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ITERATIVE SOLUTIONS TO A COUPLED SYSTEM OF NON-LINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this article we investigate the existence of extremal(maximal and minimal) solutions for the following coupled system of integro- differential equations of fractional order with the given boundary conditions of the form

$$\begin{cases} D^{\alpha}u(t) + f_1(t, v(t), I^{\beta}v(t)) = 0, \quad 0 < t < 1, \\ D^{\beta}v(t) + f_2(t, u(t), I^{\alpha}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u'(1) = 0, \; v(0) = 0, \quad v'(1) = 0. \end{cases}$$

where $1 < \alpha, \beta \leq 2$ and $f_1, f_2 : [0, 1] \times R \times R \to R$ are given functions satisfying Caratheodory conditions and D is standing for Riemann-liouville-differentiation of fractional order. We find some sufficient conditions for the existence and uniqueness of maximal and minimal solutions by using monotone iterative technique along with the method of upper and lower solutions. We also link our analysis for the problem to equivalent integral equations. We also give the error estimate and an example for the illustration of our results.

1. Introduction

The advancement in applied analysis and its significant applications especially of fractional calculus and its existence in various field of science and engineering, many researchers are taking interest to study them and its various applications. Fractional differential equations arise in the field of physics, electro chemistry, viscoelasticity, Control theory,aerodynamics,electrodynamics of complex medium, polymer rheology and image and signal processing phenomenon etc. Large number of research articles are devoted to the existence of positive solution and solutions of differential equations of fractional order for detail see [1, 2, 3, 4, 5, 6] and the reference therein. Recently many authors are taking interest in the study of coupled system of fractional order differential equations because they occur in various problems of applied analysis, [7, 8, 9, 1, 11, 12, 22] and the reference therein. Considerable work have been done to study existence and uniqueness of positive solution for the coupled system of fractional order differential equations by the help of standard fixed point theorem of Cone expansion and Banach contraction type.

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The monotone iterative technique is a powerful tool when combined with the method of upper and lower solutions and applied useful consequences may be drawn for existence of multiple solutions of fractional order differential equations, such type work can be be found in [13, 14, 15, 16, 17] the references used in them. In [18, 19, 20, 23, 24, 25, 26, 27, 28, 29] the above mentioned technique were used for initial value problems and useful results were obtained for the existence of solutions. Also some authors applied the same technique for boundary value problems and investigated the solutions of differential equations of fractional order. To the best of our information very few articles are devoted to such type of results for coupled systems of fractional order differential equations for example in [21], N. Xu and W. Liu applied the monotone iterative technique to the following coupled system of fractional differential-integral equations with two-point boundary conditions of the form

$$\begin{cases} D^{\alpha}u(t) + f(t, v(t), I^{\beta}v(t)) = 0, \quad t \in [0, 1], \\ D^{\beta}v(t) + g(t, u(t), I^{\alpha}u(t)) = 0, \quad t \in [0, 1], \\ I^{3-\alpha}u(0) = D^{\alpha-2}u(0) = u(1) = 0, \\ I^{3-\beta}v(0) = D^{\beta-2}v(0) = v(1) = 0, \end{cases}$$

where $2 < \alpha,\ \beta \leq 3$ and $f, g : [0, 1] \times R \times R \to R$ are satisfying Caratheodory conditions and $D^{\alpha}, D^{\beta}, I^{\alpha}, I^{\beta}$ are standing for Riemann-Liouville fractional derivative and fractional integration respectively.

Motivated by the above work, we study the following coupled system of nonlinear fractional order differential equations with mixed type boundary conditions by applications of monotone iterative technique along with the method of upper and lower solutions to investigate the following integro-differential equations for the existence and uniqueness results of extremal solutions as

$$\begin{cases} D^{\alpha}u(t) + f_1(t, v(t), I^{\beta}v(t)) = 0, \quad t \in [0, 1], \\ D^{\beta}v(t) + f_2(t, u(t), I^{\alpha}u(t)) = 0, \quad t \in [0, 1], \\ u(0) = 0,\ u'(1) = 0,\ v(0) = 0,\ v'(1) = 0. \end{cases} \tag{1}$$

where $1 < \alpha,\ \beta \leq 2$ and $f_1,\ f_2 : I \times R \times R \to R$ are satisfying Caratheodory conditions and $D^{\alpha}, D^{\beta}, I^{\alpha}, I^{\beta}$ are standing for Riemann-Liouville fractional derivative and fractional integration respectively. We use various tools of functional analysis to obtain sufficient conditions for existence and uniqueness of maximal and minimal solutions. Further we provide an example to illustrate our results and error estimate.

2. Preliminaries

We give some basic definitions and known results of fractional calculus and functional analysis which will play important rule in the studies of maximal and minimal solutions of this paper, for this we refer to [2, 3, 4, 13, 14, 15, 16, 17, 19].

Definition 2.1. The arbitrary order integral for a function $y \in L^1([a, b], R_+)$ is given by

$$I^{\alpha}_{a+}y(t) = \int_a^t \frac{(t-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}ds$$

where $\alpha \in R_+$ and 'Γ' Gamma function provided that the integral converges at right side.

Definition 2.2. The Riemann-liouville fractional order derivative of a function y on the interval $[a,b]$ is defined by

$$D^{\alpha}_{a+}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}y(s)\,ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ represents the integer part of α, provided that the right side is point wise defined on $(0,\infty)$.

Lemma 2.1. [6], The following result holds for fractional differential equations $y \in C(0,1)\bigcap L(0,1)$

$$I^{\alpha}D^{\alpha}y(t) = y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + ... + c_nt^{\alpha-n},$$

for arbitrary $c_i \in R, \quad i = 1,2,...,n$.

Definition 2.3.[20], Let X, Y be Banach spaces satisfying the property of partial order and$S \subset X, T : S \to Y$ be an operator. Then T is increasing operator if for all $u, v \in S$ with $u \leq v$, if T satisfies $Tu \leq Tv$. Similarly T is decreasing operator if for all $u, v \in S$ such that $u \geq v$ we have $Tu \geq Tv$.

Definition 2.4. [20], For a Banach space$X$ such that $S \subset X, T : S \to Y$ be an operator. If $\bar{u} \geq T\bar{u}$ holds for all $\bar{u} \in S$ is called upper solution and $\underline{u} \leq T\underline{u}$ holds for all $\underline{u} \in S$ holds is called lower solution for the operator equation $Tu = u$.

Definition 2.5. [20], A functions $f(t,x,y) : [0,1]\times R\times R \to R$ satisfies Caratheodory conditions, if satisfies the following conditions:

(i) $f(t,x,y)$ is Lebesgue measurable for t and for each $x, y \in R$,
(ii) $f(t,x,y)$ is continuous for x, y for all most $t \in [0,1]$.

Lemma 2.2.[20], For a partial order Banach space X, for each $n \in Z^+, u_n \leq v_n$, if $u_n \to u$ and $v_n \to v$, then $u \leq v$.

Definition 2.6. [20], Functions $(u_0, v_0) \in C[0,1]\times C[0,1]$are called a lower solution of (1) if they satisfy

$$\begin{cases} D^{\alpha}u_0(t) + f_1(t, v_0(t), I^{\beta}v_0(t)) \leq 0, & t \in [0,1], \\ D^{\beta}v_0(t) + f_2(t, u_0(t), I^{\alpha}u_0(t)) \leq 0, & t \in [0,1], \\ u_0(0) \leq 0,\ u_0'(1) \leq 0,\ v_0(0) \leq 0, & v_0'(1) \leq 0. \end{cases}$$

Definition 2.7. [20], Similarly $(\bar{u}_0, \bar{v}_0)$ are called upper solution of(1), if they obey

$$\begin{cases} D^{\alpha}\bar{u}_0(t) + f_1(t, \bar{v}_0(t), I^{\beta}\bar{v}_0(t)) \geq 0, & t \in [0,1], \\ D^{q}\bar{v}_0(t) + f_2(t, \bar{u}_0(t), I^{\alpha}\bar{u}_0(t)) \geq 0, & t \in [0,1], \\ \bar{u}_0(0) \geq 0,\ \bar{u}_0'(0) \geq 0,\ \bar{v}_0(0) \geq 0, \bar{v}_0'(1) \geq 0. \end{cases}$$

Definition 2.8. Let $(u(t), v(t)) \in [u_0, v_0] \times [u_0, v_0]$ is any system of solutions of (1) and there exist an iterative sequences

$$u_0 \leq u_1 \leq u_2 \leq \cdots u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0$$

such that $\lim_{n\to\infty} u_n(t) = u^*(t),\ \lim_{n\to\infty} v_n(t) = v^*(t)$ on compact subset of $[0,1]$ and the limit functions $u^*,\ v^*$ satisfy (1). Further $u^*, v^* \in [u_0, v_0]$. That is $u_n \leq u,\ v \leq v_n,\ n = 1,2,3,...$, by taking limit $n \to \infty$, we have $u^* \leq u,\ v \leq v^*$. Then $(u^*(t), v^*(t))$ is said to be extremal solutions of (1) in $[u_0, v_0] \times [u_0, v_0]$.

Now we are going to introduce some data dependence results and symbols that used throughout in this paper.

Let $X = \{C[0,1], \|u\| = \max_{t\in[0,1]}|u(t)|\}$, clearly X is a Banach space. Let K be a cone in X having a property of partial ordering $u \leq v$ for $u, v \in X$ such that

$v - u \in K$, then X is called a partial order Banach space. For any $u_0, \bar{u}_0 \in X$ are called lower and upper solution respectively of(1) with $u_0 \leq \bar{u}_0$, by assigning $S = [u_0, \bar{u}_0]$.

3. Main Results

Theorem 3.1. *If* $f_i(i = 1, 2) : I \times R \times R \to R$ are satisfying Caratheodory conditions, then (1) has integral representation of the form given by

$$\begin{cases} u(t) = \int_0^1 G_\alpha(t,s) f_1(s, v(s), I^\beta v(s)) ds, \ t \in [0,1], \\ v(t) = \int_0^1 G_\beta(t,s) f_2(s, u(s), I^\alpha u(s)) ds, \ t \in [0,1], \end{cases} \tag{2}$$

where $G_\alpha, G_\beta$are Green's functions given by

$$G_\alpha(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3}$$

$$G_\beta(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-2} - (t-s^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-2}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Proof. Now by applying I^α on both sides of (1) , we obtain

$$u(t) = -I^\alpha f_1(t, v(t), I^\alpha v(t)) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2}, \, c_1, \, c_2 \in R, \tag{5}$$

on the application of the boundary conditions

$$u(0) = 0 \text{ and } u'(1) = 0,$$

we have $c_2 = 0$, and $c_1 = -\frac{1}{\alpha-1} I^{\alpha-1} f_1(1, v(s), I^\beta v(t))$, thus (5) becomes

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\alpha-1} I^\alpha f_1(1, v(1), I^\beta v(1)) - I^\alpha f_1(t, v(t), I^\beta v(t)) \\ &= \frac{1}{(\alpha-1)\Gamma(\alpha-1)} \int_0^1 t^{\alpha-1}(1-s))^{\alpha-2} f_1(s, v(s), I^\beta v(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t ((t-s))^{\alpha-1} f_1(s, v(s), I^\beta v(s)) ds \\ &= \int_0^1 G_\alpha(t,s) f_1(s, v(s), I^\beta v(s)) ds. \end{aligned}$$

Similarly repeating the above process with the second equation of the system, we obtain the second part of (1)

$$v(t) = \int_0^1 G_\beta(t,s) f_2(s, u(s), I^\alpha u(s)) ds.$$

Hence, proof is received. □

Theorem 3.2. Let $G = (G_\alpha, G_\beta)$ be the Green functions of (1)then they satisfying the following properties

(1) $G(t,s) \geq 0$, for all $t, s \in [0,1]$;
(2) $\int_0^1 G(t,s) ds \leq (\frac{1}{(\alpha-1)\Gamma(\alpha+1)}, \frac{1}{(\beta-1)\Gamma(\beta+1)})$, for all $t \in [0,1]$.

Proof. For all $t, s \in [0,1]$ we have

$$\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} \geq 0, \text{as } 0 \leq s \leq t \leq 1,$$

thus $G_\alpha(t,s) \geq 0$ for all $t,s \in [0,1]$,$G_\beta(t,s) \geq 0$ for all $t,s \in [0,1]$.
Now

$$\int_0^1 G_\alpha(t,s)ds = \int_0^1 \frac{t^{\alpha-1}(1-s))^{\alpha-2}}{\Gamma(\alpha)}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \leq \frac{1}{(\alpha-1)\Gamma(\alpha+1)},$$

similarly

$$\int_0^1 G_\beta(t,s)ds \leq \frac{1}{(\beta-1)\Gamma(\beta-1)}.$$

$$\text{Thus} \quad \int_0^1 G(s,t)ds \leq (\frac{1}{(\alpha-1)\Gamma(\alpha+1)}, \frac{1}{(\beta-1)\Gamma(\beta+1)}).$$

Thus proof is completed. □

Now equation(1)can be written in integral equation form as

$$\begin{aligned} u(t) &= \int_0^1 G_\alpha(t,s)f_1(s,v(s),I^\beta v(s))ds \\ &= \int_0^1 G_\alpha(t,s)f_1\bigg(s, \int_0^1 G_\beta(s,x)f_2(x,u(x),I^\alpha u(x))dx, I^\beta\left[\int_0^1 G_\beta(s,x)f_2(x,u(x),I^\alpha u(x))dx\right]\bigg)ds. \end{aligned} \tag{6}$$

For further investigation we give the following assumptions and notations:

- (A_1) $f_i(i=1,2):[0,1]\times R\times R \to R$ satisfy Caratheodory conditions;
- (A_2) For any $u_1,u_2,v_1,v_2 \in S$ with $u_i \leq v_i, i=1,2$ there exist constants $\Pi_i > 0$, for $i=1,2,3,4$, such that
  $0 \leq f_1(t,v_1,v_2) - f_1(t,u_1,u_2) \leq \Pi_1(v_1-u_1)+\Pi_2(v_2-u_2), \ \ t \in [0,1]$
  $0 \leq f_2(t,v_1,v_1) - f_2(t,u_1,u_2) \leq \Pi_3(v_1-u_1)+\Pi_4(v_2-u_2), \ \ t \in [0,1]$;
- (A_3) $(u_0,v_0) \in X\times X$ and $(\bar{u}_0,\bar{v}_0) \in X \times X$ are lower and upper solution respectively with

$$u_0 \leq \bar{u}_0, \quad v_0 \leq \bar{v}_0.$$

We use

$$\begin{aligned} \Upsilon = &\frac{\Pi_1\Pi_3}{(\beta-1)\Gamma(\beta+1)} + \frac{\Pi_1\Pi_4}{(\alpha-1)(\beta-1)\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &+ \frac{\Pi_2\Pi_3}{(\beta-1)^2\Gamma(\beta+1)} + \frac{\Pi_2\Pi_4}{(\alpha-1)(\beta-1)\Gamma(\alpha+1)\Gamma^2(\beta+1)}. \end{aligned}$$

Theorem 3.3. Let assumptions $(A_1)-(A_3)$ holds and $\Upsilon < 1$, then there exist maximal and minimal solutions $(\bar{u}^*,\bar{v}^*),(u^*,v^*)$ for (1), by using $\bar{u}_0^*, u_0^*$ as initial iterations and get the following iterative sequences for each $n \in Z^+$,

$$\begin{aligned} u_n(t) = \int_0^1 G_\alpha(t,s)f_1\bigg(&s, \int_0^1 G_\beta(s,x)f_2(x,u_{n-1}(x),I^\alpha u_{n-1}(x))dx, \\ &I^\beta\left[\int_0^1 G_\beta(s,x)f_2(x,u_{n-1}(x),I^\alpha u_{n-1}(x))dx\right]\bigg)ds, \end{aligned} \tag{7}$$

$$u_n^*(t) = \int_0^1 G_\alpha(t,s) f_2\Big(s, \int_0^1 G_\beta(s,x) f_2(x, u_{n-1}^*(x), I^\alpha u_{n-1}^*(x))dx, \\ I^\beta \left[\int_0^1 G_\beta(s,x) f_2(x, u_{n-1}^*(x), I^\alpha u_{n-1}^*(x))dx\right]\Big)ds, \tag{8}$$

and we have

$$u_0 \le u_1 \le u_2 \le \ldots \le u_n \le \ldots \le u_n^* \le \ldots u_1^* \le u_0^*,$$

$$u^*(t) = \lim_{n\to\infty} u_n(t), \;\; v^*(t) = \int_0^1 G_\beta(t,s) f_2(s, u^*(s), I^\alpha u^*(s))ds,$$

$$\bar{u}^*(t) = \lim_{n\to\infty} u_n^*(t), \;\; \bar{v}^*(t) = \int_0^1 G_\beta(t,s) f_2(s, \bar{u}^*(s), I^\alpha \bar{u}^*(s))ds,$$

further the error estimate of the minimal solutions is given by

$$\|u^*(t) - u_n(t)\| \le \frac{\Upsilon^n}{1-\Upsilon}\|u_1(t) - u_0(t)\|$$

and maximal solutions is given by

$$\|u_n^*(t) - \bar{u}^*(t)\| \le \frac{\Upsilon^n}{1-\Upsilon}\|u_0^*(t) - u_1^*(t)\|.$$

Proof. First define an operator $T : S \to X$ by

$$Tu(t) = \int_0^1 G_\alpha(t,s) f_1\Big(s, \int_0^1 G_\beta(s,x) f_2(x, u(x), I^\alpha u(x))dx, I^\beta \left[\int_0^1 G_\beta(s,x) f_2(x, u(x), I^\alpha u(x))dx\right]\Big)ds.$$

Clearly T is continuous as $G_\alpha(t,s), G_\beta(t,s), f_i, i = 1,2$ are continuous. As the fixed point of T is the solution of(6) and(6) is equivalent to (2). Hence we need to prove the existence of the fixed point for T. To obtain this we proceed as, let $(u_0, v_0) \in X \times X$ is lower solution (1), then

$$v_0(t) \le \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s))ds, \;\; u_0(t) \le \int_0^1 G_\beta(t,s) f_1(s, v_0(s), I^\beta v_0(s))ds.$$

Since $I^\beta v_0(t) \le I^\beta \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s))ds$. In view of (A_2), we get

$$f_1(t, v_0(t), I^\beta v_0(t)) \le f_1\Big(t, \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s))ds, I^\beta \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s))ds\Big),$$

thus

$$u_0(t) \le \int_0^1 G_\alpha(t,s) f_1(s, v_0(s), I^\beta v_0(s))ds$$

$$\le \int_0^1 G_\alpha(t,s) f_1\Big(t, \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s)ds, I^\beta \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s)ds\Big)ds$$

$$\Rightarrow u_0(t) \le \int_0^1 G_\alpha(t,s) f_1\Big(t, \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s))ds, I^\beta \int_0^1 G_\beta(t,s) f_2(s, u_0(s), I^\alpha u_0(s)ds\Big)ds$$

$$\Rightarrow u_0(t) \le Tu_0(t)$$

so u_0 is lower solution of$T$. Now by in view of (A_2) for any $u, v \in S$ with$u \le v \Rightarrow Tu \le Tv$ which implies that T is increasing operator, so$T$ can be written from the iterative sequence (7) as $u_n = Tu_{n-1}$ for all $n \in Z^+$. Due to (A_3) we have

$$u_0 \le u_1 \le u_2 \le \ldots \le u_n. \tag{9}$$

For any $u, v \in S$ with $u \leq v \Rightarrow I^{\alpha}u \leq I^{\alpha}v$ we have $f_2(t, u, I^{\alpha}u) \leq f_2(t, v, I^{\alpha}v)$ and $G_{\beta}(t, s) \geq 0$ we obtain

$$\int_0^1 G_{\beta}(s, x) f_2(x, u(x), I^{\alpha}u(x))dx \leq \int_0^1 G_{\beta}(s, x) f_2(s, v(s), I^{\alpha}v(s))dx$$

$$\Rightarrow I^{\beta}\left[\int_0^1 G_{\beta}(s, x) f_2(x, u(x), I^{\alpha}u(x))dx\right] \leq I^{\beta}\left[\int_0^1 G_{\beta}(s, x) f_2(s, v(s), I^{\alpha}v(s))dx\right].$$

In view of (A_3) we have

$$\begin{aligned}
&0 \leq f_1\Big(t, \int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx\Big) \\
&- f_1\Big(t, \int_0^1 G_{\beta}(t, x) f_2(x, u(x), I^{\alpha}u(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(x, u(x), I^{\alpha}u(x))dx\Big) \\
&\leq \Pi_1\Big(\int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx\Big) \\
&- \Pi_1\Big(\int_0^1 G_{\beta}(t, x) f_2(x, u(x), I^{\alpha}u(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(x, u(x), I^{\alpha}u(x))dx\Big) \\
&\leq \Pi_1\int_0^1 G_{\beta}(s, x)[\Pi_3(v - u) + \Pi_4(I^{\alpha}v - I^{\alpha}u)]dx \\
&+ \frac{\Pi_2}{\Gamma(\beta)}\int_0^t (t - s)^{\beta - 1}\int_0^1 G_{\beta}(s, x)[\Pi_3(v - u) + \Pi_4(I^{\alpha}v - I^{\alpha}u)]dxds \\
&\Rightarrow \Big\|f_1\Big(t, \int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(x, v(x), I^{\alpha}v(x))dx\Big) \\
&- f_1\Big(t, \int_0^1 G_{\beta}(t, x) f_2(t, u(x), I^{\alpha}u(x))dx, I^{\beta}\int_0^1 G_{\beta}(t, x) f_2(t, u(x), I^{\alpha}u(x))dx\Big)\Big\| \\
&\leq \left[\frac{\Pi_1\Pi_3}{(\beta - 1)\Gamma(\beta + 1)} + \frac{\Pi_1\Pi_4}{(\beta - 1)\Gamma(\alpha + 1)\Gamma(\beta + 1)}\right]\|v - u\| \\
&+ \left[\frac{\Pi_2\Pi_3}{(\beta - 1)\Gamma^2(\beta + 1)} + \frac{\Pi_2\Pi_4}{(\beta - 1)\Gamma^2(\beta + 1)\Gamma(\alpha + 1)}\right]\|v - u\|,
\end{aligned}$$

using this we have

$$\|Tv - Tu\| \leq \Upsilon\|v - u\|. \tag{10}$$

Thus from (9) and (10) we have

$$\begin{aligned}
&\|u_2 - u_1\| = \|Tu_1 - Tu_0\| \leq \Upsilon\|u_1 - u_0\|, \\
&\|u_3 - u_2\| = \|Tu_2 - Tu_1\| \leq \Upsilon^2\|u_1 - u_0\|, \\
&\dots, \\
&\|u_{n+1} - u_n\| = \|Tu_n - Tu_{n-1}\| \leq \Upsilon^n\|u_1 - u_0\|,
\end{aligned} \tag{11}$$

therefore for each$n, m \in Z^+$we have

$$\|u_{n+m} - u_n\| \leq \|u_{n+m} - u_{n+m-1}\| + \|u_{n+m-1} - u_{n+m-2}\| + \dots \|u_{n+1} - u_n\|,$$

$$\Rightarrow \|u_{n+m} - u_n\| \leq \frac{\Upsilon^n(1 - \Upsilon^m)}{1 - \Upsilon}\|u_1 - u_0\|,$$

for $0 < \Upsilon < 1$, we have $\|u_{n+m} - u_n\| \to 0$ as $n \to \infty$. Thus $u_n$is a cauchy sequence in S.

Let $u^*(t) = \lim_{n\to\infty} u_n(t)$. So $Tu^* = u^*$. Hence (1) has a pairs of solutions $(u^*(t), v^*(t))$ and $v^*(t) = \int_0^1 G_\beta(t,s) f_2(s, u^*(s), I^\alpha u^*(s))ds$.
Let $m \to \infty$ in (11), then we have $\|u^* - u_n\| \le \frac{\Upsilon^n}{1-\Upsilon}\|u_1 - u_0\|$. Similar if we use u_0^* is initial iteration for upper solution then we have $(\bar{u}^*, \bar{v}^*)$ of (1) and $\bar{u}^*(t) = \lim_{n\to\infty} u_n^*(t),\ \bar{v}^*(t) = \int_0^1 G_\beta(t,s) f_2(s, \bar{u}^*(s), I^\alpha \bar{u}^*(s))ds$. So it is clear to observe that

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots u_n^* \le \ldots \le u_2^* \le u_1^* \le u_0^*.$$

Further to prove maximal and minimal solution of (1) for this need let us take $(\bar{u}^*, \bar{v}^*)$ is maximal and (u^*, v^*) is minimal solution, then let for any $w(t) \in S$ with $Tw = w$, we have $u_n \le w \le u_n^*$. As T is increasing operator so by Lemma(2.2), $Tu_n \le Tw \le Tu_n^*$ as $n \to \infty$ we get $u^*(t) \le w(t) \le \bar{u}^*(t)$. Thus $u^*(t), \bar{u}^*(t)$ are the minimal and maximal fixed point of T respectively. Thus $(u^*(t), v^*(t))$ and $(\bar{u}^*(t), \bar{v}^*(t))$ are the minimal and maximal solutions of(1) respectively. □

Theorem 3.4. Under the assumptions $(A_1) - (A_3)$ with $\Upsilon < 1$, then the extremal (maximal and minimal) solutions of BVP (1) are unique.

Proof. Let $x_0, y_0 \in X$ be minimal and maximal solution of operator equation $Tw = w$ respectively such that $x_0 \le Tx_0,\ y_0 \ge Ty_0,\ t \in [0,1]$.
Assume that $x_0(t),\ y_0(t)$ be the initial iterations respectively such that $x_n \to x^*,\ y_n \to y^*$ as $n \to \infty$, also $Tx^* = x^*,\ Ty^* = y^*$.
We need to prove that $x^* = u^*$ and $y^* = v^*$. As $x_0 \le x^*$ and T is increasing operator so we have $x_n = T^n x_0 \le T^n x^*$, for each $n \in Z^+$. Thus $u_0 \le u_1 \le u_2 \le \ldots \le u_n \le \ldots \le x^*$. Now by using mathematical inductions and from (10) one can get $\|x^* - u_n\| = \|T^n x^* - T^n u_0\| \le \Upsilon^n \|x^* - u_0\| \to 0$ as $n \to \infty$.
Thus $\lim_{n\to\infty} \|x^* - u_n\| = 0 \Rightarrow x^* = \lim_{n\to\infty} u_n = u^* \Rightarrow u^* = x^*$, similarly we can show that $v^* = y^*$. Similarly we can do for maximal solution. Thus uniqueness of minimal and maximal solution has been proved. □

Let $(\bar{u}, \bar{v}), (\bar{u^*}, \bar{v^*})$ are unique minimal and maximal solutions then error estimate may be calculated as

$$\bar{u}_n(t) = \int_0^1 G_\alpha(t,s) f_1\left(s, \int_0^1 G_\beta(s,x) f_2(x, \bar{u}_{n-1}(x), I^\alpha \bar{u}_{n-1}(x))dx\right)ds,\ n = 1,2,3,...$$

$$\bar{u^*}_n(t) = \int_0^1 G_\alpha(t,s) f_1\left(s, \int_0^1 G_\beta(s,x) f_2(x, \bar{u^*}_{n-1}(x), I^\alpha \bar{u^*}_{n-1}(x))dx\right)ds,\ n = 1,2,3,...$$

as $\bar{u}_0 \le \bar{u}_1 \le ... \le \bar{u}_n \le ...\bar{u}_n^* \le ... \le \bar{u}_2^* \le \bar{u}_1^* \le \bar{u}_0^*$,

$$\bar{u}^*(t) = \lim_{n\to\infty} \bar{u}_n(t),\ \ \bar{v}^*(t) = \int_0^1 G_\beta(t,s) f_2(s, \bar{u}^*(s), I^\alpha \bar{u}^*(s))ds,$$

$$\bar{u}^*(t) = \lim_{n\to\infty} \bar{u}_n(t),\ \ \bar{v}^*(t) = \int_0^1 K_2(t,s) f_2(s, \bar{u}^*(s), I^\alpha \bar{u}^*(s))ds,$$

we obtain the following error estimates for lower and upper solutions as

$$\|\bar{u} - \bar{u}_n\| \le \frac{\Upsilon^n}{1-\Upsilon}|\bar{u}_1 - \bar{u}_0\|,\ \ |\bar{u}_n^* - \bar{u}^*\| \le \frac{\Upsilon^n}{1-\Upsilon}|\bar{u}_0^* - \bar{u}_1^*\|.$$

4. Examples

Example 4.1. Consider the following coupled system of boundary values problem

$$\begin{cases} D^{\frac{3}{2}}u(t) + \left(\frac{1-t}{4}\right)^2 v(t) + \left(\frac{1-t^3}{6}\right) I^{\frac{3}{2}}v(t) = 0, & t \in [0,1], \\ D^{\frac{3}{2}}v(t) + \left(\frac{t^2-1}{3}\right)^2 u(t) + \left(\frac{t-1}{2}\right)^3 I^{\frac{3}{2}}u(t) = 0, & ,t \in [0,1], \\ u(0) = u'(1) = 0, \quad v(0) = v'(1) = 0. \end{cases} \tag{12}$$

Since

$$f_1(t, v, I^{\frac{3}{2}}v) = \left(\frac{1-t}{4}\right)^2 v(t) + \left(\frac{1-t^3}{6}\right) I^{\frac{3}{2}}v(t),$$
$$f_2(t, u, I^{\frac{3}{2}}u) = \left(\frac{t^2-1}{3}\right)^2 u(t) + \left(\frac{t-1}{2}\right)^3 I^{\frac{3}{2}}u(t),$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$. For any $u(t) \leq v(t)$, we have

$$0 \leq f_1(t, v, I^{\frac{3}{2}}v) - f_1(t, u, I^{\frac{3}{2}}u) \leq \frac{1}{16}(v-u) + \frac{1}{6}(I^{\frac{3}{2}}v - \frac{3}{2}u),$$
$$0 \leq f_2(t, v, I^{\frac{3}{2}}v) - f_2(t, u, I^{\frac{3}{2}}u) \leq \frac{1}{9}(v-u) + \frac{1}{9}(I^{\frac{3}{2}}v - \frac{3}{2}u),$$

which implies that

$$\Pi_1 = \frac{1}{16}, \ \Pi_2 = \frac{1}{6}, \ \Pi_3 = \frac{1}{9}, \ \Pi_4 = \frac{1}{8}$$

and $\Upsilon = 0.638522 < 1$ is satisfied. Clearly $(0,0)$ is the unique solutions of (12). Let us take $(u_0, v_0) = (-2,-2)$ and $(u_0^*, v_0^*) = (2,2)$ as initial iteration for lower and upper solutions respectively and the iterative sequences by taking n is large enough as

$$u^*(t) = u_9(t), \ \ v^*(t) = \int_0^1 G_\beta(t,s)\left[\left(\frac{s^2-1}{3}\right)^2 u_9(s) + \left(\frac{s-1}{2}\right)^3 I^{\frac{3}{2}}u_9(s)\right] ds,$$
$$\bar{u}^*(t) = u_9^*(t), \ \bar{v}^*(t) = \int_0^1 G_\beta(t,s)\left[\left(\frac{s^2-1}{3}\right)^2 u_9^*(t) + \left(\frac{s-1}{2}\right)^3 I^{\frac{3}{2}}u_9^*(s)\right] ds.$$

The error estimates are

$$\|u(t) - u_9(t)\| \leq \frac{\Upsilon^9}{1-\Upsilon}\|u_1(t) - u_0(t)\| \leq \frac{(.638522)^9}{1-0.638522} \max_{t\in[0,1]} |u_1(t)+2| \simeq 5.8\times 10^{-2},$$

similarly

$$\|u(t) - u_9^*(t)\| \leq \frac{\Upsilon^9}{1-\Upsilon}\|u_1^*(t) - u_0^*(t)\| \leq \frac{(0.638522)^9}{1-0.638522)} \max_{t\in[0,1]} |u_1^*(t)-2\| \simeq 4.8\times 10^{-2}.$$

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