

**NON-STANDARD FINITE DIFFERENCE SCHEMES FOR
SOLVING FRACTIONAL ORDER HYPERBOLIC PARTIAL
DIFFERENTIAL EQUATIONS WITH RIESZ FRACTIONAL
DERIVATIVE**

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ABSTRACT. In this paper, the Mickens non-standard discretization method which effectively preserves the dynamical behavior of linear differential equations is adapted to solve numerically the fractional order hyperbolic partial differential equations. The fractional derivative is described in the Riesz sense. Special attention is given to study the stability analysis and the convergence of the proposed method. Numerical studies for the model problems are presented to confirm the accuracy and the effectiveness of the proposed method. The obtained results are compared with exact solutions and the standard finite difference method.

1. INTRODUCTION

Recently, fractional calculus has gained an increasing popularity due to the wide range of applications in fields including engineering, chemistry, finance, physics, seismology and so on ([2], [4], [6], and the references cited therein). In most cases, the solution of fractional differential equations (FDEs) cannot be obtained in terms of a finite number of elementary functions, it is therefore fundamental to devise numerical methods in order to practically evaluate approximate solutions by means of difference schemes or other alternative approaches ([1], [5], [9], [10], [16]-[22]). The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular.

The genesis of non-standard finite difference (NSFD) modeling procedures began with the 1989 publication of Mickens [11]. Extensions and a summary of the known results up to 1994 are given in Mickens [14]. We shall apply Mickens' non-standard method ([11]-[15]) to the fractional order differential equations, which is increasingly

2010 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.

Key words and phrases. Non-Standard finite difference schemes; Fractional hyperbolic partial differential equations; Riesz fractional derivative; Convergence and stability analysis.

Submitted April 21, 2015.

used to model problems in a number of research areas including dynamical systems, mechanical systems, signal processing, control, chaos, chaos synchronization and others. Some of these applications can be found in [3], [8] and the references cited therein.

The hyperbolic partial differential equations model the vibrations of structures (e.g. buildings, beams and machines) and are the basis for fundamental equations of atomic physics ([6], [17], [18], [22]).

In this work, we illustrate that the non-standard discretization is another numerical way to solve the fractional differential equations while preserving their crucial non-local property.

This paper is organized as follows: In section 2, some well-known mathematical preliminaries on fractional differential equations and non-standard rules are given, which will be used in our study. We apply the Mickens non-standard discretization scheme to the fractional order hyperbolic equation described in Riesz sense in section 3. In section 4, we study the convergence and the stability of the presented method. Numerical test examples are presented to show the efficiency of the method in section 5. In section 6, conclusion is given.

2. PRELIMINARIES AND NSFD RULES

In this section, some basic definitions and the main rules of the non-standard discretization methods are presented.

Definition 1 The Riemann-Liouville fractional derivative is defined as:

$${}_0\mathcal{D}_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad (1)$$

where $n-1 \leq \alpha < n$.

Definition 2 The Grünwald-Letnikov fractional derivative is defined as:

$${}_0\mathcal{D}_x^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{[x/h]} w_k^{(\alpha)} f(x-hk), \quad x \geq 0, \quad (2)$$

where $[x/h]$ means the integer part of x/h and $w_k^{(\alpha)}$ are the normalized Grünwald weights which are defined by $w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$.

Definition 3 The Riesz fractional derivative is defined as follows [7]:

$${}_x R^\alpha f(x, t) = \frac{-\sec(\frac{\alpha\pi}{2})}{\Gamma(2-\alpha)} \left(\rho \frac{d^2}{d\theta^2} \int_{X_a}^\theta \frac{f(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} + \sigma \frac{d^2}{d\theta^2} \int_\theta^{X_b} \frac{f(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x}, \quad (3)$$

where $1 < \alpha \leq 2$, $X_a \leq x \leq X_b$, $0 < t \leq T$, $\rho \geq 0$, $\sigma \geq 0$ and $\rho + \sigma = 1$.

NSFD Rules

In this part, we would like to introduce several comments related to NSFD schemes which were firstly proposed by Mickens [11]. This class of schemes and their formulations center on two issues. First, how should discrete representations for derivatives be determined, and second, what are the proper forms to be used for nonlinear terms.

The forward Euler method is one of the simplest discretization schemes. In this method the derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{h}$, where h is the step size. However, in the Mickens schemes this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h)}$, where $\phi(h)$ is a continuous function of step size h , and the function $\phi(h)$ satisfies the following conditions:

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \rightarrow 0.$$

Examples of functions $\phi(h)$ that satisfy these conditions are [13]:

$$\phi(h) = h, \quad \sinh h, \quad e^h - 1, \quad \frac{1-e^{-\lambda h}}{\lambda}, \quad \text{etc., } \dots$$

Note that in taking the $\lim h \rightarrow 0$ to obtain the derivative, the use of any of these $\phi(h)$ will lead to the usual result for the first derivative

$$\frac{dy}{dt} = \lim_{h \rightarrow 0} \frac{y[t + \phi_1(h)] - y(t)}{\phi_2(h)} = \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h},$$

where $\phi_1(h)$, $\phi_2(h)$ are continuous functions of step size h .

A scheme is called nonstandard if at least one of the following conditions is satisfied:

- 1- Nonlocal approximation is used.
- 2- Discretization of derivative is not traditional and use a nonnegative function.

In addition to this replacement, if there are nonlinear terms in the differential equation [13], these are replaced by

$$y^2(t) \rightarrow \begin{cases} y_n(t)y_{n+1}(t), \\ y_{n-1}(t)y_n(t). \end{cases}$$

In dimensions two and above, nonlinear terms such as $y(t)x(t)$ are either replaced by

$$y(t)x(t) \rightarrow \begin{cases} y_n(t)x_{n+1}(t), \\ y_{n+1}(t)x_n(t). \end{cases}$$

One can say that there is no appropriate general method to choose the function $\phi(h)$ or to choose which nonlinear terms are to be replaced, some special techniques may be found in [14].

3. DESCRIPTION OF NSFD SCHEME

Consider the fractional order hyperbolic partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = {}_x R^\alpha u(x, t) + f(u, x, t), \quad a < x < b, \quad 0 < t < T, \quad (4)$$

with initial conditions

$$u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad a < x < b, \quad (5)$$

and boundary conditions

$$u(a, t) = d_1(t), \quad u(b, t) = d_2(t), \quad 0 < t < T, \quad (6)$$

where $1 < \alpha \leq 2$.

We assume that the function $f(u, x, t)$ is nonlinear and satisfies the Lipschitz condition i.e.,

$$|f(u_1, x, t) - f(u_2, x, t)| \leq L |u_1 - u_2|,$$

Then by using the generalized of the Riesz fractional derivative we can rewrite equation (4) in the following form

$$\frac{\partial^2 u}{\partial t^2} = \frac{A}{\Gamma(2-\alpha)} \left(\rho \frac{d^2}{d\theta^2} \int_a^\theta \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} + \sigma \frac{d^2}{d\theta^2} \int_\theta^b \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x} + f(u, x, t), \quad (7)$$

where $A = -\sec(\frac{\alpha\pi}{2})$.

In the following, the NSFD notions and the shifted Grünwald formula are introduced. First, the relationship between the Grünwald – Letnikov and Riemann – Liouville fractional derivatives [8] to approximate the integrals in Riesz fractional derivative definition is given as follows:

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_a^\theta \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x} = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^j \binom{\alpha}{j} u(x-jh, t), \quad (8)$$

and

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_\theta^b \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x} = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{\lfloor \frac{b-x}{h} \rfloor} (-1)^j \binom{\alpha}{j} u(x+jh, t). \quad (9)$$

For the numerical approximation of equations (8) and (9), pick an integers $m, T > 0$, define the space step size $h = \frac{b-a}{m}$, and select a time step size τ . Now, define $x_m = mh$, $t_n = n\tau$, where $N = T/\tau$ and $u_m^n = u(x_m, t_n)$, $f_m^n = f(u_m^n, x_m, t_n)$, for $m = 0, 1, 2, \dots, N$ and $n = 0, 1, 2, \dots, N$.

From the shifted Grünwald formula [16] for the α -fractional derivative approximation, we can rewrite (8) and (9) in the following forms

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_a^\theta \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x} \approx h^{-\alpha} \sum_{k=0}^{m+1} \omega_{m+1-k}^\alpha u(x_m, t_n), \quad (10)$$

and

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_\theta^b \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x} \approx h^{-\alpha} \sum_{k=m-1}^N \omega_{k-(m-1)}^\alpha u(x_m, t_n). \quad (11)$$

where $\omega_k^\alpha = (-1)^k \binom{\alpha}{k}$, $k = 0, 1, \dots$, or recursively

$$\omega_0^\alpha = h^{-\alpha} \quad \text{and} \quad \omega_k^\alpha = \left[1 - \frac{1+\alpha}{k} \right] \omega_{k-1}^\alpha, \quad k = 0, 1, \dots \quad (12)$$

Now, we apply the Mickens discretization scheme to the equation (4) by replacing the step size h by a function of h say $\phi(h)$ and the step size τ by a function of τ say $\psi(\tau)$. Substituting (10), (11) into (4), and using NSFD we get:

$$\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{(\psi(\tau))^2} = \rho A (\phi(h))^{-\alpha} \sum_{k=0}^{m+1} \omega_{m+1-k}^\alpha u_m^n + \sigma A (\phi(h))^{-\alpha} \sum_{k=m-1}^N \omega_{k-(m-1)}^\alpha u_m^n + f_m^n, \quad (13)$$

where $m = 1, \dots, N-1$, $n = 1, \dots, N-1$, $\phi(h)$ and $\psi(\tau)$ have the properties [13]:

$$\psi(\tau) = \tau + O(\tau^2) \quad \text{and} \quad \phi(h) = h + O(h^2).$$

Put $r = \frac{(\psi(\tau))^2}{(\phi(h))^\alpha}$ in(13), then we claim:

$$u_m^{n+1} = 2u_m^n - u_m^{n-1} + \rho Ar \sum_{k=0}^{m+1} \omega_{m+1-k}^\alpha u_m^n + \sigma Ar \sum_{k=m-1}^N \omega_{k-(m-1)}^\alpha u_m^n + (\psi(\tau))^2 f_m^n. \quad (14)$$

Now, we must choose a suitable $\psi(\tau)$ and $\phi(h)$ to ensure that the discrete representation in (14) converges to the corresponding continuous derivative as $\tau \rightarrow 0$ and $h \rightarrow 0$. Among the various denominator functions, we can take into account the following $\psi(\tau) = e^{\tau^2} - 1$ and $\phi(h) = e^h - 1$.

let $U^n = (u_{m-1}^n, u_{m-2}^n, \dots, u_1^n)^T$, and

$$B^n = \begin{pmatrix} \rho Ar \omega_0^\alpha u_m^n + \rho Ar \omega_m^\alpha u_0^n \\ \rho Ar \omega_{m-1}^\alpha u_0^n \\ \vdots \\ \rho Ar \omega_3^\alpha u_0^n \\ \rho Ar \omega_2^\alpha u_0^n \end{pmatrix} + \begin{pmatrix} \sigma Ar \omega_2^\alpha u_m^n \\ \sigma Ar \omega_3^\alpha u_m^n \\ \vdots \\ \sigma Ar \omega_{m-1}^\alpha u_m^n \\ \sigma Ar \omega_m^\alpha u_m^n + \sigma Ar \omega_0^\alpha u_0^n \end{pmatrix},$$

$$F^n = ((\psi(\tau))^2 f(u_{m-1}^{n-1}, x_{m-1}, t_n), \dots, ((\psi(\tau))^2 f(u_1^{n-1}, x_1, t_n))^T,$$

$$P^n = \rho Ar \begin{pmatrix} \omega_1^\alpha & \omega_2^\alpha & \omega_3^\alpha & \dots \\ \omega_0^\alpha & \omega_1^\alpha & \omega_2^\alpha & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \omega_0^\alpha & \omega_1^\alpha \end{pmatrix} + \sigma Ar \begin{pmatrix} \omega_1^\alpha & \omega_0^\alpha & 0 & \dots \\ \omega_2^\alpha & \omega_1^\alpha & \omega_0^\alpha & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega_3^\alpha & \omega_2^\alpha & \omega_1^\alpha \end{pmatrix} + 2I_{m-1}, \quad (15)$$

where I_{m-1} is The unit matrix of order $m - 1$.

From (2), we have $u_m^0 = g_1(x_m)$ and $\frac{u_m^1 - u_m^0}{\varphi(\tau)} = g_2(x_m)$, for $m = 0, 1, \dots, N$. System (15) can be written in the following matrix form $U^1 = \varphi(\tau)g_2(x_m) + U^0$, and for $n > 1$, then we claim:

$$U^{n+1} = P^n U^n - U^{n-1} + B^n + F^n. \quad (16)$$

The following lemmas will be used to derive the stability condition of the scheme (16).

Lemma 1 The procedures that used to replace the integrals in Riesz definition by finite sums in equations (10), (11) of order $O(h)$:

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_a^\theta \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x_m} = h^{-\alpha} \sum_{k=0}^{m+1} \omega_{m+1-k}^\alpha u(x_m, t_n) + O(h),$$

and

$$\frac{1}{\Gamma(2-\alpha)} \left(\frac{d^2}{d\theta^2} \int_\theta^b \frac{u(\zeta, t) d\zeta}{(\theta-\zeta)^{\alpha-1}} \right)_{\theta=x_{m+1}} = h^{-\alpha} \sum_{k=m-1}^N \omega_{k-(m-1)}^\alpha u(x_m, t_n) + O(h).$$

Proof. See for example [7].

Lemma 2 The coefficients $\omega_k^\alpha = (-1)^k \binom{\alpha}{k}$, for $1 < \alpha < 2$, and $k = 0, 1, \dots$, satisfy the following conditions:

$$(1) \omega_0^\alpha = 1.$$

$$(2) \omega_1^\alpha < 0 \text{ and } \omega_k^\alpha > 0 \text{ for any } k > 1.$$

$$(3) \omega_1^\alpha + \omega_2^\alpha + \dots = -\omega_0^\alpha.$$

Proof. See for example [7].

4. CONVERGENCE AND STABILITY OF THE METHOD

4.1. The Convergence of the Method

Let us denote by $\bar{U}^n = (\bar{u}_{m-1}^n, \bar{u}_{m-2}^n, \dots, \bar{u}_1^n)^T$, the exact solution of the system (16) and the error vector at level $t = t_{n+1}$ is denoted by $e^{n+1} = \bar{U}^{n+1} - U^{n+1}$, $e^0 = 0$, assume that $\bar{A} \geq A$ and $\bar{\alpha} \geq \alpha$.

Theorem 1 The system (16) is convergent and $|u_m^n - \bar{u}_m^n| = O(O(\psi) + O(\phi))$ for any m, n where

$$(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho + \sigma)}.$$

Proof. From (16), we have $e^{n+1} = P^n e^n - e^{n-1} + F_e^n + (O(\psi) + O(\phi))$, where

$$\begin{aligned} F_e^n &= (|((\psi(\tau))^2 f(u_{m-1}^n, x_{m-1}, t_n) - (\psi(\tau))^2 f(\bar{u}_{m-1}^n, x_{m-1}, t_n)|, \dots, |((\psi(\tau))^2 f(u_1^n, x_1, t_n) - \\ &\quad ((\psi(\tau))^2 f(\bar{u}_1^n, x_1, t_n)|)^T \\ &\leq ((\psi(\tau))^2 L_{m-1}^n e_{m-1}^n, \dots, (\psi(\tau))^2 L_1^n e_1^n)^T = \Delta F^n e^n, \\ &\text{when } \Delta F^n = \text{diag}(((\psi(\tau))^2 L_{m-1}^n, \dots, (\psi(\tau))^2 L_1^n)^T). \end{aligned}$$

Noting that $|L_m^n| \leq L$, for any m, n . Then

$$\|P^n\|_\infty = \max_{1 \leq m \leq N-1} \left(|\sigma Ar \omega_{N-(m-1)-1}^\alpha| + \dots + |\sigma Ar \omega_3^\alpha| + |\rho Ar \omega_0^\alpha + \sigma Ar \omega_2^\alpha| + |\rho Ar \omega_1^\alpha + \sigma Ar \omega_1^\alpha + 2| + |\rho Ar \omega_2^\alpha + \sigma Ar \omega_0^\alpha| + |\rho Ar \omega_3^\alpha| + \dots + |\rho Ar \omega_m^\alpha| \right).$$

For $\phi(h)$ given, we choose $\psi(\tau)$ to satisfy the following

$$\rho Ar \omega_1^\alpha + \sigma Ar \omega_1^\alpha + 2 \geq 0$$

$$\text{i.e., } (\psi(\tau))^2 \leq -2/((\rho Ar \omega_1^\alpha)/(\phi(h))^\alpha + (\sigma Ar \omega_1^\alpha)/(\phi(h))^\alpha). \quad \text{for any } m, n.$$

$$\text{From } \frac{2(\phi(h))^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho + \sigma)} \geq -2/((\rho Ar \omega_1^\alpha)/(\phi(h))^\alpha + (\sigma Ar \omega_1^\alpha)/(\phi(h))^\alpha),$$

$$\text{we have the condition } (\psi(\tau))^2 \leq -2/((\rho Ar \omega_1^\alpha)/(\phi(h))^\alpha + (\sigma Ar \omega_1^\alpha)/(\phi(h))^\alpha),$$

holds when $(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho + \sigma)}$. Under this condition, we claim

$$\|P^n\|_\infty = 2 + \rho Ar \sum_{m=0}^{k+1} \omega_m^\alpha + \sigma Ar \sum_{m=0}^{N-(k-1)} \omega_m^\alpha,$$

$$\|P^n\|_\infty = 2 + \rho Ar \sum_{m=k+2}^{\infty} \omega_m^\alpha + \sigma Ar \sum_{m=N-(k-2)}^{\infty} \omega_m^\alpha \leq 2.$$

This is true, by using lemma 2, property 3 of ω_m^α and since $\sum_{m=k+2}^{\infty} \omega_m^\alpha, \sum_{m=N-(k-2)}^{\infty} \omega_m^\alpha \leq 0$. So when $(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho + \sigma)}$, and for any constant $C > 0$ independent of $\phi(h)$, $\psi(\tau)$, we obtain

$$\begin{aligned} \|e^{n+1}\|_\infty &\leq \|P^n + \Delta F^n\|_\infty \|e^n\|_\infty + \|e^{n-1}\|_\infty + C((\psi(\tau))^2 + \phi(h)) \\ &\leq (2 + (\psi(\tau))^2 L) \|e^n\|_\infty + \|e^{n-1}\|_\infty + C((\psi(\tau))^2 + \phi(h)), \end{aligned}$$

let $s_1 = (2 + (\psi(\tau))^2 L)$, and $s_2 = C((\psi(\tau))^2 + \phi(h))$, then

$$\begin{aligned} \|e^{n+1}\|_\infty &\leq s_1 \|e^n\|_\infty + \|e^{n-1}\|_\infty + s_2 \\ &\leq s_1 (s_1 \|e^{n-1}\|_\infty + \|e^{n-2}\|_\infty + s_2) + \|e^{n-1}\|_\infty + s_2 \\ &= (s_1^2 + 1) \|e^{n-1}\|_\infty + s_1 \|e^{n-2}\|_\infty + s_2 (s_1 + 1) \\ &\leq (s_1^2 + 1) (s_1 \|e^{n-2}\|_\infty + \|e^{n-3}\|_\infty + s_2) + s_1 \|e^{n-2}\|_\infty + s_2 (s_1 + 1) \\ &= (s_1^3 + 2s_1) \|e^{n-2}\|_\infty + (s_1^2 + 1) \|e^{n-3}\|_\infty + s_2 (s_1^2 + s_1 + 2) \\ &\leq \dots \leq C_k \|e^{n-k}\|_m + C_{k-1} \|e^{n-k-1}\|_m + s_2 (C_{k-2} + C_{k-3} + C_{k-4} + C_{k-5}), \end{aligned}$$

such that $C_k = s_1 C_{k-1} + C_{k-2}$, $k = 2, 3, \dots, N$, $C_0 = 1$, $C_1 = s_1$.

Then we have

$$\begin{aligned} \|e^{n+1}\|_\infty &\leq C_{N-1} \|e^1\|_m + C_{N-2} \|e^0\|_m + s_2 (C_{N-2} + C_{N-3} + C_{N-4} + C_{N-5}) \\ &\leq C_{N-1} \|e^0\|_m + C_{N-2} \|e^0\|_m + s_2 (C_{N-2} + C_{N-3} + C_{N-4} + C_{N-5}) \\ &\leq C \|e^0\|_m + O((\psi(\tau))^2 + \phi(h)) \approx C \|e^0\|_m + (O(\psi) + O(\phi)). \end{aligned}$$

Since, from the second initial condition we have $\|e^1\|_m \leq \|e^0\|_m$ and $C = C_{N-1} + C_{N-2}$.

4.2. The Stability of the Method

Let W^{n+1} and U^{n+1} be two different numerical solutions of (16) with initial values given by W^0 and U^0 , respectively.

Theorem 2 If $(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{A\bar{\alpha}(\rho + \sigma)}$, the the system (16) is stable and $|W^{n+1} - U^{n+1}| \leq C_1 |W^0 - U^0|$ for any n , and for any constant $C_1 > 0$, independent of $\phi(h)$, $\psi(\tau)$.

Proof. Let $W^{n+1} - U^{n+1} = \varepsilon^{n+1}$, from (16) we have $\varepsilon^{n+1} = P^n \varepsilon^n - \varepsilon^{n-1} + F_\varepsilon^n$, where

$$\begin{aligned} F_\varepsilon^n &= ((\psi(\tau))^2 f(u_{m-1}^n, x_{m-1}, t_n) - (\psi(\tau))^2 f(\omega_{m-1}^n, x_{m-1}, t_n), \dots, |(\psi(\tau))^2 f(u_1^n, x_1, t_n) - \\ &\quad ((\psi(\tau))^2 f(\omega_1^n, x_1, t_n))|^T \\ &\leq ((\psi(\tau))^2 L_{m-1}^n \varepsilon_{m-1}^n, \dots, (\psi(\tau))^2 L_1^n \varepsilon_1^n)^T = \Delta F^n \varepsilon^n. \end{aligned}$$

From the convergence theorem, we have $\|P^n + \Delta F^n\|_\infty \leq (2 + (\psi(\tau))^2 L)$, when

$$(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{A\bar{\alpha}(\rho + \sigma)},$$

then $\|\varepsilon^{n+1}\|_\infty \leq \|P^n + \Delta F^n\|_\infty \|\varepsilon^n\|_\infty + \|\varepsilon^{n-1}\|_\infty \leq (2 + (\psi(\tau))^2 L) \|\varepsilon^n\|_\infty + \|\varepsilon^{n-1}\|_\infty$.

Let $s = (2 + (\psi(\tau))^2 L)$, then

$$\begin{aligned} \|\varepsilon^{n+1}\|_\infty &\leq s \|\varepsilon^n\|_\infty + \|\varepsilon^{n-1}\|_\infty \\ &\leq s (s \|\varepsilon^{n-1}\|_\infty + \|\varepsilon^{n-2}\|_\infty) + \|\varepsilon^{n-1}\|_\infty \\ &= (s^2 + 1) \|\varepsilon^{n-1}\|_\infty + s \|\varepsilon^{n-2}\|_\infty \\ &\leq (s^2 + 1) (s \|\varepsilon^{n-2}\|_\infty + \|\varepsilon^{n-3}\|_\infty) + s \|\varepsilon^{n-2}\|_\infty \\ &= (s^3 + 2s) \|\varepsilon^{n-2}\|_\infty + (s^2 + 1) \|\varepsilon^{n-3}\|_\infty \\ &\leq \dots \leq C_k \|\varepsilon^{n-k}\|_m + C_{k-1} \|\varepsilon^{n-k-1}\|_m, \end{aligned}$$

such that $C_k = s C_{k-1} + C_k$, $k = 2, 3, \dots, N$, $C_0 = 1$, $C_1 = s$.

Then $\|\varepsilon^{n+1}\|_\infty \leq C_{N-1} \|\varepsilon^1\|_m + C_{N-2} \|\varepsilon^0\|_m \leq C_{N-1} \|\varepsilon^0\|_m + C_{N-2} \|\varepsilon^0\|_m \leq C \|\varepsilon^0\|_m$.

Since, from the second initial condition we have $\|\varepsilon^1\|_m \leq \|\varepsilon^0\|_m$, and $C = C_{N-1} + C_{N-2}$.

Therefore, for the system (16), if there is a perturbation in U^0 , the small change would not cause large error in the numerical solution. Then the system (16) is stable when

$$(\psi(\tau))^2 \leq \frac{2(\phi(h))^{\bar{\alpha}}}{A\bar{\alpha}(\rho + \sigma)}.$$

5. NUMERICAL EXAMPLES

Example 5.1. Consider the following fractional order partial differential equation.

$$\frac{\partial^2 u(x, t)}{\partial t^2} = {}_x R^\alpha u(x, t), \quad 0 < x < 1, \quad 0 < t \leq 1, \quad 1 < \alpha \leq 2,$$

the initial and boundary conditions are:

$$u(x, 0) = \sin(2\pi x), \quad u_t(x, 0) = 2\pi \sin(2\pi x), \quad u(0, t) = u(1, t) = 0.$$

Let $\rho = 1$, $\sigma = 0$, $\psi(\tau) = \sinh(\tau^2)$ and $\phi(h) = e^h - 1$.

When $\alpha = 2$, the exact solution is: $u(x, t) = \sin 2\pi x (\cos 2\pi t + \sin 2\pi t)$.

In table 1, the error between integer order and fractional order solutions at $\alpha = 1.8$ are given where $h = 0.05$ and $\tau = 0.0025$ where the maximum error is 0.007. Figure (1), shows the behavior of the exact and the NSFD solutions with $h = 0.05$ and $\tau = 0.0025$. To study the behaviour of the solutions figures (2-5), show the 3D solutions when $\alpha = 2$, $\alpha = 1.8$, $\alpha = 1.6$ and $\alpha = 1.4$, respectively. Figures (6) and (7) show the unstable solutions behaviour when $h = 0.157$ and $\tau = 0.001$, where the value of $\psi(\tau)$ is larger than the stability bound.

TABLE 1. The error between the integer u_{int} and the fractional u_{frac} solutions when $h = 0.05$ and $\tau = 0.0025$.

x	$u_{int} - u_{frac}$
0.0500	0.00426472
0.1000	0.00556466
0.1500	0.00678550
0.2000	0.00747101
0.2500	0.00748019
0.3000	0.00678600
0.3500	0.00544468
0.4000	0.00358144
0.4500	0.00137521
0.5000	0.00096021
0.5500	0.00319760
0.6000	0.00511890
0.6500	0.00653671
0.7000	0.00731273
0.7500	0.00737136
0.8000	0.00670714
0.8500	0.00538530
0.9000	0.00353540
0.9500	0.00133866
1.000	0.00000000

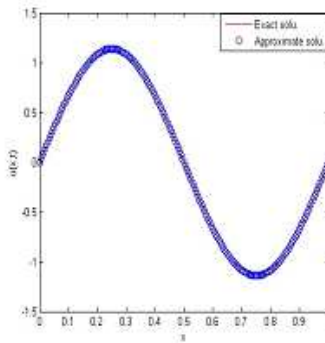


FIGURE 1. Comparison between the exact and the NSFD solutions with $h = 0.05$ and $\tau = 0.0025$.

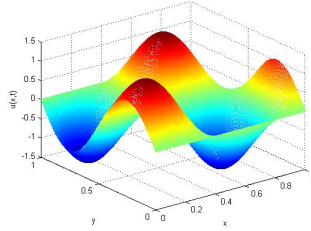


FIGURE 2. 3D-solutions when $\alpha = 2$ with $h = 0.05$ and $\tau = 0.0025$.

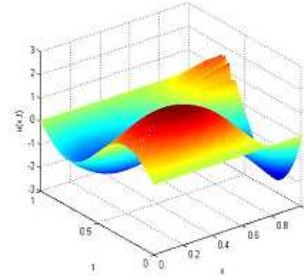


FIGURE 3. 3D-solutions when $\alpha = 1.8$ with $h = 0.05$ and $\tau = 0.0025$.

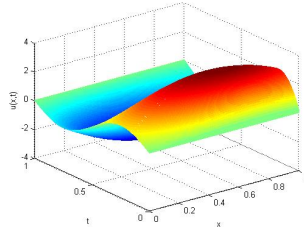


FIGURE 4. 3D-solutions when $\alpha = 1.6$ with $h = 0.05$ and $\tau = 0.0025$.

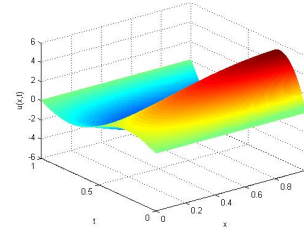


FIGURE 5. 3D-solutions when $\alpha = 1.4$ with $h = 0.05$ and $\tau = 0.0025$.

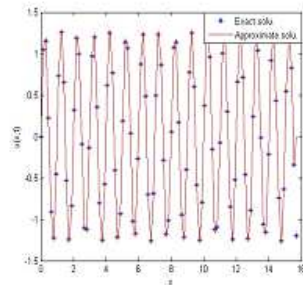


FIGURE 6. NSFD unstable solutions when $h = 0.157$ and $\tau = 0.001$.

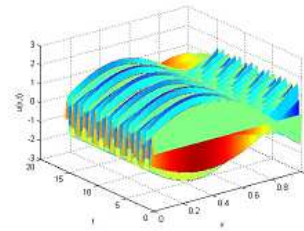


FIGURE 7. 3D-NSFD unstable solutions when $h = 0.157$ and $\tau = 0.001$.

Example 5.2. Consider the one-dimensional fractional hyperbolic partial differential equation.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = {}_x R^\alpha u(x,t) \quad , \quad 0 < x < 5 \quad , \quad 0 < t \leq 1 \quad , \quad 1 < \alpha \leq 2,$$

the initial and boundary conditions are:

$$u(x,0) = \sin x, \quad u_t(x,0) = 0, \quad u(0,t) = 0, \quad u(5,t) = \sin(5) \cos(t),$$

Let $\rho = 1$, $\sigma = 0$, $\psi(\tau) = e^{\tau^2} - 1$ and $\phi(h) = e^h - 1$.

When $\alpha = 2$, the exact solution is: $u(x,t) = \sin x \cos t$.

The numerical studies for the above model problem can be presented as follows: In table 2, the error between the integer order and the fractional order solutions at $\alpha = 1.6$ are given, where $h = 0.2$ and $\tau = 0.001$ where the maximum error is 0.000006. In order to test the numerical scheme, we describe in figure (8) the analytical and approximate solutions for $h = 0.2$, $\tau = 0.001$ and $\alpha = 1.6$. To study the behaviour of the solutions figure (9) shows the 3D solutions for $\alpha = 1.6$. Figures(10) and(11) show the unstable solutions behaviour when $h = 0.008$ and $\tau = 0.01$.

TABLE 2. The error between the integer and the fractional solutions when $h = 0.2$, $\tau = 0.001$ and $\alpha = 1.6$.

x	$u_{int} - u_{frac}$
0.2000	0.00000479
0.4000	0.00000475
0.6000	0.00000520
0.8000	0.00000564
1.0000	0.00000595
1.2000	0.00000608
1.4000	0.00000600
1.6000	0.00000570
1.8000	0.00000518
2.0000	0.00000448
2.2000	0.00000360
2.4000	0.00000258
2.6000	0.00000147
2.8000	0.00000030
3.0000	0.00000087
3.2000	0.00000201
3.4000	0.00000307
3.6000	0.00000400
3.8000	0.00000477
4.0000	0.00000534

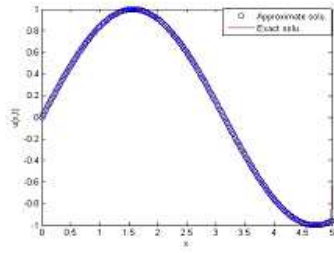


FIGURE 8. Comparison between the analytical and the NSFD solutions at $h = 0.2$, $\tau = 0.001$ and $\alpha = 1.6$.

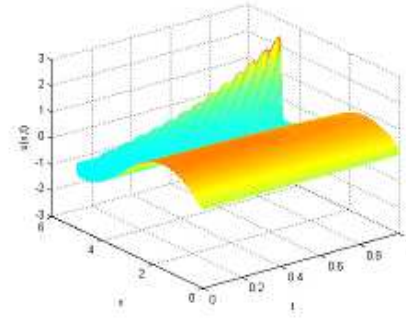


FIGURE 9. 3D-solutions with $h = 0.2$ and $\tau = 0.001$.

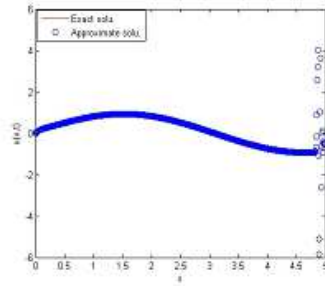


FIGURE 10. NSFD unstable solutions when $h = 0.008$ and $\tau = 0.01$.

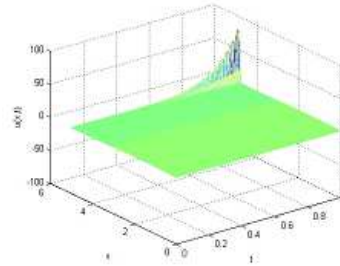


FIGURE 11. 3D- EFDA unstable solutions when $h = 0.008$ and $\tau = 0.01$.

Example 5.3. Consider the following fractional hyperbolic partial differential equation.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = {}_x R^\alpha u(x,t) + f(u,x,t) \quad 0 < x < 2, \quad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

where $f(u,x,t) = -u - 2\sin(t)$,

the initial and boundary conditions are: $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = x^2$, $u(0,t) = u(2,t) = 4\sin(t)$.

Let $\rho = 1$, $\sigma = 0$, $\psi(\tau) = \sinh(\tau^2)$ and $\phi(h) = e^h - 1$.

When $\alpha = 2$, the exact solution is: $u(x,t) = x^2 \sin(t)$.

In table 3, the exact solutions at $\alpha = 2$, NSFD solutions at $\alpha = 1.7$, and the error between the two results are given where the maximum error is 0.00004, with $h = 0.2$, $\tau = 0.002$ and $\alpha = 1.7$. In order to test the numerical scheme, we describe in figure (12) the analytical and approximate solutions for $\alpha = 1.7$, $h = 0.2$ and $\tau = 0.002$. To study the behaviour of the solutions figure (13), shows the 3D solutions.

TABLE 3. The exact and the NSFD solutions.

x_i	$\alpha = 2$	$\alpha = 1.7$	$u_{int} - u_{frac}$
0.2000	0.00151963	0.00152611	0.00000648
0.4000	0.00607854	0.00608749	0.00000896
0.6000	0.01367671	0.01368845	0.00001174
0.8000	0.02431415	0.02432921	0.00001506
1.0000	0.03799086	0.03800984	0.00001898
1.2000	0.05470683	0.05473039	0.00002356
1.4000	0.07446208	0.07449088	0.00002880
1.6000	0.09725659	0.09729132	0.00003473
1.8000	0.12309037	0.12313173	0.00004136
2.0000	0.15196342	0.15196342	0.00000000

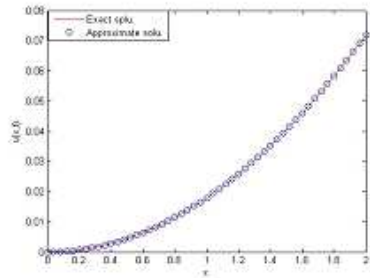


FIGURE 12. Comparison between the analytical and the NSFD when $h = 0.2$ and $\tau = 0.002$.

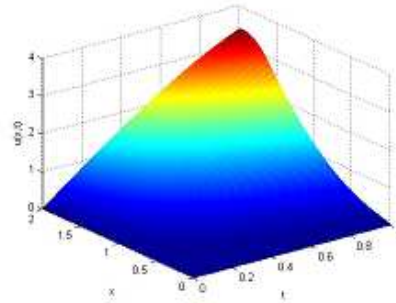


FIGURE 13. 3D-solutions with $h = 0.2$ and $\tau = 0.002$.

Example 5.4. Consider the following fractional hyperbolic partial differential equation.

$$\frac{\partial^2 u(x, t)}{\partial t^2} = -0.5 \cos(\alpha\pi/2) {}_x R^\alpha u(x, t) + f(u, x, t), \quad 1 < \alpha \leq 2,$$

with $\alpha = 1.5$, where $f(u, x, t) = \frac{2u}{t^2+1} - (t^2 + 1)\left(\frac{16x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{6x^{3-\alpha}}{\Gamma(4-\alpha)}\right)$, the initial and boundary conditions are:

$$u(x, 0) = x^2(8 - x), \quad u_t(x, 0) = 0 \quad \text{and} \quad u(0, t) = u(8, t) = 0, \quad \text{where} \quad 0 \leq x \leq 8, \quad T = 1.$$

Let $\rho = 1$, $\sigma = 0$, $\psi(\tau) = e^{\tau^2} - 1$ and $\phi(h) = e^h - 1$. When $\alpha = 2$, the exact solution is: $u(x, t) = x^2(8 - x)(t^2 + 1)$.

In the following, a comparison between NSFD and the standard finite difference (SFD) solutions with $\alpha = 1.5$, $h = 0.04$, $T = 1$, and $\tau = 0.005$, are presented in table 4. Figure (14) shows the behavior of the numerical and the exact solutions and figure (15) shows 3D-simulation of the numerical solutions, where the maximum error using the NSFD scheme is 1×10^{-3} . The results shows that NSFD gives good results than SFD where the maximum error of the SFD is 1×10^{-2} . Table 5 shows that the NSFD is more accurate.

TABLE 4. Comparison between NSFD and SFD solutions when $h = 0.04$ and $T = 1$.

x	NSFD	SFD
0.0000	0.00000000	0.00000000
0.4000	0.00003040	0.00023054
0.8000	0.00011520	0.00101979
1.2000	0.00024480	0.00220045
1.6000	0.00040960	0.00369367
2.0000	0.00060000	0.00541511
2.4000	0.00080640	0.00727923
2.8000	0.00101920	0.00920005
3.2000	0.00122880	0.01109142
3.6000	0.00142560	0.01286710
4.0000	0.00160000	0.01444079
4.4000	0.00174240	0.01572615
4.8000	0.00184320	0.01663686
5.2000	0.00189280	0.01708654
5.6000	0.00188160	0.01698883
6.0000	0.00180000	0.01625734
6.4000	0.00163840	0.01480572
6.8000	0.00138720	0.01254756
7.2000	0.00103680	0.00939648
7.6000	0.00057760	0.00526609
8.0000	0.00000000	0.00000000

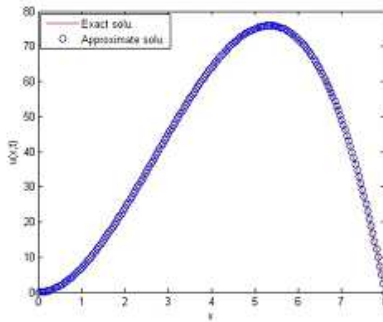


FIGURE 14. The exact solution and the NSFD solution $h = 0.4$ and $T = 1$.

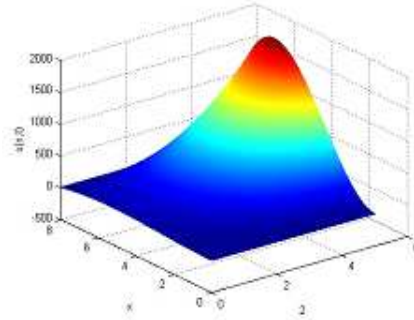


FIGURE 15. The exact solution and the NSFD solution in 3D.

Remark: If we choose $\phi(h)$ and $\psi(\tau)$ such that $(\psi(\tau))^2 > \frac{2h^\alpha(\phi(h))^\alpha}{A\bar{\alpha}(\rho+\sigma)}$, i.e., the stability condition is not satisfied, the behavior of the solution is given in figures (16) and (17), clear that the numerical solution is unstable. In addition, NSFD-algorithm is in general more stable than FD-algorithm.

TABLE 5. Comparison between NSFD and SFD.

N	SFD	NSFD
200	convergent	convergent
375	convergent	convergent
500	convergent	convergent
750	divergent	convergent
τ	SFD	NSFD
0.005	convergent	convergent
0.05	divergent	convergent
0.5	failed	convergent

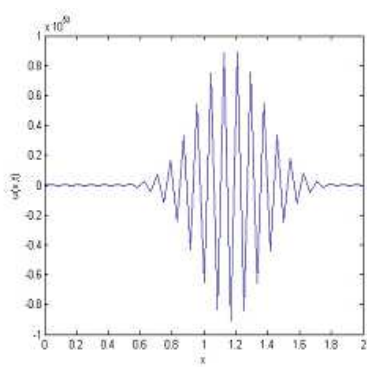


FIGURE 16. The numerical solution is unstable.

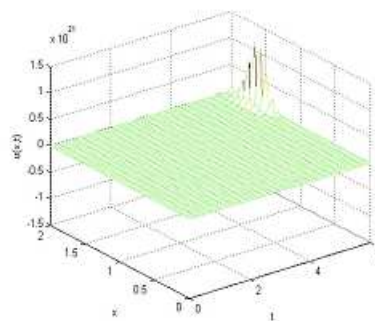


FIGURE 17. The numerical solution is unstable in 3D.

6. CONCLUSIONS

In this paper, we have used NSFD with Riesz fractional definition to construct simple and high accuracy algorithms for solving the linear fractional hyperbolic partial differential equations. The stability and the convergence of the method are proved. Numerical test examples are given and the results obtained by the method are compared with the exact solutions. In example 3, the results obtained by the method are compared with the SFD. The comparison shows that NSFD is more accurate than SFD. Summarizing these results, we can say that NSFD in its general form gives a reasonable calculations, easy to use and can be applied for the linear fractional differential equations. All results obtained by using MATLAB version R2013b.

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