Journal of Fractional Calculus and Applications,
Vol. 7(1) Jan. 2016, pp. 12-23.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

THE POWERS OF THE DIRAC DELTA FUNCTION BY CAPUTO FRACTIONAL DERIVATIVES

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Abstract. One of the problems in distribution theory is the lack of definitions of products and powers of distributions in general. In this paper, we choose a fixed δ-sequence without compact support and the generalized Taylor's formula based on Caputo fractional derivatives to give meaning to the distributions $\delta^k(x)$ and $(\delta')^k(x)$ for some values of k. These can be regarded as powers of Dirac delta functions.

1. Introduction

In mathematics, the Dirac delta function, or δ (singular) function, is a generalized function, on the real number line that is zero everywhere except at zero, with an integral of one over the entire real line. The delta function is sometimes thought of as an infinitely high, infinitely thin spike at the origin, with total area one under the spike, and physically represents the density of an idealized point mass or point charge. It was introduced by theoretical physicist Paul Dirac in 1920. In the context of signal processing it is often referred to as the unit impulse symbol (or function). It is clear to see that this δ function contradicts with the integral theory in terms of Lebesgue sense, and hence it cannot be properly defined within the framework of classical function theory.

Around 1950, Schwartz established the theory of distributions by treating singular functions as linear and continuous functionals on the testing function space. Let $\mathcal{D}(R)$ be the space (Schwartz) [1] of infinitely differentiable functions with compact support in R, and let $\mathcal{D}'(R)$ be the space of distributions (linear and continuous functionals) defined on $\mathcal{D}(R)$. Further, we shall define a sequence $\phi_1(t), \phi_2(t), \cdots, \phi_n(t), \cdots$ converges to zero in $\mathcal{D}(R)$ if all these functions vanish outside a certain fixed bounded interval, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. The functional δ is defined as

$$(\delta, \phi) = \phi(0)$$

2010 *Mathematics Subject Classification.* 46F10, 26A33.
Key words and phrases. Distribution, delta function, delta sequence, Caputo fractional derivative and generalized Taylor's formula.
Submitted June 19, 2015.

where $\phi \in \mathcal{D}(R)$. Clearly, δ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta \in \mathcal{D}'(R)$.
The definition of the product of a distribution and an infinitely differentiable function is the following (see for example [1]).
Definition 1 Let f be a distribution and let g be an infinitely differentiable function. Then the product fg is defined by

$$(fg, \phi) = (f, g\phi)$$

for all testing functions $\phi \in \mathcal{D}(R)$.
It follows from Definition 1 that

$$t^k \delta^{(m)}(t) = \begin{cases} (-1)^k k! \binom{m}{k} \delta^{(m-k)}(t), & \text{if } k \leq m, \\ 0, & \text{otherwise} \end{cases}$$

for $k, m = 0, 1, 2, \cdots$.
It seems impossible to define δ^2, although one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions in elementary particle physics [2]. A definition for product of distributions is given using delta sequences in [3]. However, δ^2 as a product of δ with itself is shown not to exist in mathematical sense. Embacher, Grübl, and Oberguggenberger [4] studied products of distributions containing the δ functions in 1992 and found applications to quantum electrodynamics. In perturbative computations of quantum-mechanical path integrals in curvilinear coordinates, people encounter Feynman diagrams involving multiple temporal integrals over products of distributions, which are undefined. In addition, there are terms proportional to powers of the δ functions at the origin coming from the measure of path integration [5]. Furthermore, products of distributions, including powers of the δ functions, are in great demand for certain types of partial differential equations [6] and path integrals in quantum mechanics [7], which require complex computations. Recently, Li and Li [8] used the following δ-sequence

$$\delta_n(t) = \left(\frac{n}{\pi}\right)^{1/2} e^{-nt^2}, \ t \in R$$

and the neutrix limit to define powers of the distributions δ and δ'.
On the other hand, fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalizes integer-order differentiation and n-fold integration. The beginning of fractional calculus is considered to be the Leibniz's letter to L'Hôspital in 1695, where the notation for differentiation of non-integer orders was discussed.
We let Y_α be the convolution kernel of order $\alpha \in R^+$ for fractional integrals, given by

$$Y_\alpha = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{loc}(R^+),$$

where Γ is the well-known Euler Gamma function, and

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Definition 2 The fractional integral (or, the Riemann-Liouville) $D_{0,t}^{-\alpha}$ of fractional order $\alpha \in R^+$ of function $\phi(t)$ is defined by

$$D_{0,t}^{-\alpha}\phi(t) = Y_\alpha * \phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\phi(\tau)d\tau,$$

where we set the initial time to zero.
As an example, we have the following for $\gamma > -1$

$$D_{0,t}^{-\alpha}t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma},$$

$$D_{0,t}^{-\alpha}e^t = t^\alpha\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+\gamma+1)}t^k.$$

The following properties of Y_α, $D_{0,t}^{-\alpha}$, and the fractional derivatives can be found in [9] and [10].

Property 1

(i) The convolution property $Y_\alpha * Y_\beta = Y_{\alpha+\beta}$ holds for $\alpha > 0$ and $\beta > 0$, which implies that $D_{0,t}^{-\alpha}D_{0,t}^{\beta} = D_{0,t}^{-\alpha-\beta}$.

(ii) Consistency property with the integer-order integral: $\lim_{\alpha\to m} D_{0,t}^{-\alpha}\phi(t) = D_{0,t}^{-m}\phi(t)$, where $\alpha > 0$, $m \in Z^+$ and

$$D_{0,t}^{-m}\phi(t) = \int_0^t\int_0^{t_{m-1}}\cdots\int_0^{t_1}\phi(\tau)d\tau dt_1\cdots dt_{m-1} = \frac{1}{(m-1)!}\int_0^t(t-\tau)^{m-1}\phi(\tau)d\tau.$$

Definition 3 The Riemann-Liouville derivative of fractional order α of function $\phi(t)$ is defined as

$${}_{RL}D_{0,t}^{\alpha}\phi(t) = \frac{d^m}{dt^m}D_{0,t}^{-(m-\alpha)}\phi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t(t-\tau)^{m-\alpha-1}\phi(\tau)d\tau,$$

where $m-1 < \alpha < m \in Z^+$.
It follows that

$${}_{RL}D_{0,t}^{\alpha}c = \frac{ct^\alpha}{\Gamma(1-\alpha)},$$

$${}_{RL}D_{0,t}^{\alpha}t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha},$$

where c is a constant and $\lambda > -1$.

Definition 4 The Caputo derivative of fractional order α of function $\phi(t)$ is defined as

$${}_{C}D_{0,t}^{\alpha}\phi(t) = D_{0,t}^{-(m-\alpha)}\frac{d^m}{dt^m}\phi(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\phi^{(m)}(\tau)d\tau,$$

where $m-1 < \alpha < m \in Z^+$.
It follows that

$${}_{C}D_{0,t}^{\alpha}t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, & m-1<\alpha<m\in Z^+,\ p>m-1,\ p\in R, \\ 0, & m-1<\alpha<m\in Z^+,\ p\le m-1,\ p\in Z^+. \end{cases}$$

To make this paper self-contained as much as possible, we provide a short survey on generalized Taylor's formulas in the following, which is mainly given in [8].
The ordinary Taylor's formula has been generalized by many authors. Riemann

[11] had already written a formal version of the generalized Taylor's series for a real number r:

$$\phi(t+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}\ {}_{RL}D_{0,t}^{m+r}\phi(t),$$

where for $\alpha<0$, ${}_{RL}D_{0,t}^{\alpha}\phi(t)=D_{0,t}^{-\alpha}\phi(t)$ is the Riemann-Liouville fractional integral of order $-\alpha$ in Definition 2. Moreover, ${}_{RL}D_{0,t}^{0}\phi(t)=D_{0,t}^{0}\phi(t)=\phi(t)$.
The proof of validity of the Riemann expansion above for certain classes of functions was undertaken by Hardy [12], both for finite and infinite initial time (we set it to zero in this paper, as mentioned in Definition 2).
On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesian ([13] and [14]). For ϕ having all of the required continuous derivatives, they derived that

$$\phi(t)=\sum_{k=0}^{m-1}\frac{\mathcal{D}^{(\alpha_k)}\phi(0)}{\Gamma(1+\alpha_k)}x^{\alpha_k}+\frac{1}{\Gamma(1+\alpha_k)}\int_0^t(t-x)^{\alpha_m-1}\mathcal{D}^{(\alpha_k)}\phi(x)dx,$$

where $t>0$, $\alpha_0,\alpha_1,\cdots,\alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\le 1$, $k=1,2,\cdots,m$ and $\mathcal{D}^{(\alpha_k)}\phi=D_{0,t}^{\alpha_k-\alpha_{k-1}-1}\,{}_{RL}D_{0,t}{}^{1+\alpha_{k-1}}\phi$.
Trujillo, Rivero, and Bonilla [15] established the following generalized Taylor's formula under certain conditions for ϕ and $\alpha\in[0,1]$:

$$\phi(t)=\sum_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}\ t^{(j+1)\alpha-1}+R_n(t), \tag{1}$$

where

$$R_n(t)=\frac{{}_{RL}\hat{D}_{0,t}^{(n+1)\alpha}\phi(\zeta)}{\Gamma((n+1)\alpha+1)}t^{(n+1)\alpha},\quad 0\le\zeta\le t,$$

and

$$c_j=\Gamma(\alpha)[x^{1-\alpha}\,{}_{RL}\hat{D}_{0,t}^{j\alpha}]\phi(0^+),\quad \forall j=0,1,\cdots,n,$$

and the sequential fractional derivative is denoted by

$${}_{RL}\hat{D}_{0,t}^{j\alpha}={}_{RL}D_{0,t}^{\alpha}\cdots{}_{RL}D_{0,t}^{\alpha},\quad j\text{-times and } j\in Z^{+}.$$

We would also like to mention that there is another version of fractional Taylor's series in the Riemann-Liouville form in [16], which is a particular case of Equation (1).
The following theorem due to Odibat and Shawagfeh in 2007 can be found in [17].
Theorem 1 (Generalized Taylor's Theorem) Suppose that ${}_{C}\hat{D}_{0,t}^{k\alpha}\phi(t)\in C(a,b]$ for $k=0,1,2,\cdots m+1$, where $0<\alpha<1$, then we have

$$\phi(t)=\sum_{i=0}^{m}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}({}_{C}\hat{D}_{a,t}^{i\alpha}\phi)(a)+\frac{({}_{C}\hat{D}_{a,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)}(t-a)^{(m+1)\alpha}$$

with $a\le\zeta\le t,\forall t\in(a,b]$, where ${}_{C}\hat{D}_{0,t}^{i\alpha}={}_{C}D_{0,t}^{\alpha}\,{}_{C}D_{0,t}^{\alpha}\cdots{}_{C}D_{0,t}^{\alpha}$.
In particular, we have for $a=0$,

$$\phi(t)=\sum_{i=0}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}({}_{C}\hat{D}_{0,t}^{i\alpha}\phi)(0)+\frac{({}_{C}\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha}. \tag{2}$$

Note that for $\alpha = 1$,

$$\phi(t) = \sum_{i=0}^{m} \frac{(t-a)^i}{i!}\phi^{(i)}(a) + \frac{\phi^{(m+1)}(\zeta)}{(m+1)!}(t-a)^{m+1},$$

which is the classical Taylor's formula.
Remark: One understands, in fractional calculus, that ${}_C\hat{D}_{0,t}^{i\alpha} = {}_CD_{0,t}^{\alpha}\,{}_CD_{0,t}^{\alpha} \cdot \cdots {}_CD_{0,t}^{\alpha} \neq {}_CD_{0,t}^{i\alpha}$ in general. Generally speaking, it is easier to compute ${}_CD_{0,t}^{i\alpha}\phi(t)$ than ${}_C\hat{D}_{0,t}^{i\alpha}\phi(t)$. The following two theorems shown by Li and Deng in [18] describe, under certain circumstances, that ${}_CD_{0,t}^{i\alpha}\phi(t) = {}_C\hat{D}_{0,t}^{i\alpha}\phi(t)$.
Theorem 2 If $\phi(t) \in C^1[0,T]$ for $T > 0$, then

$${}_CD_{0,t}^{\alpha_2}\,{}_CD_{0,t}^{\alpha_1}\phi(t) = {}_CD_{0,t}^{\alpha_1}\,{}_CD_{0,t}^{\alpha_2}\phi(t) = {}_CD_{0,t}^{\alpha_1+\alpha_2}\phi(t), \quad t \in [0,T],$$

where $\alpha_1,\, \alpha_2 \in R^+$ and $\alpha_1 + \alpha_2 \le 1$.

In particular, we obtain

$$\begin{aligned} {}_C\hat{D}_{0,t}^{2\cdot 0.5} &= {}_CD_{0,t}^{0.5}\,{}_CD_{0,t}^{0.5}\phi(t) = \phi'(t), \ \text{ and} \\ {}_C\hat{D}_{0,t}^{2\cdot 0.3} &= {}_CD_{0,t}^{0.3}\,{}_CD_{0,t}^{0.3}\phi(t) = {}_CD_{0,t}^{0.6}\phi(t). \end{aligned}$$

Theorem 3 If $\phi(t) \in C^m[0,T]$ for $T > 0$, then

$${}_CD_{0,t}^{\alpha}\phi(t) = {}_CD_{0,t}^{\alpha_n} \cdots {}_CD_{0,t}^{\alpha_2}\,{}_CD_{0,t}^{\alpha_1}\phi(t), \ \ t \in [0,T]$$

where $\alpha = \sum_{i=1}^{n}\alpha_i,\ \alpha_i \in (0,1]$, $m-1 \le \alpha < m \in Z^+$ and there exists $i_k < n$ such that $\sum_{j=1}^{i_k}\alpha_j = k$ for $k = 1, 2, \cdots, m-1$.
Using this theorem, we get as an example,

$$\begin{aligned} &{}_CD_{0,t}^{101\cdot 0.5}\phi(t) = {}_CD_{0,t}^{0.5}\,{}_CD_{0,t}^{0.5}\cdots{}_CD_{0,t}^{0.5}\phi(t) = {}_C\hat{D}_{0,t}^{101\cdot 0.5}\phi(t), \ \text{ if } \ \phi(t) \in C^{51}[0,T], \\ &{}_CD_{0,t}^{100\cdot\frac{1}{3}}\phi(t) = {}_CD_{0,t}^{\frac{1}{3}}\,{}_CD_{0,t}^{\frac{1}{3}}\cdots{}_CD_{0,t}^{\frac{1}{3}}\phi(t) = {}_C\hat{D}_{0,t}^{100\cdot\frac{1}{3}}\phi(t), \ \text{ if } \ \phi(t) \in C^{34}[0,T]. \end{aligned}$$

In this paper, we will adopt a new δ-sequence and the generalized Taylor's formula of equation (2) to define powers of the distributions $\delta^k(t)$ and $(\delta')^k(t)$ for some values of $k \in R$ due to simplicity of the coefficients in the equation, which can be further simplified by Theorems 2 and 3 under certain conditions. Without a doubt, using other Taylor's formulas given above to define powers of the distributions will require more complicated computations for the coefficients, although it is doable.

2. The distributions $\delta^k(t)$ and $(\delta')^k(t)$

In order to study the distribution $\delta^k(t)$, we choose the following δ-sequence without compact support

$$\delta_n(t) = \frac{n}{\pi}\,\frac{1}{(nt)^2+1} \ \text{ for } t \in R,$$

which implies that

$$(\delta(t), \phi(t)) = \lim_{n\to\infty}(\delta_n(t), \phi(t)) = \lim_{n\to\infty}\frac{n}{\pi}\int_{-\infty}^{\infty}\frac{1}{(nt)^2+1}\phi(t)dt = \phi(0). \tag{3}$$

We define for all $k > 1/2$

$$(\delta^k(t), \phi(t)) := N - \lim_{n\to\infty}\int_{-\infty}^{\infty}\left(\frac{n}{\pi}\right)^k \frac{1}{((nt)^2+1)^k}\phi(t)dt \tag{4}$$

where N is the neutrix having domain $N' = \{1, 2, 3, \cdots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \cdots)$$

and all functions of n that converge to zero in the normal sense as n tends to infinity (see [19] and [20]).

Remark: The reason we define equation (4) only for $k > 1/2$ is that the integral

$$\int_{-\infty}^{\infty} \frac{1}{(y^2+1)^k} dy$$

diverges for all $k \leq 1/2$.

Clearly, we have from equation (4)

$$(\delta^0(t), \phi(t)) = N - \lim_{n\to\infty} (\delta_n^0(t), \phi(t)) = (1, \phi(t)) = \int_{-\infty}^{\infty} \phi(t) dt \ \text{ for } \ \phi(t) \in \mathcal{D}(R),$$

which infers that $\delta^0(t) = 1$.

For $1/2 < k < 1$, we make the substitution $y = nt$ in equation (4) and come to

$$(\delta^k(t), \phi(t)) = N - \lim_{n\to\infty} \frac{1}{\pi^k} \frac{1}{n^{1-k}} \int_{-\infty}^{\infty} \frac{\phi(y/n)}{(y^2+1)^k} dy = 0,$$

since ϕ is a bounded function. Thus, $\delta^k(t) = 0$.

Furthermore, it follows from equation (3) that $\delta^1(t) = \delta(t)$. As for $k > 1$, we obtain from equation (4)

$$\begin{aligned}
&(\delta^k(t), \phi(t)) \\
&= N - \lim_{n\to\infty} \left(\int_0^{\infty} \left(\frac{n}{\pi}\right)^k \frac{1}{((nt)^2+1)^k} \phi(t) dt + \int_0^{\infty} \left(\frac{n}{\pi}\right)^k \frac{1}{((nt)^2+1)^k} \phi(-t) dt \right) \\
&:= N - \lim_{n\to\infty} (I_1 + I_2).
\end{aligned}$$

By the generalized Taylor's formula from equation (2), we have

$$\begin{aligned}
\phi(t) &= \sum_{i=0}^{m} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} ({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) + \frac{({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha} \\
&= \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} ({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) + \frac{({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha}
\end{aligned}$$

where $m\alpha = k - 1$, $m \in Z^+$ and $0 < \alpha \leq 1$ (note that it reduces to the classical Taylor's formula when $\alpha = 1$).

Thus,

$$\begin{aligned}
I_1 = \quad & \sum_{i=0}^{m-1} \frac{1}{\Gamma(i\alpha+1)} ({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) \left(\frac{n}{\pi}\right)^k \int_0^{\infty} \frac{1}{((nt)^2+1)^k} t^{i\alpha} dt \\
&+ \frac{1}{\Gamma(m\alpha+1)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \left(\frac{n}{\pi}\right)^k \int_0^{\infty} \frac{1}{((nt)^2+1)^k} t^{m\alpha} dt \\
&+ \frac{1}{\Gamma((m+1)\alpha+1)} \left(\frac{n}{\pi}\right)^k \int_0^{\infty} \frac{1}{((nt)^2+1)^k} t^{(m+1)\alpha} ({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta) dt \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}$$

Setting $y = nt$ again, we get

$$I_{11} = \sum_{i=0}^{m-1} \frac{1}{\Gamma(i\alpha+1)} ({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) \left(\frac{n}{\pi}\right)^k \frac{1}{n^{i\alpha+1}} \int_0^\infty \frac{y^{i\alpha}}{(y^2+1)^k} dy.$$

Hence

$$N - \lim_{n\to\infty} I_{11} = 0.$$

Since $\phi(t) \in \mathcal{D}(R)$, there exists a positive real number M_1 such that

$$\sup_{t\in R^+} \left| ({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(t) \right| \le M_1, \text{ for } m \in Z^+ \text{ and } 0 < \alpha \le 1$$

which infers that

$$\lim_{n\to\infty} I_{13} = 0.$$

Coming to I_{12}, we use the following formula

$$\int_0^\infty \frac{y^{k-1}}{(1+y^2)^k} dy = \frac{\Gamma^2(k/2)}{2\Gamma(k)}$$

to imply that

$$\begin{aligned} I_{12} &= \frac{1}{\Gamma(m\alpha+1)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \left(\frac{n}{\pi}\right)^k \int_0^\infty \frac{1}{((nt)^2+1)^k} t^{m\alpha} dt \\ &= \frac{1}{\pi^k\Gamma(m\alpha+1)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \int_0^\infty \frac{y^{k-1}}{(1+y^2)^k} dy \\ &= \frac{1}{\pi^k\Gamma(m\alpha+1)} \frac{\Gamma^2(k/2)}{2\Gamma(k)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \\ &= \frac{\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0). \end{aligned}$$

Therefore,

$$N - \lim_{n\to\infty} I_1 = \frac{\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) = \frac{\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{k-1}\phi)(0).$$

Following the similar calculation, we derive that

$$\begin{aligned} N - \lim_{n\to\infty} I_2 &= (-1)^{m\alpha} \frac{\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \\ &= (-1)^{k-1} \frac{\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{k-1}\phi)(0). \end{aligned}$$

Finally,

$$\begin{aligned} (\delta^k(t), \phi(t)) &:= N - \lim_{n\to\infty} (I_1 + I_2) \\ &= \frac{((-1))^{m\alpha} + 1)\Gamma^2(k/2)}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) \\ &= \frac{((-1)^{k-1} + 1)\Gamma^2(\frac{k}{2})}{2\pi^k\Gamma^2(k)} ({}_C\hat{D}_{0,t}^{k-1}\phi)(0). \end{aligned}$$

In particular for $k = 1$,

$$(\delta(t), \phi(t)) = \frac{((-1)^{1-1} + 1)\Gamma^2(\frac{1}{2})}{2\pi\Gamma^2(1)} ({}_C\hat{D}_{0,t}^{1-1}\phi)(0) = \phi(0). \tag{5}$$

It follows that

$$\delta^{2l}(t) = 0 \text{ for } l = 1, 2, 3, \cdots$$
$$\delta^{2l+1}(t) = \frac{\Gamma^2(l+1/2)}{\pi^{2l+1}[(2l)!]^2}\delta^{(2l)}(t)$$

for $l = 0, 1, 2, \cdots$.
Using

$$\Gamma(l+1/2) = \frac{1 \cdot 3 \cdot 5 \cdots (2l-1)}{2^l}\sqrt{\pi} = \frac{(2l)!}{4^l l!}\sqrt{\pi} \text{ for } l = 0, 1, 2, \cdots ,$$

We have

$$\delta^{2l+1}(t) = C_l \delta^{(2l)}(t),$$

where

$$C_l = \frac{1}{2^{4l}(l!)^2\, \pi^{2l}} \text{ for } l = 0, 1, 2, \cdots .$$

Now we can summarize to get
Theorem 4

$$\delta^0(t) = 1,$$
$$\delta^k(t) = 0 \text{ for } 1/2 < k < 1,$$
$$(\delta^k(t), \phi(t)) = \frac{((-1)^{k-1}+1)\Gamma^2(\frac{k}{2})}{2\pi^k\Gamma^2(k)}({}_C\hat{D}_{0,t}^{k-1}\phi)(0) \text{ for } k \geq 1$$

where $({}_C\hat{D}_{0,t}^{k-1}\phi)(0) = ({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) = ({}_CD_{0,t}^{\alpha} \cdot {}_CD_{0,t}^{\alpha} \cdots {}_CD_{0,t}^{\alpha}\phi)(0)$ (m-times) and $(-1)^k = \cos k\pi + i \sin k\pi$ for $k \in [0, \infty)$. In particular,

$$\delta^{2l}(t) = 0 \text{ for } l = 1, 2, 3, \cdots , \text{ and}$$
$$\delta^{2l+1}(t) = \frac{1}{2^{4l}(l!)^2\, \pi^{2l}}\delta^{(2l)}(t) \text{ for } l = 0, 1, 2, \cdots .$$

Remark:
(i) We would like to point out that Theorem 4 is a generalization of Theorem 1 obtained in [21], where the case for $k \in Z^+$ is mainly discussed.
(ii) The choice of $\alpha \in (0, 1]$ is not unique. For example, we can pick up $m = 1$, $\alpha = 0.5$ or $m = 2$, $\alpha = 0.25$ (and others) if $k = 1.5$. Generally speaking, we choose α and m in Theorem 4 to make $({}_C\hat{D}_{0,t}^{k-1}\phi)(0)$ as simple as possible. Hence $({}_C\hat{D}_{0,t}^{0.5}\phi)(0) = ({}_CD_{0,t}^{0.5}\phi)(0)$, which is Caputo derivative of order 1/2.
It follows from Theorem 4 that

$$\delta^2(t) = 0,$$
$$\delta^3(t) = \frac{1}{16\pi^2}\delta''(t),$$
$$(\delta^{1.5}(t), \phi(t)) = \frac{(i+1)\Gamma^2(0.75)}{2\pi^2\Gamma^2(1.5)}({}_CD_{0,t}^{0.5}\phi)(0) = \frac{2(i+1)\Gamma^2(0.75)}{\pi^3}({}_CD_{0,t}^{0.5}\phi)(0), \; i = \sqrt{-1}$$

since $\Gamma(1.5) = \frac{1}{2}\sqrt{\pi}$.
We use Theorems 3 and 4 to derive

$$\begin{aligned}(\delta^{\frac{103}{3}}(t), \phi(t)) &= \frac{((-1)^{\frac{100}{3}}+1)\Gamma^2(\frac{103}{6})}{2\pi^{103/3}\Gamma^2(\frac{103}{3})}({}_C\hat{D}_{0,t}^{\frac{100}{3}}\phi)(0) \\ &= \frac{(1-i\sqrt{3})\Gamma^2(\frac{103}{6})}{4\pi^{103/3}\Gamma^2(\frac{103}{3})}({}_C D_{0,t}^{\frac{100}{3}}\phi)(0).\end{aligned}$$

Similarly, we are able to define the distribution $(\delta')^k(t)$ for all $k > 1/3$, based on the derivative of the δ-sequence

$$\delta_n'(t) = \frac{2n^3}{\pi}\frac{-t}{((nt)^2+1)^2}, \quad \text{and}$$

$$\begin{aligned}&((\delta')^k(t), \phi(t)) := N - \lim_{n\to\infty}((\delta_n')^k(t), \phi(t)) \\ &= N - \lim_{n\to\infty}\frac{2^k n^{3k}}{\pi^k}\left(\int_0^\infty \frac{(-t)^k}{((nt)^2+1)^{2k}}\phi(t)dt + \int_0^\infty \frac{t^k}{((nt)^2+1)^{2k}}\phi(-t)dt\right) \\ &= N - \lim_{n\to\infty}(S_1+S_2). \qquad (6)\end{aligned}$$

Clearly, we have for $k = 0$ that

$$((\delta')^0(t), \phi(t)) = \int_{-\infty}^{\infty}\phi(t)dt = (1, \phi(t))$$

which claims that $(\delta')^0(t) = 1$.
Remark: The reason we define equation (6) only for $k > 1/3$ is that the integral

$$\int_0^\infty \frac{y^k}{(y^2+1)^{2k}}dy$$

diverges for all $k \leq 1/3$.
Setting $y = nt$ in equation (6) for $1/3 < k < 1/2$, we can prove that

$$\begin{aligned}&((\delta')^k(t), \phi(t)) := N - \lim_{n\to\infty}((\delta_n')^k(t), \phi(t)) \\ &= \lim_{n\to\infty}\frac{2^k n^{3k}}{\pi^k}\frac{1}{n^{1+k}}\left(\int_0^\infty \frac{(-1)^k y^k}{(y^2+1)^{2k}}\phi(y/n)dy + \int_0^\infty \frac{y^k}{(y^2+1)^{2k}}\phi(-y/n)dy\right) \\ &= 0 \qquad (7)\end{aligned}$$

since ϕ is a bounded function. Therefore, $(\delta')^k(t) = 0$.
For $k = 1/2$, we get from equation (6) that

$$\begin{aligned}&((\delta')^{1/2}(t), \phi(t)) := N - \lim_{n\to\infty}((\delta_n')^{1/2}(t), \phi(t)) \\ &= \lim_{n\to\infty}\frac{2^{1/2}n^{3/2}}{\pi^{1/2}}\frac{1}{n^{1+1/2}}\left(\int_0^\infty \frac{iy^{1/2}}{y^2+1}\phi(y/n)dy + \int_0^\infty \frac{y^{1/2}}{y^2+1}\phi(-y/n)dy\right) \\ &= \sqrt{\frac{2}{\pi}}(i+1)\phi(0)\int_0^\infty \frac{y^{1/2}}{y^2+1}dy.\end{aligned}$$

Using

$$\int_0^\infty \frac{y^{1/2}}{y^2+1}dy = \frac{\Gamma(3/4)\Gamma(1/4)}{2},$$

we infer that

$$(\delta')^{1/2}(t) = (i+1)\frac{\Gamma(3/4)\Gamma(1/4)}{\sqrt{2\pi}}\delta(t) \tag{8}$$

which coincides with the result obtained in [8].
For $k > 1/2$, we apply the generalized Taylor's formula from equation (2)

$$\begin{aligned}\phi(t) &= \sum_{i=0}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) + \frac{({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha} \\ &= \sum_{i=0}^{m-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0) + \frac{({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha}\end{aligned}$$

where $m\alpha = 2k-1$ ($m \in Z^+$ and $0 < \alpha \leq 1/2$), to derive that

$$\begin{aligned}S_1 = \quad & (-1)^k\frac{2^k n^{3k}}{\pi^k}\sum_{i=0}^{m-1}\frac{({}_C\hat{D}_{0,t}^{i\alpha}\phi)(0)}{\Gamma(i\alpha+1)}\int_0^\infty\frac{t^{k+i\alpha}}{((nt)^2+1)^{2k}}dt \\ & +(-1)^k\frac{2^k n^{3k}}{\pi^k}\frac{({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{\Gamma(m\alpha+1)}\int_0^\infty\frac{t^{k+m\alpha}}{((nt)^2+1)^{2k}}dt \\ & +(-1)^k\frac{2^k n^{3k}}{\pi^k}\frac{1}{\Gamma((m+1)\alpha+1)}\int_0^\infty\frac{t^{k+(m+1)\alpha}}{((nt)^2+1)^{2k}}({}_C\hat{D}_{0,t}^{(m+1)\alpha}\phi)(\zeta)dt \\ & = S_{11}+S_{12}+S_{13}.\end{aligned}$$

Following the previous calculations above, we can show that

$$N-\lim_{n\to\infty}S_{11} = 0 \quad \text{and} \quad \lim_{n\to\infty}S_{13} = 0$$

by noting that the integral

$$\int_0^\infty\frac{y^{k+(m+1)\alpha}}{(y^2+1)^{2k}} = \int_0^\infty\frac{y^{3k-1+\alpha}}{(y^2+1)^{2k}}dy$$

converges for $k > 1/2$ and $0 < \alpha \leq 1/2$.
Coming to S_{12}, we use the following formula

$$\int_0^\infty\frac{y^{3k-1}}{(y^2+1)^{2k}}dy = \frac{\Gamma(3k/2)\Gamma(k/2)}{2\Gamma(2k)}$$

to imply that

$$\begin{aligned}N-\lim_{n\to\infty}S_1 = S_{12} &= (-1)^k\frac{2^k n^{3k}}{\pi^k}\frac{({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{\Gamma(m\alpha+1)}\int_0^\infty\frac{t^{k+m\alpha}}{((nt)^2+1)^{2k}}dt \\ &= (-1)^k\frac{2^k}{\pi^k}\frac{({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{\Gamma(2k)}\int_0^\infty\frac{y^{3k-1}}{(1+y^2)^{2k}}dy \\ &= (-1)^k\frac{2^k}{\pi^k}\frac{({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{\Gamma(2k)}\frac{\Gamma(3k/2)\Gamma(k/2)}{2\Gamma(2k)} \\ &= (-1)^k\frac{2^k\Gamma(3k/2)\Gamma(k/2)({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{2\Gamma^2(2k)\pi^k}.\end{aligned}$$

Similarly, we can infer that

$$N-\lim_{n\to\infty}S_2 = (-1)^{2k-1}\frac{2^k\Gamma(3k/2)\Gamma(k/2)({}_C\hat{D}_{0,t}^{m\alpha}\phi)(0)}{2\Gamma^2(2k)\pi^k}.$$

Hence

$$\begin{aligned}((\delta')^k(t),\phi(t)) &= \frac{(-1)^k+(-1)^{2k-1}}{2}\frac{2^k\Gamma(3k/2)\Gamma(k/2)({}_C\hat{D}^{m\alpha}_{0,t}\phi)(0)}{\Gamma^2(2k)\pi^k}\\ &= \frac{(-1)^k+(-1)^{2k-1}}{2}\frac{2^k\Gamma(3k/2)\Gamma(k/2)({}_C\hat{D}^{2k-1}_{0,t}\phi)(0)}{\Gamma^2(2k)\pi^k} \qquad (9)\end{aligned}$$

where $({}_C\hat{D}^{2k-1}_{0,t}\phi)(0) = ({}_C\hat{D}^{m\alpha}_{0,t}\phi)(0) = ({}_CD^{\alpha}_{0,t}\cdot {}_CD^{\alpha}_{0,t}\cdots {}_CD^{\alpha}_{0,t}\phi)(0)$ (m-times).
In particular for $k=1$, we get

$$((\delta')^1(t),\phi(t)) = \frac{-2\Gamma(3/2)}{\sqrt{\pi}}\phi'(0) = -\phi'(0) = (\delta'(t),\phi(t)).$$

This shows that

$$(\delta')^1(t) = \delta'(t).$$

Similarly, we deduce that for $k=1/2$

$$((\delta')^{1/2}(t),\phi(t)) = (i+1)\frac{\Gamma(3/4)\Gamma(1/4)}{\sqrt{2\pi}}\phi(0)$$

which infers that

$$(\delta')^{1/2}(t) = (i+1)\frac{\Gamma(3/4)\Gamma(1/4)}{\sqrt{2\pi}}\delta(t),$$

which is identical with equation (8).
It follows from equation (9) that

$$\begin{aligned}&(\delta')^{2l}(t) = 0 \ \text{ for } \ l=1,2,\cdots,\\ &(\delta')^{2l+1}(t) = \frac{2^{2l+1}\Gamma(l+1/2)\Gamma(3l+1+1/2)}{\pi^{2l+1}[(4l+1)!]^2}\delta^{(4l+1)}(t).\end{aligned}$$

In summary,
Theorem 5

$$\begin{aligned}&(\delta')^0(t) = 1,\\ &(\delta')^k(t) = 0 \ \text{ for } \ 1/3<k<1/2,\\ &(\delta')^{1/2}(t) = (i+1)\frac{\Gamma(3/4)\Gamma(1/4)}{\sqrt{2\pi}}\delta(t),\\ &((\delta')^k(t),\phi(t)) = \frac{(-1)^k+(-1)^{2k-1}}{2}\frac{2^k\Gamma(3k/2)\Gamma(k/2)({}_C\hat{D}^{2k-1}_{0,t}\phi)(0)}{\Gamma^2(2k)\pi^k} \ \text{ for } \ k>1/2\end{aligned}$$

where $({}_C\hat{D}^{2k-1}_{0,t}\phi)(0) = ({}_C\hat{D}^{m\alpha}_{0,t}\phi)(0) = ({}_CD^{\alpha}_{0,t}\cdot {}_CD^{\alpha}_{0,t}\cdots {}_CD^{\alpha}_{0,t}\phi)(0)$ (m-times). In particular,

$$\begin{aligned}&(\delta')^{2l}(t) = 0 \ \text{ for } \ l=1,2,\cdots, \ \text{ and}\\ &(\delta')^{2l+1}(t) = \frac{2^{2l+1}\Gamma(l+1/2)\Gamma(3l+1+1/2)}{\pi^{2l+1}[(4l+1)!]^2}\delta^{(4l+1)}(t) \ \text{ for } \ l=0,1,2,\cdots.\end{aligned}$$

Remark:
(i) We should note that Theorem 5 is a generalization of Theorem 2 in [21], where the case for $k\in Z^+$ is mainly considered.
(ii) Again, the choice of $\alpha\in(0,1/2]$ in Theorem 5 is not unique. We generally choose α and m in Theorem 5 to make $({}_C\hat{D}^{2k-1}_{0,t}\phi)(0)$ as simple as possible by Theorem 3.

3. Conclusion and acknowledgement

In this paper, we use a new delta sequence to define the distributions $\delta^k(x)$ and $(\delta')^k(x)$ by the generalized Taylor's formula and the neutrix limit, which have potential applications in elementary particle physics and quantum mechanics. This research is partially supported by BURC.

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