.99\ Hz\#(23a)$$

and

$$\tilde{t}_{m2} = \frac{712.17 - 702.04}{\sqrt{2}} \cong 7.163\ s\#(23b)$$

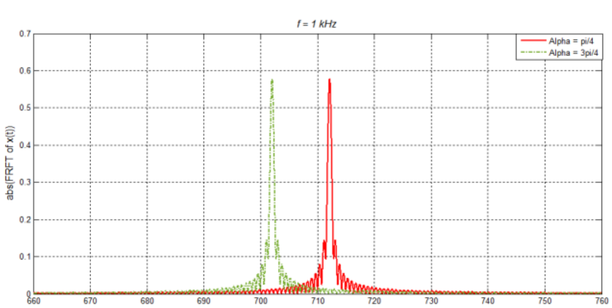

Fig. 5. $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the second frequency component for application (1)**

- ***For the third peak point***

Using MATLAB as in Fig. 6; For the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$; $u_m = 1420.46$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; $v_m = 1407.97$,

then

$$f_3 = \frac{1420.46 + 1407.97}{\sqrt{2}} \cong 2000\ Hz\#(24a)$$

and

$$\tilde{t}_{m3} = \frac{1420.46 - 1407.97}{\sqrt{2}} \cong 8.832\ s\#(24b)$$

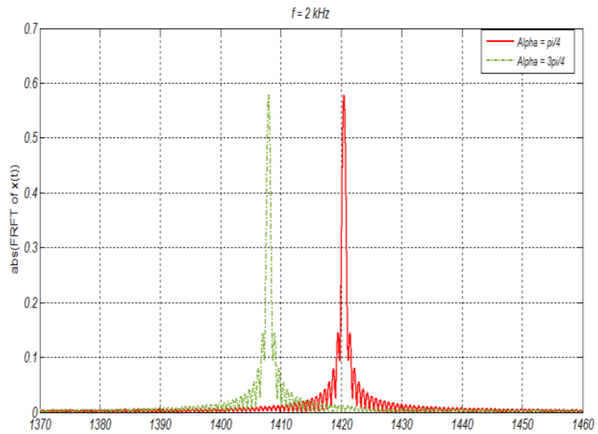

Fig. 6. $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the third frequency component for application (1)**

The summary of application (1) is given in the tables below.

Table I: Comparison between actual frequency component value and calculated one for Application (1)

	first peak	*second peak*	*third peak*
Actual frequency	$50\ Hz$	$1000\ Hz$	$2000\ Hz$
Calculated frequency	$49.99\ Hz$ s	$999.99\ Hz$	$2000\ Hz$
Error ($\frac{\lvert Actual\ frequency - Calculated\ frequency \rvert}{Actual\ frequency}$ x 100 %)	*0.02%*	*0.01%*	*0%*

Table II: Comparison between actual midpoint time and calculated one per frequency component for Application (1)

	$f_1 = 50\ Hz$	$f_2 = 1\ kHz$	$f_3 = 2\ kHz$
Actual midpoint time (t_m)	$7.5\ s$	$7.166\ s$	$8.833\ s$
Calculated midpoint time $(\tilde{t}_m)$	$7.502\ s$	$7.163\ s$	$8.832\ s$
Error $(\frac{\lvert t_m - \tilde{t}_m \rvert}{t_m}$ *x 100 %)*	*0.02%*	*0.04%*	*0.01%*

u , v

B. Application (2)

Consider low frequency time-limited signal $x_1(t)$ and two shared additive high frequency components $x_2(t)$ and $x_3(t)$, such that the function (signal) $x(t) = x_1(t) + x_2(t) + x_3(t)$; $(t_i = 0\text{ s}) \leq t \leq (t_f = 10\text{ s})$;

Where

$$x_1(t) = \sin(2f_1\pi t)\ ;0\text{ s} \leq t \leq 10\text{ s};\ f_1 = 50\ Hz\#(25a)$$

$$x_2(t) = \cos(2f_2\pi t)\ ;3.3334\text{ s} \leq t \leq 10\text{ s};\ f_2 = 1\ kHz\#(25b)$$

$$x_3(t) = \cos(2f_3\pi t)\ ;6.6668\text{ s} \leq t \leq\ 10\text{ s};\ f_3 = 2\ kHz\#(25c)$$

The frequency component $(f_2 = 1\ kHz)$ starts at $t_1 + \frac{1}{3}(t_2 - t_1)$, and the frequency component $(f_3 = 2kHz)$ starts at $t_1 + \frac{2}{3}(t_2 - t_1)$; all f_1, f_2, f_3 and their time locations are assumed to be unknown and the use of FRFT should deduce them.

The actual midpoints time have the following values.

$$t_{m1} = \frac{0 + 10}{2} = 5\ s\#(26a)$$

$$t_{m2} = \frac{3.3334 + 10}{2} = 6.6667\ s\#(26b)$$

$$t_{m3} = \frac{6.6668 + 10}{2} = 8.3334\ s\#(26c)$$

The plot in Fig. 7 shows $x(t)$ calculated at the same sampling rate that is used in application (1).

Both $\left|X_{\frac{\pi}{4}}(u)\right|$ and $\left|X_{\frac{3\pi}{4}}(v)\right|$ is drawn in Fig. 8 by using RALQ technique as in application (1).

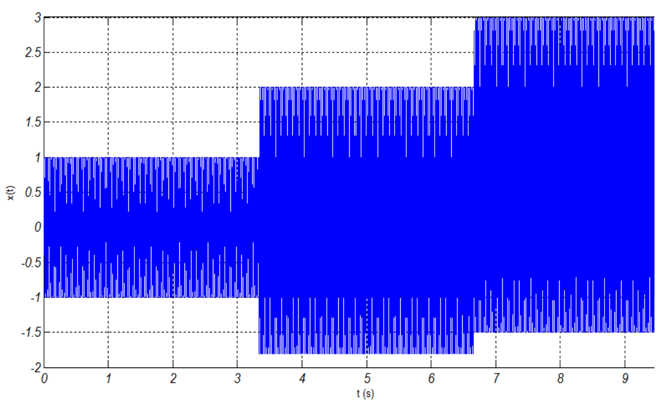

Fig. 7: The function (signal) $x(t)$ for application (2)

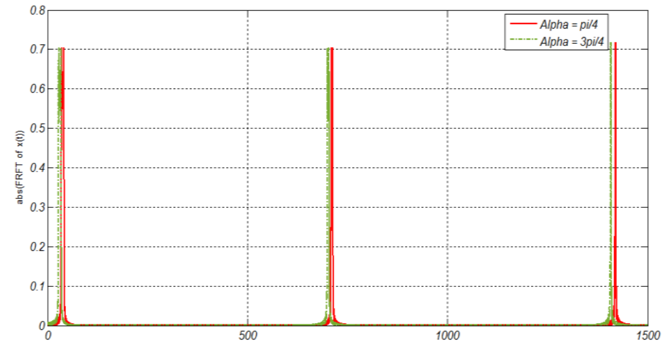

Fig. 8. $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the three frequency components for application (2)**

- ***For the first peak point***

As in Fig. 9; For the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$; $u_m = 38.89$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; $v_m = 31.82$.

According to (19)

$$f_1 = \frac{38.89 + 31.82}{\sqrt{2}} = 49.99\ Hz\#(27a)$$

According to (17)

$$\tilde{t}_{m1} = \frac{38.89 - 31.82}{\sqrt{2}} = 4.999\ s\#(27b)$$

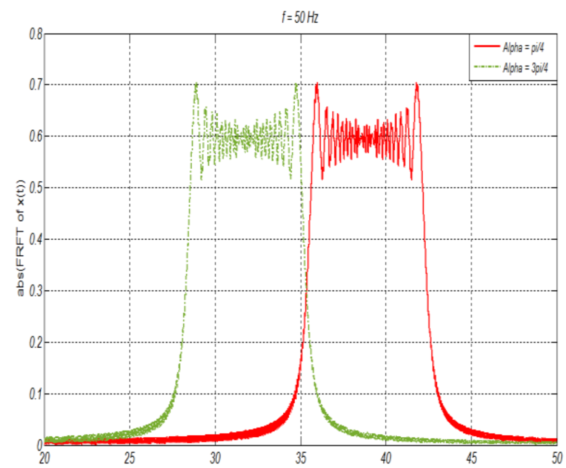

Fig. 9. $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the first frequency component for application (2)**

- ***For the second peak point***

As in Fig. 10; For the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$; $u_m = 711.82$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; $v_m = 702.39$,

then

$$f_2 = \frac{711.82 + 702.39}{\sqrt{2}} \cong 999.99\ Hz \quad (28a)$$

and

$$\tilde{t}_{m2} = \frac{711.82 - 702.39}{\sqrt{2}} \cong 6.668\ s \quad (28b)$$

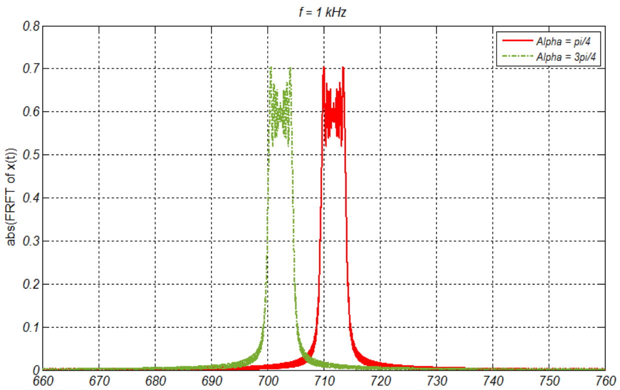

Fig. 10: $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the second frequency component for application (2)**

- ***For the third peak point***

As in Fig. 11; For the curve that represents $\left|X_{\frac{\pi}{4}}(u)\right|$; $u_m = 1420.11$, and for $\left|X_{\frac{3\pi}{4}}(v)\right|$; $v_m = 1408.32$,

then

$$f_3 = \frac{1420.11 + 1408.32}{\sqrt{2}} \cong 2000\ Hz \quad (29a)$$

and

$$\tilde{t}_{m3} = \frac{1420.11 - 1408.32}{\sqrt{2}} \cong 8.3367\ s \quad (29b)$$

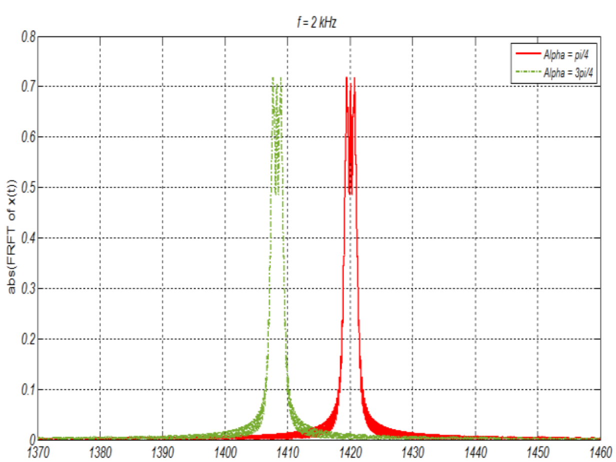

Fig. 11: $\left|X_{\frac{\pi}{4}}(u)\right|$ **and** $\left|X_{\frac{3\pi}{4}}(v)\right|$ **for the third frequency component for application (2)**

The summary of application (2) is in the table below.

Table 4: Comparison between actual frequency component value and calculated one for Application (2)

	first peak	*second peak*	*third peak*
Actual frequency	$50\ Hz$	$1000\ Hz$	$2000\ Hz$
Calculated frequency	$49.99\ Hz\ s$	$999.99\ Hz$	$2000\ Hz$
Error $\left(\frac{\lvert Actual\ frequency - Calculated\ frequency\rvert}{Actual\ frequency}\ x\ 100\ \%\right)$	*0.02%*	*0.01%*	*0%*

Table 5: Comparison between actual midpoint time and calculated one per frequency component for Application (2)

	$f_1 = 50\ Hz$	$f_2 = 1\ kHz$	$f_3 = 2\ kHz$
Actual midpoint time (t_m)	$5\ s$	$6.667\ s$	$8.333\ s$
Calculated midpoint time $(\tilde{t}_m)$	$4.999\ s$	$6.668\ s$	$8.337\ s$
Error $\left(\frac{\lvert t_m - \tilde{t}_m\rvert}{t_m}\ x\ 100\ \%\right)$	*0.02%*	*0.01%*	*0.04%*

5. Conclusions

The time-limited function (signal) has been expressed in terms of the FRFT for two different angles $\left(\alpha = \frac{\pi}{4}, \alpha = \frac{3\pi}{4}\right)$, and then proposed method has been developed to find the value and the midpoint time for each frequency component.

This method is based on the similarity property for the FRFT of the time-limited function (signal) as was proved in lemma (1), and on the rotation of axis effect of the FRFT method on the time-frequency plane as proved in lemma (2).

Numerical Applications illustrate the high accuracy for this technique in finding the value and midpoint time for frequency components of time limited signal compared to actual case.

Conflict of Interest

The authors declare no conflict of interest.

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Abbreviation and symbols

FT	**Fourier transform.**
FRFT	**Continuous fractional Fourier transform.**
FRFD	**Continuous fractional Fourier domain.**
SFRFT	**Sparse fractional Fourier transform**
STFT	**Short time Fourier transform.**
TFFR	**Time fractional Fourier domain representation.**
WT	**Wavelet transforms.**
FRWT	**Fractional Wavelet transform.**
RALQ	**Recursive adaptive Lobatto quadrature.**