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# EXISTENCE OF A MILD SOLUTION FOR NEUTRAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper, we investigate the existence of a mild solution of the neutral fractional integrodifferential equations with nonlocal initial conditions. The results are obtained by using the fractional power of operators and the Sadovskii's fixed point theorem. As an application controllability problem is studied for neutral fractional integrodifferential equaion with nonlocal condition.

## 1. INTRODUCTION

In this paper, we study the existence of mild solution for semilinear neutral fractional integrodifferential equations with nonlocal conditions in the following form

$$\begin{aligned}&{}^cD^\alpha[x(t)+F(t,x(t),x(b_1(t)),\ldots,x(b_m(t)))]+Ax(t)\\&\quad=G(t,x(t),x(a_1(t)),\ldots,x(a_n(t)))+K\left(t,x(t),\int_0^t k(t,s,x(s))ds\right),\ t\in J=[0,a],\\&x(0)+h(x)=x_0,\end{aligned}\tag{1.1}$$

where $-A$ generates an analytic semigroup, and the functions $F$, $G$, $K$, $k$ and $h$ are given functions to be defined later. The fractional derivative ${}^cD^\alpha$, $0<\alpha<1$ is understood in the Caputo sense.

Fractional differential equations have received increasing attention during recent years due to its applications in various fields of science and engineering such as viscoelasticity, electrochemistry, porous media, electromagnetic etc. [1, 2, 3, 4, 5, 6]. For more details on this theory and applications, we refer the monographs of Lakshmikantham et al. [7], Miler and Ross [8], Podlubny [9], Kilbas and Srivastava [10] and the papers of Guo and Liu [11] and N'Guerekata [12].

Theory of neutral differential equations has been studied by several authors in Banach space [13, 14, 15]. A neutral functional differential is one in which the derivatives of the past history or derivatives of functionals of the past history are

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involved as well as the present state of the system. Neutral differential equations are encountered in problems dealing with electric networks containing lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits. A good guide to the litrature for neutral functional differential equations is the Hale book [16].

On the other hand, integrodifferential equations arise in many fields such as electronic fluid dynamics, biological models and chemical kinetics. The equations of basic electric circuit analysis are well-known examples of these equations. Fractional integrodifferential equation arises in many fields of engineering such as optimal control problem and heat conduction of materials with memory, etc. Fractional neutral integrodifferential equations have been studied by many authors [17, 18].

The existence of solution to evolution equations with nonlocal conditions in Banach space was first studied by Byszewski [19]. Then it has been studied extensively by many authors, see [20, 21] and the references therein.

The result obtained is a generalization and continuation of some results reported in [22, 23, 24]. The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove our main theorem for (1.1). In Section 4, we have given an example to illustrate the theory.

## 2. Preliminaries

In this paper, $X$ will be a Banach space with the norm $\|\cdot\|$. Let $-A$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $S(t)$. This means that there exists a $M \geq 1$ such that $\|S(t)\| \leq M$. Suppose $0 \in \rho(A)$, then define the fractional power $A^\gamma$, for $0 < \gamma \leq 1$, as a closed linear operator on its domain $D(A^\gamma)$ with inverse $A^{-\gamma}$ having following basic properties.

**Theorem 2.1** (see [25]).

(i) *$X_\gamma = D(A^\gamma)$ is a Banach space with the norm $\|x\|_\gamma = \|A^\gamma x\|$, $x \in X_\gamma$.*
(ii) *$S(t) : X \to X_\gamma$ for each $t > 0$ and $A^\gamma S(t)x = S(t)A^\gamma x$ for each $x \in X_\gamma$ and $t \geq 0$.*
(iii) *For every $t > 0$, $A^\gamma S(t)$ is bounded on $X$ and there exists a positive constants $C_\gamma$ such that*

$$\|A^\gamma S(t)\| \leq \frac{C_\gamma}{t^\gamma}\,. \tag{2.1}$$

(iv) *If $0 < \beta < \gamma \leq 1$, then $D(A^\gamma) \hookrightarrow D(A^\beta)$ and the embedding is compact whenever the resolvent operator of $A$ is compact.*

Now we recall the following known definitions.

**Definition 2.1** (see [8, 9, 26]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f$ can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0\,.$$

Provided the right-hand side is pointwise defined on $[0, \infty)$ where $\Gamma(\cdot)$ is Gamma function.

**Definition 2.2** (see [8, 9, 26])**.** The Caputo derivative of order $\alpha$ with the lower limit zero for a function $f$ can be written as

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t), \quad t>0,\ 0\le n-1<\alpha<n.$$

If $f$ is an abstract function with values in $X$, then the integrals appearing in the above definitions are taken in Bochner's sense.

We assume the following conditions:

(A$_1$) $F:[0,a]\times X^{m+1}\to X$ is a continuous function, and there exists a constant $\beta\in(0,1)$ and $L_1,L_2>0$ such that the function $A^{\beta}F$ satisfies the Lipschitz condition:

$$\begin{aligned}&\|A^{\beta}F(s_1,x_0,x_1,\dots,x_m)-A^{\beta}F(s_2,\overline{x}_0,\overline{x}_1,\dots,\overline{x}_m)\|\\ &\quad\le L_1(|s_1-s_2|+\max_{i=0,\dots,m}\|x_i-\overline{x}_i\|),\end{aligned}$$

for any $0\le s_1,\ s_2\le a,\ x_i,\overline{x}_i\in X,\ i=0,1,\dots,m$; and the inequality

$$\|A^{\beta}F(t,x_0,x_1,\dots,x_m)\|\le L_2(\max_{i=0,\dots,m}\{\|x_i\|: i=0,1,\dots,m\}+1), \tag{2.2}$$

holds for any $(t,x_0,x_1,\dots,x_m)\in[0,a]\times X^{m+1}$.

(A$_2$) The function $G:[0,a]\times X^{n+1}\to X$ satisfies the following conditions:

(i) for each $t\in[0,a]$, the function $G(t,\cdot):X^{n+1}\to X$ is continuous and for each $(x_0,x_1,\dots,x_n)\in X^{n+1}$, the function $G(\cdot,x_0,x_1,\dots,x_n):[0,a]\to X$ is strongly measurable;

(ii) for each positive number $p\in N$, there is a positive function $g_p(\cdot):[0,a]\to R^{+}$ such that

$$\sup_{\|x_0\|,\dots,\|x_n\|\le p}\|G(t,x_0,x_1,\dots,x_n)\|\le g_p(t),$$

the function $s\to(t-s)^{1-\alpha}g_p(s)\in L^1([0,t],R^{+})$ and there exists a $\gamma_1>0$ such that

$$\liminf_{p\to\infty}\frac{1}{p}\int_0^t(t-s)^{1-\alpha}g_p(s)ds=\gamma_1<\infty,\quad t\in[0,a],$$

(A$_3$) The function $K:[0,a]\times X\times X\to X$ satisfies the following conditions:

(i) for each $t\in[0,a]$, the function $K(t,\cdot,\cdot):X\times X\to X$ and for each $x,y\in X$, the function $K(\cdot,x,y):[0,a]\to X$ is strongly measurable;

(ii) for each positive number $p\in N$, there is a positive function $q_p(\cdot):[0,a]\to R^{+}$ such that

$$\sup_{\|x\|\le p}\left\|K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right\|\le q_p(t),$$

the function $s\to(t-s)^{1-\alpha}q_p(s)\in L^1([0,t],R^{+})$, and there exists a $\gamma_2>0$ such that

$$\liminf_{p\to\infty}\frac{1}{p}\int_0^t(t-s)^{1-\alpha}q_p(s)ds=\gamma_2<\infty,\quad t\in[0,a].$$

(A$_4$) $a_i,b_j\in C([0,a];[0,a])$, $i=1,2,\dots,n$, $j=1,2,\dots,m$; $h\in C(E;X)$, here and hereafter $E=C([0,a];X)$, and $h$ satisfies that:

(i) There exist positive constants $L_3$ and $L_3'$ such that $\|h(x)\|\le L_3\|x\|+L_3'$ for all $x\in E$;

(ii) $h$ is completely continuous map.

At the last of this section, we recall the Sadoviskii's fixed point theorem [27], which is used to establish the existence of the mild solution of equation (1.1).

**Theorem 2.2** ([27])**.** *Let $\phi$ be a condensing operator on a Banach space $X$, i.e., $\phi$ is continuous and takes bounded sets into bounded sets, and $\mu(\phi(D)) \le \mu(D)$ for every bounded set $D$ of $X$ with $\mu(D) > 0$. If $\phi(E) \subset E$ for convex, closed and bounded set $E$ of $X$, then $\phi$ has a fixed point in $X$ (where $\mu(\cdot)$ denotes the Kuratowski's measures of noncompactness).*

## 3. Existence of mild solution

**Definition 3.1.** A continuous function $x(\cdot) : [0, a] \to X$ is said to be a mild solution of the nonlocal Cauchy problem (1.1), if the function $(t-s)^{\alpha-1}AT_\alpha(t-s)F(s, x(s), x(b_1(s)), \dots, x(b_m(s)))$, $s \in [0, a)$ is integrable on $[0, a)$ and the following integral equation is verified:

$$\begin{aligned} x(t) = {} & S_\alpha(t)[x_0 + F(0, x(0), x(b_1(0)), \dots, x(b_m(0))) - h(x)] \\ & - F(t, x(t), x(b_1(t)), \dots, x(b_m(t))) \\ & + \int_0^t (t-s)^{\alpha-1} AT_\alpha(t-s)F(s, x(s), x(b_1(s)), \dots, x(b_m(s)))ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)G(s, x(s), x(a_1(s)), \dots, x(a_n(s)))ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)\right] ds, \quad t \in [0,\, a], \end{aligned} \tag{3.1}$$

where $S_\alpha(t)x = \displaystyle\int_0^\infty \eta_\alpha(\theta)S(t^\alpha\theta)xd\theta$, $T_\alpha(t)x = \alpha\displaystyle\int_0^\infty \theta\eta_\alpha(\theta)S(t^\alpha\theta)xd\theta$ with $\eta_\alpha$ is a probability density function defined on $(0, \infty)$, that is $\eta_\alpha(\theta) \ge 0$, $\theta \in (0, \infty)$ and $\displaystyle\int_0^\infty \eta_\alpha(\theta)d\theta = 1$.

*Remark.* $\displaystyle\int_0^\infty \theta\eta_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$.

**Lemma 3.1** (see [28])**.** *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(i) *for any fixed point $x \in X$, $\|S_\alpha(t)x\| \le M\|x\|$, $\|T_\alpha(t)x\| \le \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|$;*
(ii) *$\{S_\alpha(t), t \ge 0\}$ and $\{T_\alpha(t), t \ge 0\}$ are strongly continuous;*
(iii) *for every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operator;*
(iv) *for any $x \in X$, $\beta \in (0, 1)$ and $\delta \in (0, 1)$, we have $AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x$ and*

$$\|A^\delta T_\alpha(t)\| \le \frac{\alpha C_\delta \Gamma(2-\delta)}{t^{\alpha\delta}\Gamma(1+\alpha(1-\delta))}, \quad t \in (0, a].$$

**Theorem 3.1.** *If the assumptions $(A_1)$-$(A_4)$ are satisfied and $x_0 \in X$, then the nonlocal Cauchy problem* (1.1) *has a mild solution provided that*

$$L_0 = L_1\left[(M+1)M_0 + \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right] < 1 \tag{3.2}$$

*and*

$$M\left[M_0L_2+L_3+\frac{\alpha\gamma_1}{\Gamma(\alpha+1)}+\frac{\alpha\gamma_2}{\Gamma(\alpha+1)}\right]+M_0L_2+\frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}L_2}{\beta\Gamma(1+\alpha\beta)}<1, \tag{3.3}$$

*where* $M_0 = \|A^{-\beta}\|$.

*Proof.* For the sake of brevity, we rewrite that

$$(t, x(t), x(b_1(t)), \dots, x(b_m(t))) = (t, v(t))$$

and

$$(t, x(t), x(a_1(t)), \dots, x(a_n(t))) = (t, u(t))$$

Define the operator $\phi$ on $E$ by

$$\begin{aligned}(\phi x)(t) = {} & S_\alpha(t)[x_0 + F(0, v(0)) - h(x)] - F(t, v(t)) \\ & + \int_0^t (t-s)^{\alpha-1} AT_\alpha(t-s)F(s, v(s))ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)G(s, u(s))ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)\right]ds, \quad 0 \le t \le a.\end{aligned}$$

For each positive number $p$, let

$$D_p = \{x \in E : \|x(t)\| \le p,\ 0 \le t \le a\}.$$

Then for each $p$, $D_p$ is clearly a bounded closed convex set in $E$.

From Lemma 3.1, (2.2) yields

$$\begin{aligned}&\left\|\int_0^t (t-s)^{\alpha-1} AT_\alpha(t-s)F(s, v(s))ds\right\| \\ &\le \int_0^t \|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta F(s, v(s))\|ds \\ &\le \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\int_0^t \frac{(t-s)^{\alpha-1}}{(t-s)^{\alpha-\alpha\beta}}\|A^\beta F(s, v(s))\| \\ &\le \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\|A^\beta F(s, v(s))\|ds \\ &\le \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2(\{\|x_i\| : i = 0, \dots, m\} + 1) \\ &\le \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2(p+1) \end{aligned} \tag{3.4}$$

then from Bochner's theorem [29] it follows that $(t-s)^{\alpha-1}AT_\alpha(t-s)F(s, v(s))$ is integrable on $[0, a]$, so $\phi$ is well defined on $D_p$. Similarly, from $(A_2)$(ii), we obtain

$$\begin{aligned}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)G(s, u(s))ds\right\| &\le \int_0^t \|(t-s)^{\alpha-1}T_\alpha(t-s)G(s, u(s))\|ds \\ &\le \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|G(s, u(s))\|ds \\ &\le \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}g_p(s)ds \end{aligned} \tag{3.5}$$

Again from $(\mathrm{A}_3)$(ii), we obtain

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right]ds\right\| \\
&\leq \int_0^t \left\|(t-s)^{\alpha-1}T_\alpha(t-s)\left[K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right]\right\|ds \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left\|K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right\|ds \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}q_p(s)ds \qquad (3.6)
\end{aligned}$$

We claim that there exists a positive number $p$ such that $\phi D_p \subseteq D_p$. If it is not true, then for each positive number $p$, there is a function $x_p(\cdot) \in D_p$, but $\phi x_p \notin D_p$, that is $\|\phi x_p(t)\| > p$ for some $t(p) \in [0,a]$, where $t(p)$ denotes $t$ is independent of $p$. However, on the other hand, we have

$$\begin{aligned}
p &\leq \|(\phi x_p)(t)\| \\
&\leq M[\|x_0\| + M_0L_2(p+1) + (L_3p + L_3')] + M_0L_2(p+1) \\
&\quad + \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2(p+1) + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}g_p(s)ds \\
&\quad + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}q_p(s)ds \\
&\leq M[\|x_0\| + M_0L_2(p+1) + (L_3p + L_3')] + M_0L_2(p+1) \\
&\quad + \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2(p+1) + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}g_p(s)ds \\
&\quad + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}q_p(s)ds. \qquad (3.7)
\end{aligned}$$

Dividing both sides of (3.7) by $p$ and taking the lower limit as $p \to +\infty$, we get

$$1 \leq MM_0L_2 + ML_3 + M_0L_2 + \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2 + \frac{\alpha\, M}{\Gamma(\alpha+1)}\gamma_1 + \frac{\alpha\, M}{\Gamma(\alpha+1)}\gamma_2$$

or

$$M\left[M_0L_2 + L_3 + \frac{\alpha}{\Gamma(\alpha+1)}\gamma_1 + \frac{\alpha}{\Gamma(\alpha+1)}\gamma_2\right] + M_0L_2 + \frac{C_{1-\beta}\Gamma(1+\beta)a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}L_2 \geq 1\,.$$

This contradicts (3.3). Hence, for positive $p$, $\phi D_p \subseteq D_p$.

Next we will show that the operator $\phi$ has a fixed point on $D_p$, which implies that equation (1.1) has a mild solution. To this end, we decompose $\phi$ as $\phi = \phi_1 + \phi_2$, where the operators $\phi_1$, $\phi_2$ are defined on $D_p$, respectively, by

$$(\phi_1 x)(t) = S_\alpha(t)F(0,v(0)) - F(t,v(t)) + \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F(s,v(s))ds$$

and

$$(\phi_2 x)(t) = S_\alpha(t)[x_0 - h(x)] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) G(s, u(s)) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[ K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right)\right] ds,$$

for $0 \le t \le a$, and we will verify that $\phi_1$ is a contraction while $\phi_2$ is a compact operator.

To prove that $\phi_1$ is a contraction, we take $x_1, x_2 \in D_p$, then for each $0 \le t \le a$ and by condition $(A_1)$ and (3.2), we have

$$\begin{aligned}
&\|(\phi_1 x_1)(t) - (\phi_1 x_2)(t)\| \\
&\le \|S_\alpha(t)[F(0, v_1(0)) - F(0, v_2(0))]\| + \|F(t, v_1(t)) - F(t, v_2(t))\| \\
&\quad + \left\| \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)[F(s, v_1(s)) - F(s, v_2(s))] ds \right\| \\
&\le (M+1) M_0 L_1 \sup_{0 \le s \le a} \|x_1(s) - x_2(s)\| \\
&\quad + \frac{C_{1-\beta}\Gamma(1+\beta) L_1 a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \sup_{0 \le s \le a} \|x_1(s) - x_2(s)\| \\
&\le L_1 \left[ (M+1) M_0 + \frac{C_{1-\beta}\Gamma(1+\beta) a^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \right] \sup_{0 \le s \le a} \|x_1(s) - x_2(s)\| \\
&= L_0 \sup_{0 \le s \le a} \|x_1(s) - x_2(s)\|.
\end{aligned}$$

Thus $\|\phi x_1 - \phi x_2\| \le L_0 \sup_{0 \le s \le a} \|x_1(s) - x_2(s)\|$.

So by assumption $0 < L_0 < 1$, we see that $\phi_1$ is a contraction.

To prove that $\phi_2$ is a compact, firstly we prove that $\phi_2$ is continuous on $D_p$. Let $\{x_n\} \subseteq D_p$ with $x_n \to x$ in $D_p$, then by $(A_2)$(i) and $(A_3)$(i), we have

$$G(s, u_n(s)) \to G(s, u(s)), \quad n \to \infty,$$
$$K\left(t, x_n(t), \int_0^t k(t, s, x_n(s)) ds\right) \to K\left(t, x(t), \int_0^t k(t, s, x(s)) ds\right), \quad n \to \infty.$$

Since

$$\|G(s, u_n(s)) - G(s, u(s))\| \le 2 g_p(s),$$
$$\left\| K\left(t, x_n(t), \int_0^t k(t, s, x_n(s)) ds\right) - K\left(t, x(t), \int_0^t k(t, s, x(s)) ds\right) \right\| \le 2 q_p(s).$$

By the dominated convergence theorem, we have

$$\begin{aligned}
&\|\phi_2 x_n - \phi_2 x\| \\
&= \sup_{0 \le t \le a} \left\| S_\alpha(t)[h(x) - h(x_n)] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[G(s, u_n(s)) - G(s, u(s))] ds \right. \\
&\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[ K\left(s, x_n(s), \int_0^s k(s, \tau, x_n(\tau)) d\tau\right) \right. \\
&\qquad \left.\left. - K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right)\right] ds \right\| \to 0,
\end{aligned}$$

as $n \to \infty$, i.e. $\phi_2$ is continuous.

Next, we prove that $\{\phi_2 x : x \in D_p\}$ is a family of equicontinuous functions. To see this we fix $t_1 > 0$ and let $t_2 > t_1$ and $\varepsilon > 0$, be enough small. Then

$$\begin{aligned}
&\|(\phi_2 x)(t_2) - (\phi_2 x)(t_1)\| \\
&\leq \|S_\alpha(t_2) - S_\alpha(t_1)\| \, \|x_0 - h(x)\| \\
&\quad + \int_0^{t_1-\varepsilon} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) - (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s)\| \, \|G(s, u(s))\| ds \\
&\quad + \int_{t_1-\varepsilon}^{t_1} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) - (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s)\| \, \|G(s, u(s))\| ds \\
&\quad + \int_{t_1}^{t_2} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s)\| \, \|G(s, u(s))\| ds \\
&\quad + \int_0^{t_1-\varepsilon} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) - (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s)\| \\
&\quad \times \left\| K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right) \right\| ds \\
&\quad + \int_{t_1-\varepsilon}^{t_1} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) - (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s)\| \\
&\quad \times \left\| K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right) \right\| ds \\
&\quad + \int_{t_1}^{t_2} \|(t_2 - s)^{\alpha-1} T_\alpha(t_2 - s)\| \, \left\| K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right) \right\| ds.
\end{aligned}$$

We see that $\|(\phi_2 x)(t_2) - (\phi_2 x)(t_1)\|$ tends to zero independently of $x \in D_p$ as $t_2 \to t_1$, with $\varepsilon$ sufficiently small since the compactness of $S_\alpha(t)$ for $t > 0$ (see [25]) implies the continuity of $S_\alpha(t)$ for $t > 0$ in $t$ in the uniform operator topology. Similarly, using the compactness of the set $h(D_p)$ we can prove that the function $\phi_2 x$, $x \in D_p$ are equicontinuous at $t = 0$. Hence, $\phi_2$ maps $D_p$ into a family of eqiucontinuous functions.

It remains to prove that $V(t) = \{(\phi_2 x)(t) : x \in D_p\}$ is relatively compact in $X$. $V(0)$ is relatively compact in $X$. Let $0 < t \leq a$ be fixed and $0 < \varepsilon < t$, arbitrary $\delta > 0$, for $x \in D_p$, we define

$$\begin{aligned}
(\phi_2^{\varepsilon,\delta} x)(t) &= \int_\delta^\infty \eta_\alpha(\theta) S(t^\alpha \theta)[x_0 - h(x)] d\theta \\
&\quad + \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta (t - s)^{\alpha-1} \eta_\alpha(\theta) S((t - s)^\alpha \theta) G(s, u(s)) d\theta ds \\
&\quad + \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta (t - s)^{\alpha-1} \eta_\alpha(\theta) S((t - s)^\alpha \theta) \\
&\quad \times \left[ K\left(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau\right) \right] d\theta ds \\
&= S(\varepsilon^\alpha \delta) \int_\delta^\infty \eta_\alpha(\theta) S(t^\alpha \theta - \varepsilon^\alpha \delta)[x_0 - h(x)] d\theta \\
&\quad + \alpha S(\varepsilon^\alpha \delta) \int_0^{t-\varepsilon} \int_\delta^\infty \theta (t - s)^{\alpha-1} \eta_\alpha(\theta) S((t - s)^\alpha \theta - \varepsilon^\alpha \delta) G(s, u(s)) d\theta ds
\end{aligned}$$

$$+\alpha S(\varepsilon^\alpha\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\varepsilon^\alpha\delta)$$
$$\times\left[K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right]d\theta ds.$$

Since $S(\varepsilon^\alpha\delta)$, $\varepsilon^\alpha\delta>0$ is a compact operator, then the set $V^{\varepsilon,\delta}(t)=\{(\phi_2^{\varepsilon,\delta}x)(t):x\in D_p\}$ is relatively compact in $X$ for every $\varepsilon$, $0<\varepsilon<t$ and for all $\delta>0$. Moreover, for every $x\in D_p$, we have

$$\begin{aligned}
&\left\|\left(\phi_2 x\right)(t)-\left(\phi_2^{\varepsilon,\delta}x\right)(t)\right\|\\
&\le\left\|\int_0^\delta \eta_\alpha(\theta)\,S(t^\alpha\theta)\left[x_0-h(x)\right]d\theta\right\|\\
&\quad+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)G\left(s,u(s)\right)d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)G\left(s,u(s)\right)d\theta\,ds\right.\\
&\quad\left.-\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)G\left(s,u(s)\right)d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)\left[K\left(s,x(s),\int_0^s k\left(s,\tau,x\left(\tau\right)\right)d\tau\right)\right]d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)\left[K\left(s,x(s),\int_0^s k\left(s,\tau,x\left(\tau\right)\right)d\tau\right)\right]d\theta\,ds\right.\\
&\quad\left.-\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)\left[K\left(s,x(s),\int_0^s k\left(s,\tau,x\left(\tau\right)\right)d\tau\right)\right]d\theta\,ds\right\|\\
&\le\left\|\int_0^\delta \eta_\alpha(\theta)\,S(t^\alpha\theta)\left[x_0-h(x)\right]\,d\theta\right\|\\
&\quad+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)G\left(s,u(s)\right)d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)G\left(s,u(s)\right)d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)\left[K\left(s,x(s),\int_0^s k\left(s,\tau,x\left(\tau\right)\right)d\tau\right)\right]d\theta\,ds\right\|\\
&\quad+\alpha\left\|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)\,S\left((t-s)^\alpha\theta\right)\left[K\left(s,x(s),\int_0^s k\left(s,\tau,x\left(\tau\right)\right)d\tau\right)\right]d\theta\,ds\right\|\\
&\le M\left[\|x_0\|+L_3\|x\|+L_3'\right]\int_0^\delta\eta_\alpha(\theta)\,d\theta\\
&\quad+\alpha M\left(\int_0^t(t-s)^{\alpha-1}g_p(s)ds\right)\int_0^\delta\theta\eta_\alpha(\theta)\,d\theta+\alpha M\left(\int_{t-\varepsilon}^t(t-s)^{\alpha-1}g_p(s)ds\right)\int_0^\infty\theta\eta_\alpha(\theta)\,d\theta\\
&\quad+\alpha M\left(\int_0^t(t-s)^{\alpha-1}q_p(s)ds\right)\int_0^\delta\theta\eta_\alpha(\theta)\,d\theta+\alpha M\left(\int_{t-\varepsilon}^t(t-s)^{\alpha-1}q_p(s)ds\right)\int_0^\infty\theta\eta_\alpha(\theta)\,d\theta
\end{aligned}$$

,

$$\leq M\left[\|x_0\| + L_3\|x\| + L_3'\right]\int_0^\delta \eta_\alpha(\theta)\,d\theta$$
$$+ \alpha M\left(\int_0^t (t-s)^{\alpha-1} g_p(s)ds\right)\int_0^\delta \theta\eta_\alpha(\theta)\,d\theta + \frac{\alpha M}{\Gamma(\alpha+1)}\left(\int_{t-\varepsilon}^t (t-s)^{\alpha-1} g_p(s)ds\right)$$
$$+ \alpha M\left(\int_0^t (t-s)^{\alpha-1} q_p(s)ds\right)\int_0^\delta \theta\eta_\alpha(\theta)\,d\theta + \frac{\alpha M}{\Gamma(\alpha+1)}\left(\int_{t-\varepsilon}^t (t-s)^{\alpha-1} q_p(s)ds\right)$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$, $t > 0$. Hence, the set $V(t)$, $t > 0$ is also relatively compact in $X$.

Thus, by Arzela-Ascoli Theorem $\phi_2$ is a compact operator. Those arguments enable us to conclude that $\phi = \phi_1 + \phi_2$, is a condensing map $D_p$, and by the fixed point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for $\phi$ on $D_p$. Therefore, the nonlocal Cauchy problem (1.1) has a mild solution, and the proof is completed.

## 4. Application

As an application of theorem, we shall consider the system (1.1) with control parameter such as:

$$\begin{aligned} &{}^cD^\alpha[x(t) + F(t, x(t), x(b_1(t)), \dots, x(b_m(t)))] + Ax(t) \\ &\quad = Cw(t) + G(t, x(t), x(a_1(t)), \dots, x(a_n(t))) + K\left(t, x(t), \int_0^t k(t,s,x(s))ds\right), \\ &\qquad\qquad t \in J = [0, a], \\ &x(0) + h(x) = x_0, \end{aligned} \tag{4.1}$$

where the control function $w(\cdot)$ is given in $L^2(J, W)$ – the Banach space of admissible control function with $W$ as a Banach space and $C$ is a bounded linear operator from $W$ into $X$. The mild solution of the system (4.1) is given by

$$\begin{aligned} x(t) = &S_\alpha(t)\left[x_0 + F(0, x(0), x(b_1(0)), \dots, x(b_m(0))) - h(x)\right] \\ &- F(t, x(t), x(b_1(t)), \dots, x(b_m(t))) \\ &+ \int_0^t (t-s)^{\alpha-1} AT_\alpha(t-s) F(s, x(s), x(b_1(s)), \dots, x(b_m(s)))\,ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) G(s, x(s), x(a_1(s)), \dots, x(a_n(s)))\,ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Cw(s)ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))\,d\tau\right)\right] ds, t \in [0,\ a] \end{aligned}$$

**Definition 4.1.** The system (4.1) is said to be controllable on the interval $J$ if for every $x_0, x_1 \in X$, there exists a control $w \in L^2(J, W)$ such that the solution $x(\cdot)$ of (4.1) satisfies $x(0) + h(x) = x_0$ and $x(a) = x_1$.

($A_5$) The linear operator $Q$ from $W$ into $X$ defined by

$$Qw = \int_0^a (a-s)^{\alpha-1} T_\alpha(a-s) Cw(s)ds$$

has an induced inverse operator $\tilde{Q}^{-1}$ which takes values in $L^2\left(J,W\right)/\ker Q$ and there exists a positive constant $M_1$ such that $\|C\tilde{Q}^{-1}\| \leq M_1$.

**Theorem 4.1.** *If the assumptions ($A_1$)-($A_5$) are satisfied then the system* (4.1) *is controllable on $J$ provided that*

$$L_0 = L_1\left[\left(M+1\right)M_0 + \frac{C_{1-\beta}\,\Gamma(1+\beta)a^{\alpha\beta}}{\beta\,\Gamma\left(1+\alpha\beta\right)}\right] < 1 \tag{4.2}$$

$$\begin{aligned} &M\left[M_0L_2 + L_3 + \frac{\alpha}{\Gamma\left(\alpha+1\right)}\left(\gamma_1+\gamma_2\right)\right] + M_0L_2 + \frac{C_{1-\beta}\,\Gamma(1+\beta)a^{\alpha\beta}}{\beta\,\Gamma\left(1+\alpha\beta\right)}L_2 \\ &\quad + \frac{MM_1}{\Gamma\left(\alpha+1\right)}\left[MM_0L_2 + ML_3 + M_0L_2 + \frac{C_{1-\beta}\,\Gamma(1+\beta)a^{\alpha\beta}}{\beta\,\Gamma\left(1+\alpha\beta\right)}L_2\right]a^{\alpha} \\ &\quad + \frac{\alpha\,M^2M_1a^{\alpha}\left(\gamma_1+\gamma_2\right)}{\left(\Gamma\left(\alpha+1\right)\right)^2} < 1. \end{aligned} \tag{4.3}$$

*Proof.* Using the assumption ($A_5$), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} w(t) = \tilde{Q}^{-1}\Big[&x_1 - S_\alpha(t)\{x_0 + F(0,v(0)) - h(x)\} + F(a,v(a)) \\ &- \int_0^a (a-s)^{\alpha-1}AT_\alpha(a-s)F(s,v(s))ds \\ &- \int_0^a (a-s)^{\alpha-1}T_\alpha(a-s) \\ &\times \left\{G(s,u(s)) + K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right\}ds\Big](t) \end{aligned}$$

We shall show that when using this control the operator

$$\begin{aligned} \left(\psi x\right)(t) = {}& S_\alpha\left(t\right)\left\{x_0 + F\left(0,v(0)\right) - h\left(x\right)\right\} - F\left(t,v\left(t\right)\right) \\ &+ \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F\left(s,v(s)\right)ds \\ &+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\ &\times \left\{Cw(s) + G(s,u(s)) + K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right\}ds, \quad t\in J \end{aligned}$$

has a fixed point $x(\cdot)$. Then this fixed point $x(\cdot)$ is a mild solution of the problem (4.1), and we can easily verify that $x(a) = \psi(x)(a) = x_1$. This means that the control $w$ steers the system from the initial state $x_0$ to $x_1$ in time $a$, which implies that the system is controllable. Our aim is to prove that there exists a positive number $p$ such that $\psi D_p \subseteq D_p$.

If this is not true, then for each positive number $p$, there exists a function $x_p(\cdot) \in D_p$, but $\psi x_p \notin D_p$, that is $\|\psi x_p(t)\| > p$ for some $t(p) \in [0,a]$, from

$$\begin{aligned} p &< \|\psi(x_p)(t)\| \\ &\leq M[\|x_0\| + M_0L_2(p+1) + (L_3p + L_3')] + M_0L_2(p+1) \\ &\quad + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}a^{\alpha\beta}L_2(p+1) + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}g_p(s)ds \\ &\quad + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}q_p(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha M M_1}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}[\|x_1\|+M\{\|x_0\|+M_0L_2(p+1)+(L_3p+L_3')\} \\
&+M_0L_2(p+1)+\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}a^{\alpha\beta}L_2(p+1) \\
&+\frac{\alpha M}{\Gamma(1+\alpha)}\int_0^a (a-\tau)^{\alpha-1}g_p(\tau)d\tau+\frac{\alpha M}{\Gamma(1+\alpha)}\int_0^a (a-\tau)^{\alpha-1}q_p(\tau)d\tau]ds
\end{aligned}
$$

Dividing on both sides by $p$ and taking the lower limit as $p\to+\infty$, we get

$$
\begin{aligned}
1 \le\, & MM_0L_2+ML_3+M_0L_2+\frac{C_{1-\beta}\,\Gamma(1+\beta)}{\beta\,\Gamma\,(1+\alpha\beta)}a^{\alpha\beta}L_2+\frac{\alpha\,M}{\Gamma\,(1+\alpha)}\,(\gamma_1+\gamma_2) \\
&+\frac{MM_1}{\Gamma\,(1+\alpha)}\left[MM_0L_2+ML_3+M_0L_2+\frac{C_{1-\beta}\,\Gamma(1+\beta)}{\beta\,\Gamma\,(1+\alpha\beta)}a^{\alpha\beta}L_2\right]a^{\alpha} \\
&+\frac{\alpha M^2M_1}{(\Gamma\,(1+\alpha))^2}\,(\gamma_1+\gamma_2)\,a^{\alpha}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left(MM_0L_2+ML_3+M_0L_2+\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}a^{\alpha\beta}L_2\right)\left(1+\frac{MM_1}{\Gamma(1+\alpha)}a^{\alpha}\right) \\
&\quad+\frac{\alpha M}{\Gamma\,(1+\alpha)}\,(\gamma_1+\gamma_2)\left(1+\frac{MM_1}{\Gamma\,(1+\alpha)}a^{\alpha}\right)\ge 1.
\end{aligned}
$$

However, this contradicts (4.3). Hence for positive number $p$, $\psi D_p\subseteq D_p$. In order to apply Sadovskii's fixed point theorem, we decompose $\psi=\psi_1+\psi_2$, where the operators $\psi_1$, $\psi_2$ are defined on $D_p$, by

$$
(\phi_1x)\,(t)=S_\alpha\,(t)\,F\,(0,v(0))-F\,(t,v\,(t))+\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F\,(s,v(s))\,ds
$$

and

$$
\begin{aligned}
&(\phi_2x)\,(t) \\
&=S_\alpha\,(t)\,[x_0-h\,(x)]+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)G\,(s,u(s))\,ds \\
&\quad+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)\right]ds \\
&\quad+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)C\tilde{Q}^{-1}\,[x_1-S_\alpha\,(a)\,\{x_0-h\,(x)+F\,(0,v(0))\}+F\,(a,v\,(a)) \\
&\quad-\int_0^a (a-s)^{\alpha-1}AT_\alpha(a-s)F(s,v(s))ds \\
&\quad-\int_0^a (a-s)^{\alpha-1}T_\alpha(a-s)\{G(s,u(s)) \\
&\quad+K(s,x(s,\int_0^s k(s,\tau,x(\tau))d\tau)\}ds](s)ds
\end{aligned}
$$

for $t\in J$. By similar manner as we have done it in Theorem 3.1, we can prove that $\psi_1$ verify a contraction condition and also verify that $\psi_2$ is a compact operator. Hence it is omitted. □

## 5. Conclusion

In this paper, the existences of the mild solutions of the neutral fractional integrodifferential equations with nonlocal initial conditions are discussed. We have used the fractional power of operators and the Sadovskii's fixed point theorem to establish the existence results. In the last, we have given an example to illustrate the application of the abstract results.

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