Journal of Fractional Calculus and Applications,
Vol. 7(1) Jan. 2016, pp. 103-106.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

A NOTE ON THE RELATIONSHIP BETWEEN THE HAUSDORFF DIMENSION AND THE ORDER OF GRÜNWALD'S DEFINITION FOR FRACTIONAL CALCULUS

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Abstract. Given an iterated function system over a function of real n-dimensional space, we establish a relationship between the Hausdorff dimension s and the Grünwald's definition of order q for fractional calculus.

$$s = 1 + \sum_k \frac{1}{q_k}$$

1. Introduction

In this paper, we introduce a connection between the Hausdorff dimension s and the order q of Grünwald's definition for fractional calculus. We extend our previous work from [6] over a fractal domain of real n-dimensional space.

2. Preliminaries

We present basic definitions and results on fractional calculus from Agarwal [1], Grünwald ([4]) and Oldham and Spannier ([8]); on fractals from Edgar ([2]), Falconer ([3]) and Hutchinson ([7]); and on general measure theory from Halmos ([5]) and Mattila ([9]).

Definition 1 A (finite) *partition* π of the interval $[a, b]$ is a finite collection of points $\{x_0, x_1, \dots, x_{N-1}\}$, called partition points, such that $a = x_0 < x_1, \dots, < x_{N-1} = b$. The length of the interval $[x_{j-1}, x_j]$ is denoted by $\Delta x_j = x_j - x_{j-1}$. The collection of all (finite) partitions of the interval $[a, b]$ is denoted by $\Pi[a, b]$.

Definition 2 A partition π' is called a *refinement* of the partition π if every partition point of $x_j \in \pi$ also belongs to π'.

Remark The process of integration or differentiation, denoted by $D_x^q f(x)$, to any order $q \in \mathbb{R}$ with respect to x of the function $f : [a, b] \to \mathbb{R}$ is given in [2] by Grünwald. If $q < 0$, $q = 0$ or $q > 0$ then we say that the process is a differentiation, the identity map or an integration, respectively. When $q = 1$, the Grünwald's definition reduces to the ordinary integral of Riemann.

2010 *Mathematics Subject Classification.* 26A33, 28A80, 26A03, 28A78, 28A35.
Key words and phrases. Hausdorff dimension, Grünwald definition, fractional calculus, order.
Submitted June 19, 2015.

Definition 3 (*Grünwald definition*) Let $f : [a,b] \to \mathbb{R}$ is a continuous function on the interval closed $[a,b] \subset \mathbb{R}$. The derivative or integration to order q is given by

$$D_x^{-q} f(x) = \lim_{N\to\infty} \left\{ \frac{1}{\Gamma(q)} \sum_{k=0}^{N-1} \frac{\Gamma(k+q)}{\Gamma(k+1)} f\left(x_{N-k}^*\right) \Delta^q x_{N-k} \right\}, \tag{1}$$

where we denote the gamma function at $x \in \mathbb{R}$ by $\Gamma(x)$.

Definition 4 A map $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contractive map* if there exist an r, called the *contraction ratio* with $0 < r < 1$, such that for all $x, y \in X$, $|S(x) - S(y)| = r|x-y|$.

Definition 5 Let I be an index set, possibly infinite. The collection of contractive maps $\{S_k : X \to X | k \in I\}$ on a closed interval $X \subset \mathbb{R}$ is called an *iterated function system* or IFS.

Definition 6 The IFS $\{S_k\}$ is said to satisfy the *open set condition* iff there exists a nonempty open set U for which we have $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$ and $U \supseteq S_i(U)$ for all i.

Definition 7 Let $\{S_k\}$ be an IFS. We denote a list of contraction ratios by $(r_1, r_2, \ldots, r_N)$. If $\sum_k r_k^s = 1$ then we call s the *similarity dimension* of the IFS.

3. Extension to Many Variables

Let $f : [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ is a continuous function on the closed region $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. The derivative or integration to order $Q = (-q_1, \cdots, -q_n)$, denoted by $D^Q_{x_1,\cdots,x_n} f(x_1, \ldots, x_n)$, is given by

$$\begin{aligned}\lim_{N_1,\cdots,N_n\to\infty} \prod_{k=1}^{n} \frac{1}{\Gamma(q_k)} \sum_{j_1=0}^{N_1-1} \cdots \sum_{j_n=0}^{N_n-1} \left(\prod_{k=1}^{n} \frac{\Gamma(j_k+q_k)}{\Gamma(j_k+1)} \right) \\ \times f(x^*_{N_1-j_1}, \cdots, x^*_{N_n-j_n}) \Delta^{q_1} x_{N_1-j_1} \ldots \Delta^{q_n} x_{N_n-j_n}\end{aligned} \tag{2}$$

where we denote the gamma function at $x \in \mathbb{R}$ by $\Gamma(x)$.

4. The Extended Grünwald Definition and Iterated Function Systems

In this section, we define a partition and subsequent refinements of Grünwald's definition for integration as an iterated function system on the interval $[0,1]$. Also, we show that it satisfies the open set condition for integration order $q > 0$.

Definition 1 Let $N_k \geq 2$ and $q_k \geq 1$ for $k = 1, \ldots, n$. Then the *Grünwald IFS, denoted by* $\mathcal{G}$, is the collection of maps $\{S_{j_1,\ldots,j_n}(\vec{x})\}_{j_1,\ldots,j_n=1}^{N_1,\ldots,N_n}$, where each map is defined by

$$S_{j_1,\ldots,j_n}(\vec{x}) = \begin{pmatrix} \frac{1}{N_1^{q_1}} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & \frac{1}{N_n^{q_n}} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \frac{j_1-1}{N_1} \\ \vdots \\ \frac{j_n-1}{N_n} \end{pmatrix}. \tag{3}$$

Theorem 1 If $q > 0$ then the Grünwald IFS $\mathcal{G}$ satisfies the open set condition.

Proof. Without loss of generality, let $N \geq 2$, $q \geq 1$ and $\mathcal{G}_0 = [0,1] \times \cdots \times [0,1]$. We denote the m^{th} iteration of the region $[0,1] \times \cdots \times [0,1]$ by $\mathcal{G}_m$. We show by induction on m. Suppose $m = 1$. Then for any $j_1, \ldots, j_n$, the contraction $S_{j_1,\ldots,j_n}(\vec{x})$ sends the region $(0,1) \times \cdots \times (0,1)$ to a sub-region $\left(\frac{j_1-1}{N_1}, \frac{1}{N_1^{q_1}} + \frac{j_1-1}{N_1}\right) \times \cdots \times$

$\left(\frac{j_k-1}{N_k}, \frac{1}{N_k^{q_k}} + \frac{j_k-1}{N_k}\right) \times \cdots \times \left(\frac{j_n-1}{N_n}, \frac{1}{N_n^{q_n}} + \frac{j_n-1}{N_n}\right)$. We observe that the surrounding sub-regions are the result of sending the region $(0,1) \times \cdots \times (0,1)$ to sub-regions that do not intersect, and with all of them contained in $(0,1) \times \cdots \times (0,1)$ for all possible values of j_k. Now, assume that the Grünwald IFS satisfies the open set condition at $m = j$. Applying the contractive map $S_{j_1,\ldots,j_n}(\vec{x})$ to the region $(0,1) \times \cdots \times (0,1)$ j-times, we obtain the sub-region

$$\begin{aligned} &\left(\sum_{l=0}^{j} \frac{j_1-1}{N_1^{1+lq_1}}, \frac{1}{N_1^{jq_1}} + \sum_{l=0}^{j} \frac{j_1-1}{N_1^{1+lq_1}}\right) \times \ldots \\ &\cdots \times \left(\sum_{l=0}^{j} \frac{j_k-1}{N_k^{1+lq_k}}, \frac{1}{N_k^{jq_k}} + \sum_{l=0}^{j} \frac{j_k-1}{N_k^{1+lq_k}}\right) \times \ldots \\ &\cdots \times \left(\sum_{l=0}^{j} \frac{j_n-1}{N_n^{1+lq_n}}, \frac{1}{N_n^{jq_n}} + \sum_{l=0}^{j} \frac{j_n-1}{N_n^{1+lq_n}}\right). \end{aligned} \tag{4}$$

Again, we observe that the surrounding sub-regions send the region $(0,1)$ to non-intersecting subreations that are also contained in $(0,1)\times\cdots\times(0,1)$. The inductive step

$$S^{j+1}_{j_1,\ldots,j_n}(\vec{x}) = \begin{pmatrix} \frac{1}{N_1^{q_1}} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{N_n^{q_n}} \end{pmatrix} \begin{pmatrix} \frac{x_1}{N_1^{jq_1}} + \sum_{l=0}^{j} \frac{j_1-1}{N_1^{1+lq_1}} \\ \vdots \\ \frac{x_n}{N_n^{jq_n}} + \sum_{l=0}^{j} \frac{j_k-1}{N_n^{1+lq_n}} \end{pmatrix} + \begin{pmatrix} \frac{j_1-1}{N_1} \\ \vdots \\ \frac{j_n-1}{N_n} \end{pmatrix} \tag{5}$$

provides the conclusion for $n \geq 1$. For $q < 1$, we observe that there is sub-region overlap under the contractive maps $\{S_{j_1,\ldots,j_n}(\vec{x})\}_{j_1,\ldots,j_n=1}^{N_1,\ldots,N_n}$. However, using the Grünwald definition, we can always adjust the dimensions of the sub-regions thereby satisfying the open set condition.

5. Order and the Hausdorff Dimension

In [6], we showed how the integration order is related to the Hausdorff dimension. We now extend these results to functions of many variables for all orders.

Theorem 2 The Hausdorff dimension of the Grünwald IFS $\mathcal{G}$ is $s = \sum_k \frac{1}{q_k}$.

Proof. At the m^{th} iteration, $\mathcal{G}_m$ can be covered by N_k^m intervals of length $N_k^{-mq_k}$. Thus, the similarity dimension for each k is required to satisfy the following condition.

$$\sum_{j_k} r_{j_k}^{s_k} = N_k^m N^{-ms_kq_k} = 1 \tag{6}$$

Note that the Hausdorff dimension for any interval $[a,b]$ of $\mathbb{R}$ is one ([2]). In [5], the dimension of the product space $A \times B$ is $dimA + dimB$. Since the Grünwald definition for the integration process is defined on a product space, we have the following theorem.

Theorem 3 The Hausdorff dimension of $\mathcal{G} \times \mathbb{R}$ is $s = 1 + \sum_k \frac{1}{q_k}$.

Proof. Since the range $\mathbb{R}$ is one-dimensional and the dimension of the domain is given by Theorem 2, the dimension of the product space is the sum of these dimensions as in [5].

Acknowledgements

The authors are grateful for the support of Ms. Linda Hardy during the writing of this paper.

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