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EXISTENCE AND ULAM STABILITIES FOR HADAMARD FRACTIONAL INTEGRAL EQUATIONS IN FRÉCHET SPACES

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Abstract. In this paper, we present some results concerning the existence and Ulam Stabilities of solutions for some functional integral equations of Hadamard fractional order. We use an extension of the Burton-Kirk fixed point theorem in Fréchet spaces.

1. Introduction

Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, and bio-engineering and others. However, many researchers remain unaware of this field. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [1, 6, 7], Baleanu *et al.* [9], Kilbas *et al.* [21], Miller and Ross [23], Lakshmikantham *et al.* [22], Samko *et al.* [29]. Butzer *et al.* [11] investigate properties of the Hadamard fractional integral and derivative. In [12], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators. In [25], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [29] and the references therein. Butzer *et al.* [11] investigate properties of the Hadamard fractional integral and derivative. In [12], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators. In [25], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [29] and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [30]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [17]. Thereafter, this

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type of stability is called the Ulam-Hyers stability. In 1978, Rassias [26] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [16, 18]. Bota-Boriceanu and Petrusel [10], Petru *et al.* [24], and Rus [27, 28] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [13], and Jung [20] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. More details from historical point of view, and recent developments of such stabilities are reported in [19, 27].

Recently some interesting results on the existence and Ulam stabilities of the solutions of some classes of differential equations have been obtained by Abbas *et al.* [2, 3, 4, 5]. This paper deals with the existence and Ulam stabilities of olutions of the ollowing Hadamard fractionl integral equations of the form

$$\begin{aligned} u(t,x) &= \mu(t,x) + f(t,x,({}^H I_\sigma^r u)(t,x), u(t,x)) \\ &\quad + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \tfrac{t}{s}\right)^{r_1-1} \left(\log \tfrac{x}{y}\right)^{r_2-1} \qquad (1.1) \\ &\quad \times g(t,x,s,y,u(s,y)) \tfrac{dyds}{sy};\ (t,x) \in J := [1,+\infty) \times [1,b], \end{aligned}$$

where $b > 1,\ \sigma = (1,1),\ r = (r_1, r_2),\ r_1, r_2 \in (0,\infty),\ {}^H I_\sigma^r$ is the Hadamard integral of order $r,\ \mu : J \to \mathbb{R},\ f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ g : J' \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $J' = \{(t,x,s,y) \in J^2 : s \le t,\ y \le x\}$ and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Our investigations are conducted in Fréchet spaces with an application of the fixed point theorem of Burton-Kirk for the existence of solutions of the integral equation (1.1), and we prove that all solutions are generalized Ulam-Hyers-Rassias stable.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([1,+\infty) \times [1,b])$; for $b > 1$, we denote the space of Lebesgue-integrable functions $u : [1,+\infty) \times [1,b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_1^\infty \int_1^b |u(t,x)| dx dt.$$

By $C := C(J)$ we denote the space of all continuous functions from J into $\mathbb{R}$.

Definition 2.1. [15, 21] *The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,a],\mathbb{R})$, is defined as*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds.$$

Example 2.2. *The Hadamard fractional integral of order $q > 0$ for the function $w : [1, e] \to \mathbb{R}$, defined by $w(x) = (\log x)^{\beta-1}$ with $\beta > 0$, is*

$$({}^H I_1^q w)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+q)}(\log x)^{\beta+q-1}.$$

Definition 2.3. *Let $r_1,\ r_2 \geq 0,\ \sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in L^1(J, \mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression*

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{w(s,t)}{st}dtds.$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \quad for\ every\ x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad for\ all\ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [14].

Set

$$J_p := [1, p] \times [1, b];\ p \in \mathbb{N}\backslash\{0, 1\}.$$

For each $p \in \mathbb{N}\backslash\{0, 1\}$ we consider following set, $C_p = C(J_p)$, and we define in C the semi-norms by

$$\|u\|_p = \sup_{(t,x)\in J_p}\|u(t, x)\|.$$

Then C is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Definition 2.4. *A set $M \subset C$ is bounded if and only if*

$$\forall p \in \mathbb{N}\backslash\{0, 1\},\ \exists \ell_p > 0 : \forall u \in M,\ \|u\|_p \leq \ell_p,$$

and $M = \{u(t, x);\ (t, x)) \in J\} \subset C$ is relatively compact if and only if for all $p \in \mathbb{N}\backslash\{0, 1\}$, the family $\{u(t, x)|_{(t,x)\in J_p}\}$ is equicontinuous and uniformly bounded on J_p.

Now, we consider the Ulam stability for the Hadamard integral equation (1.1). Let us define the mapping $N : C \to C$, such that,

$$\begin{aligned}(Nu)(t,x) &= \mu(t,x) + f(t,x,({}^H I_\sigma^r u)(t,x), u(t,x)) \\ &+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left(\log\tfrac{t}{s}\right)^{r_1-1}\left(\log\tfrac{x}{y}\right)^{r_2-1} \qquad (2.1)\\ &\times g(t,x,s,y,u(s,y))\tfrac{dyds}{sy};\ (t,x) \in J.\end{aligned}$$

Let ϵ be a positive real number and $\phi : J_p \to [0,\infty)$ be a measurable and bounded function. We consider the following inequalities:

$$\|u(t,x) - (Nu)(t,x)\|_p \le \epsilon; \quad \text{for a.a. } (t,x) \in J_p. \tag{2.2}$$

$$\|u(t,x) - (Nu)(t,x)\|_p \le \phi(t,x); \quad \text{for a.a. } (t,x) \in J_p. \tag{2.3}$$

$$\|u(t,x) - (Nu)(t,x)\|_p \le \epsilon\phi(t,x); \quad \text{for a.a. } (t,x) \in J_p. \tag{2.4}$$

Definition 2.5. [6, 27] *The equation (1.1) is Ulam-Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality (2.2) there exists a solution v of the equation (1.1) with*

$$\|u(t,x) - v(t,x)\|_p \le \epsilon c_N; \ (t,x) \in J_p.$$

Definition 2.6. [6, 27] *The equation (1.1) is generalized Ulam-Hyers stable if there exists $c_N : C([0,\infty),[0,\infty))$ with $c_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality (2.2) there exists a solution v of the equation (1.1) with*

$$\|u(t,x) - v(t,x)\|_p \le c_N(\epsilon); \ (t,x) \in J_p.$$

Definition 2.7. [6, 27] *The equation (1.1) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{N,\phi} > 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality (2.4) there exists a solution v of the equation (1.1) with*

$$\|u(t,x) - v(t,x)\|_p \le \epsilon c_{N,\phi}\phi(t,x); \ (t,x) \in J_p.$$

Definition 2.8. [6, 27] *The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{N,\phi} > 0$ such that for each solution u of the inequality (2.3) there exists a solution v of the equation (1.1) with*

$$\|u(t,x) - v(t,x)\|_p \le c_{N,\phi}\phi(t,x); \ (t,x) \in J_p.$$

Remark 2.9. *It is clear that:*
(i) Definition 2.5 $\Rightarrow$ Definition 2.6,
(ii) Definition 2.7 $\Rightarrow$ Definition 2.8,
(iii) Definition 2.7 for $\phi(.,.) = 1 \Rightarrow$ Definition 2.5.

One can have similar remarks for the inequalities (2.2) and (2.4). So, the Ulam stabilities of the fractional differential equations are some special types of data dependence of the solutions of fractional differential equations.

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Theorem 2.10. [8] *Let $(X, \|.\|_n)$ be a Fréchet space and let $A, B : X \to X$ be two operators such that*

- (a) *A is a compact operator;*
- (b) *B is a contraction operator with respect to a family of seminorms $\{\|.\|_n\}$;*
- (c) *the set $\left\{x \in X : x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right),\ \lambda \in (0,1)\right\}$ is bounded.*

Then the operator equation $A(u) + B(u) = u$ has a solution in X.

3. Existence and Ulam stabilities results

Now, we are concerned with the existence and the Ulam stability of solutions for the integral equation (1.1).

Set

$$J'_p = \{(t,x,s,y) : 1 \le s \le t \le p,\ 1 \le y \le x \le b\};\ p \in \mathbb{N}\backslash\{0,1\}.$$

The following hypotheses will be used in the sequel.

(H_1) There exist continuous functions $l,k : J_p \to \mathbb{R}_+$, such that

$$|f(t,x,u_1,v_1) - f(t,x,u_2,v_2)| \le \frac{l(t,x)|u_1-u_2| + k(t,x)|v_1-v_2|}{1+|u_1-u_2|+|v_1-v_2|};$$

for each $(t,x) \in J_p$ and each $u_1,u_2,v_1,v_2 \in \mathbb{R}$.

(H_2) There exist continuous functions $P,Q,\varphi : J'_p \to \mathbb{R}_+$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|g(t,x,s,y,u)| \le \frac{P(t,x,s,y) + Q(t,x,s,y)|u|}{1+|u|};$$

for $(t,x,s,y) \in J',\ u \in \mathbb{R}$, and

$$|g(t_1,x_1,s,y,u) - g(t_2,x_2,s,y,u)| \le \varphi(s,y)(|t_1-t_2| + |x_1-x_2|)$$
$$\times\psi(|u|);\ (t_1,x_1,s,y),(t_2,x_2,s,y) \in J'_p,\ u \in \mathbb{R}.$$

(H_3) There exist continuous functions $P_1,Q_1 : J_p \to [0,\infty)$, such that for each $(t,s),(t,x) \in J_p$, we have

$$P(t,x,s,y,w) \le \phi(t,x)P_1(s,y),\ \textit{and}\ Q(t,x,s,y,w) \le \phi(t,x)Q_1(s,y).$$

Theorem 3.1. *Assume that the hypotheses (H_1) and (H_2) hold. If*

$$\ell := k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.1}$$

where

$$k_p = \sup_{(t,x)\in J_p} k(t,x),\ l_p = \sup_{(t,x)\in J_p} l(t,x);\ p \in \mathbb{N}\backslash\{0,1\},$$

then the Hadamard integral equation (1.1) has at least one solution in the space C. Furthermore, if the hypothesis (H_3) holds, then the equation (1.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Let us define the operators $A,B : C \to C$ defined by

$$(Au)(t,x) = \int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1}\frac{g(t,x,s,y,u(s,y))}{sy\Gamma(r_1)\Gamma(r_2)}dyds;\ (t,x) \in J, \tag{3.2}$$

$$(Bu)(t,x) = \mu(t,x) + f(t,x,(^HI^r_\sigma u)(t,x),u(t,x));\ (t,x) \in J. \tag{3.3}$$

We shall show that operators A and B satisfied all the conditions of Theorem 2.10. The proof will be given in several steps.

Step 1. *A is compact.*
To this aim, we must prove that A is continuous and it transforms every bounded set into a relatively compact set. Let $M \subset C$ be a bounded set of C. The proof will

be given in several claims.

Claim 1. *A is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}\backslash\{0,1\}}$ be a sequence in M such that $u_n \to u$ in M. Then, for each $(t,x) \in J_p;\ p \in \mathbb{N}\backslash\{0,1\}$, we have

$$\begin{aligned} & |(Au_n)(t,x) - (Au)(t,x)| \\ \leq\ & \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \tfrac{t}{s}\right|^{r_1-1} \left|\log \tfrac{x}{y}\right|^{r_2-1} \\ & \times |g(t,x,s,y,u_n(s,y)) - g(t,x,s,y,u(s,y))| dyds \\ \leq\ & \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \tfrac{t}{s}\right|^{r_1-1} \left|\log \tfrac{x}{y}\right|^{r_2-1} \\ & \times |g(t,x,s,y,u_n(s,y)) - g(t,x,s,y,u(s,y))| dyds. \end{aligned} \tag{3.4}$$

Then, since $u_n \to u$ as $n \to \infty$ and g is continuous, (3.4) gives

$$\|A(u_n) - A(u)\|_p \to 0 \quad \text{as } n \to \infty.$$

Claim 2. *A maps bounded sets into bonded sets in C.*
For arbitrarily fixed $(t,x) \in J_p$ and $u \in M$, we have

$$\begin{aligned} |(Au)(t,x)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times |g(t,x,s,y,u(s,y))| dyds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times \frac{P(t,x,s,y) + Q(t,x,s,y)|u(s,y)|}{1 + |u(s,y)|} dyds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times (P(t,x,s,y) + Q(t,x,s,y)) dyds \\ &\leq P_p + Q_p, \end{aligned}$$

where

$$P_p = \sup_{(t,x)\in J_p} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{P(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)} dyds,$$

and

$$Q_p = \sup_{(t,x)\in J_p} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{Q(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)} dyds.$$

Thus

$$\|A(u)\|_p \leq P_p + Q_p := \ell'_p.$$

Claim 3. *A maps bounded sets into equicontinuous sets in C.*
Let $(t_1,x_1), (t_2,x_2) \in J_p$, $t_1 < t_2,\ x_1 < x_2$ and let $u \in M$, thus we have

$$\begin{aligned} & |(Au)(t_2,x_2) - (Au)(t_1,x_1)| \\ \leq\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \Big| \int_1^{t_2} \int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1} \\ & \times [g(t_2,x_2,s,y,u(s,y)) - g(t_1,x_1,s,y,u(s,y))] dyds \Big| \end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}g(t_1,x_1,s,y,u(s,y))dyds$$
$$-\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_1}{s}\right|^{r_1-1}\left|\log\frac{x_1}{y}\right|^{r_2-1}g(t_1,x_1,s,y,u(s,y))dyds\Big|$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_1}{s}\right|^{r_1-1}\left|\log\frac{x_1}{y}\right|^{r_2-1}g(t_1,x_1,s,y,u(s,y))dyds$$
$$-\int_1^{t_1}\int_1^{x_1}\left|\log\frac{t_1}{s}\right|^{r_1-1}\left|\log\frac{x_1}{y}\right|^{r_2-1}g(t_1,x_1,s,y,u(s,y))dyds\Big|.$$

Thus

$$|(Au)(t_2,x_2)-(Au)(t_1,x_1)|$$
$$\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}$$
$$\times\Big|g(t_2,x_2,s,y,u(s,y))-g(t_1,x_1,s,y,u(s,y))\Big|dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_1^{x_1}\Big|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{y}\right)^{r_2-1}\Big|$$
$$\times\Big|g(t_1,x_1,s,y,u(s,y))\Big|dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}|g(t_1,x_1,s,y,u(s,y))|dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_1^{x_1}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}|g(t_1,x_1,s,y,u(s,y))|dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}|g(t_1,x_1,s,y,u(s,y))|dyds.$$

Hence

$$|(Au)(t_2,x_2)-(Au)(t_1,x_1)|$$
$$\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}$$
$$\times\varphi(s,y)(|t_1-t_2|+|x_1-x_2|)\psi(\ell_p)dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_1^{x_1}\Big|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{y}\right)^{r_2-1}\Big|$$
$$\times(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t_2}\int_1^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds.$$

From the continuity of functions P, Q, φ and as $t_1\longrightarrow t_2$ and $x_1\longrightarrow x_2$, the right-hand side of the above inequality tends to zero. As a consequence of claims 1-3 and from the Arzelá-Ascoli theorem, we can conclude that A is continuous and compact.

Step 2. *B is a contraction.*
Consider $v, w \in C$. Then, by (H_1), for any $p \in \mathbb{N}\backslash\{0,1\}$ and each $(t,x) \in J_P$, we have

$$\begin{aligned} |(Bv)(t,x) - (Bw)(t,x)| &\leq l(t,x)|^H I_\sigma^r (v-w)(t,x)| + k(t,x)|(v-w)(t,x)| \\ &\leq \left(k(t,x) + \frac{l(t,x)(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)|v-w|. \end{aligned}$$

Thus,

$$\|(B(v) - B(w)\|_p \leq \left(k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\|v-w\|_p.$$

By (3.1), we conclude that B is a contraction.

Step 3. *the set $\mathcal{E} := \left\{u \in C(J) : u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right),\ \lambda \in (0,1)\right\}$ is bounded.*
Let $u \in C$, such that $u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right)$ for some $\lambda \in (0,1)$. Then, for any $p \in \mathbb{N}\backslash\{0,1\}$ and each $(t,x) \in J_p$, we have

$$\begin{aligned} |u(t,x)| &\leq \lambda|A(u)| + \lambda\left|B\left(\frac{u}{\lambda}\right)\right| \\ &\leq |\mu(t,x)| + |f(t,x,0,0)| + k(t,x) + l(t,x) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1} \\ &\quad \times\frac{P(t,x,s,y) + Q(t,x,s,y)}{sy}dyds \\ &\leq \mu_p + f_p + k_p + l_p + P_p + Q_p, \end{aligned}$$

where

$$\mu_p = \sup_{(t,x)\in[1,p]\times[1,b]} \mu(t,x),\ \ f_p = \sup_{(t,x)\in[1,p]\times[1,b]} |f(t,x,0,0)|;\ \ p \in \mathbb{N}\backslash\{0,1\}.$$

Thus,

$$\|u\|_p \leq \mu_p + f_p + k_p + l_p + P_p + Q_p =: \ell_p^*.$$

Hence, the set $\mathcal{E}$ is bounded.
As a consequence of Steps 1-3 and from an application of Theorem 2.10 we deduce that N has a fixed point u which is a solution of the integral equation (1.1).

Step 4. *The generalized Ulam-Hyers-Rassias stability.*
Set

$$P_{1p} = \sup_{(s,y)\in J_p} P_1(s,y),\ \ and\ Q_{1p} = \sup_{(s,y)\in J_p} Q_1(s,y).$$

Let u be a solution of the inequality (2.3) and v be a solution of the equation (1.1). Then

$$\begin{aligned} v(t,x) &= \mu(t,x) + f(t,x,^H I_\sigma^r v(t,x), v(t,x)) \\ &\quad + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left(\log\tfrac{t}{s}\right)^{r_1-1}\left(\log\tfrac{x}{y}\right)^{r_2-1} \\ &\quad \times g(t,x,s,y,v(s,y))\tfrac{dyds}{sy};\ \ (t,x) \in J := [1,+\infty)\times[1,b], \end{aligned}$$

From the inequality (2.3) and the hypothesis (H_3), for each $(t,x) \in J_p$, we have

$$\begin{aligned}
&|u(t,x)-v(x,y)| \leq |u(t,x)-(Nu)(t,x)| \\
&+ |(Nu)(t,x)-(Nv)(t,x)| \\
&\leq \phi(x,y) + |f(t,x,({}^H I_\sigma^r u)(t,x,),u(t,x)) - f(t,x,{}^H I_\sigma^r v(t,x),v(t,x))| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,v(s,y))| \frac{dyds}{sy} \\
&\leq \phi(x,y) + l(t,x)|({}^H I_\sigma^r u)(t,x) - {}^H I_\sigma^r v(t,x)| \\
&+ k(t,x)|u(t,x)-v(t,x)| \\
&+ \frac{2\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} \phi(t,x) \\
&\times (P_1(s,y)+Q_1(s,t))dyds \\
&\leq \phi(x,y) + \ell_p|u(t,x)-v(t,x)| \\
&+ \frac{2\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} (P_{1p}+Q_{1p})dyds \\
&\leq \phi(t,x) + \ell_p|u(t,x)-v(t,x)| \\
&+ \frac{2(P_{1p}+Q_{1p})\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} dyds.
\end{aligned}$$

Thus, for each $(t,x) \in J_p$, we obtain

$$\begin{aligned}
|u(t,x)-v(x,y)| &\leq \frac{\phi(t,x)}{1-\ell_p}\left(1 + \frac{2(P_{1p}+Q_{1p})}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} dyds\right) \\
&\leq \frac{1}{1-\ell_p}\left(1 + \frac{2(P_{1p}+Q_{1p})(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\phi(t,x) \\
&:= c_{N,\phi}\phi(t,x).
\end{aligned}$$

Hence, for each $(t,x) \in J_p$, we get

$$|u(t,x)-v(x,y)| \leq c_{N,\phi}\phi(x,y).$$

Consequetly, the equation (1.1) is generalized Ulam-Hyers-Rassias stable.

4. An Example

Consider the following Hadamard fractional order integral equation of the form

$$u(t,x) = \frac{xe^{3-2t}}{1+t+x^2} + \frac{xe^{-t-2}}{c_p(1+e^{-2p}|({}^H I_\sigma^r u)(t,x)| + e^{-p}|u(t,x)|)}$$

$$+\int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{r_1-1} \left(\log\frac{x}{y}\right)^{r_2-1} \frac{g(t,x,s,y,u(s,y))}{\Gamma(r_1)\Gamma(r_2)} dyds;\ (t,x) \in [1,+\infty)\times[1,e], \tag{4.1}$$

where $c_p = e^{-p} + \frac{e^{-2p}p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}$; $p \in \mathbb{N}\backslash\{0,1\}$, $r = (r_1, r_2) \in (0,\infty)\times(0,\infty)$ and

$$g(t,x,s,y,u) = \frac{xs^{\frac{-3}{4}}(1+|u|)\sin\sqrt{t}\sin s}{(1+x^2+t^2)(1+|u|)}; if\ (t,x,s,y) \in J',\ and\ u \in \mathbb{R},$$

and

$$J' = \{(t,x,s,y) : 1 \le s \le t\ and\ 1 \le x \le y \le e\}.$$

Set

$$\mu(t,x) = \frac{xe^{3-2t}}{1+t+x^2},\ f(t,x,u,v) = \frac{xe^{-t-2}}{c_p(1+e^{-2p}|u|+e^{-p}|v|)};\ p \in \mathbb{N}\backslash\{0,1\}.$$

The function f is continuous and satisfies assumption (H_1), with $k(t,x) = \frac{xe^{-t-2-p}}{c_p}$, $l(t,x) = \frac{xe^{-t-2-2p}}{c_p}$, $k_p = \frac{e^{-2-p}}{c_p}$ and $l_p = \frac{e^{-2-2p}}{c_p}$. Also, the function g is continuous and satisfies assumption (H_2), with

$$P(t,x,s,y) = Q(t,x,s,y) = \frac{xs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+x^2+t^2};\ (t,x,s,y) \in J'.$$

Also, the function g is continuous and satisfies assumption (H_3), with

$$P_1(s,y) = Q_(s,y) = s^{\frac{-3}{4}}\sin s,\ P_{1p} = Q_{1p} = p^{\frac{-3}{4}},$$

and

$$\phi(t,x) = \frac{x\sin\sqrt{t}}{1+x^2+t^2}.$$

Finally, We shall show that condition (3.1) holds with $b = e$. Indeed, for each $p \in \mathbb{N}\backslash\{0,1\}$, we get

$$k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{c_p}\left(e^{-2-p} + \frac{e^{-2-2p}p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) = e^{-2} < 1.$$

Hence by Theorem 3.1, the equation (4.1) has a solution defined on $[1,+\infty)\times[1,e]$ and (4.1) is generalized Ulam-Hyers-Rassias stable.

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