Journal of Fractional Calculus and Applications
Vol. 7(2) July 2016, pp. 88-100.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR COUPLED SYSTEMS OF FRACTIONAL Δ-DIFFERENCE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we establish the solvability of coupled systems of two-point fractional Δ-difference boundary value problems. To this aim we use the nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems for existence results and by imposing Lipschitzian conditions on nonlinearities uniqueness of solutions will be concluded. In this paper, Green functions play crucial role for linking considered fractional Δ-difference boundary value problems and fixed point techniques in relevant Banach spaces. At the end we present some numerical examples to illustrate the obtained main results.

1. Introduction

The theory of fractional calculus has been recognized in the recent decades as an effective tool for studying both theoretical and computational approaches of mathematical based systems, particularly memory dependent ones. In this way fractional differential equations that generalizes integer-order differential equations plays pioneering role. Every interested follower can find great number of research works dealing with nonlinear fractional boundary value problems, see [12],[13]. Although in view point of comparison, investigation about discrete fractional boundary value problems has very poor historical perspective, see for instance [2]-[5],[7].
F. Atici and P. Eloe in [3], considered the two-point fractional Δ-difference boundary value problem

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t = 1,2,...,b+1, \\ y(\nu-2) = 0, \ y(\nu+b+1) = 0, \end{cases} \tag{1}$$

where $1 < \nu \leq 2$ is a real number and, $b \geq 2$ is an integer and Δ^{ν} denotes fractional Δ-difference operator of order $\nu > 0$. $f : [\nu, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous. The authors using Guo-Krasnoselskii fixed point theorem obtained at least one positive solution for (1) on a cone in a Banach space.

2010 *Mathematics Subject Classification.* 34A08, 34B15, 34B18.
Key words and phrases. Discrete fractional calculus, Boundary value problems, Fixed point theorem, Positive solutions.
Submitted January 9, 2016.

In this paper we consider the coupled system of two-point fractional Δ-difference boundary value problems

$$\begin{cases} \begin{pmatrix} \Delta_{a^+}^{\alpha} u(t) \\ \Delta_{a^+}^{\alpha} v(t) \end{pmatrix} + \begin{pmatrix} f(t+\alpha-1, v(t+\alpha-1) \\ g(t+\alpha-1, u(t+\alpha-1)) \end{pmatrix} = 0, \\ \\ \begin{pmatrix} u(\alpha+a-2) \\ u(\alpha+b+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(\alpha+a-2) \\ v(\alpha+b+1) \end{pmatrix}, \end{cases} \tag{2}$$

such that $1 < \alpha \leq 2,\ t \in [a, b+1]_{\mathbb{N}_{\alpha-1}} = \{a, a+1, ..., b, b+1\},\ a, b \in \mathbb{Z}$ such that $a \geq 1, b \geq 2$ and $f, g : [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions. We will apply couple of different fixed point theorems for obtaining the same outcome as obtained in [3] . In addition for demonstrating uniqueness of solutions we apply so called Lipschitzian conditions to obtain a contraction mapping. In this position the organization of the paper can be stated as follows: in section 2, we give some standard definition and technical lemmas related to the discrete fractional calculus and some functional analysis. In section 3, the main existence and uniqueness results of the paper in theoretical manner will be concluded and in the last section illustrating obtained main results some numerical examples for each fixed point technique will be represented.

2. Technical Background

We divide this section into the couple of steps. First we give some necessary preliminaries of fractional Δ-difference calculus.

Definition 2.1. *Fractional falling function $t^{\underline{\alpha}}$ is defined by:*

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \in \mathbb{R}\backslash\{..., \alpha-3, \alpha-2, \alpha-1\},\ \alpha \in \mathbb{R}, \tag{3}$$

such that

(*i*) $t^{\underline{\alpha}} = 0$, *provided that* $\{t+1-\alpha\} \in \mathbb{Z}_{\leq 0},\ \alpha \in \mathbb{R}$,
(*ii*) $t^{\underline{0}} = 1$.

Lemma 2.2. [2] *Suppose all the following fractional falling functions are well defined. Then*

(P_1) $\Delta_t t^{\underline{\alpha}} = \alpha t^{\underline{\alpha-1}}$,
(P_2) $(t-\alpha)t^{\underline{\alpha}} = t^{\underline{\alpha+1}}$
(P_3) $\alpha^{\underline{\alpha}} = \Gamma(\alpha+1)$,

where $\alpha \in \mathbb{R}$ and denoted by Δ_t the forward difference operator with respect to the variable t.

Now we define fractional Δ-difference operators as below.

Definition 2.3. [1] *The left sided fractional Δ-sum of order $\alpha > 0$ for function f is defined as*

$$\Delta_{a^+}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s), \tag{4}$$

where $\sigma(s) = s+1$.

Definition 2.4. [1] *The left sided fractional* Δ*-difference of order* α *for function* f *is given by*

$$\Delta_{a^+}^{\alpha} f(t) = \Delta_t^n \Delta_{a^+}^{-(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \Delta_t^n \left(\sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{\underline{n-\alpha-1}} f(s) \right), \quad (5)$$

such that $n = [\alpha] + 1$ *and denoted by* ∇_t *the backward difference operator with respect to the variable* t.

Remark 2.5. *The fractional left and right sided* Δ*-sums of order* $\alpha > 0$*, defined by (4) have the following properties:*

(*i*) $\Delta_{a^+}^{-\alpha}$ *maps functions defined on* $\mathbb{N}_a$ *to functions defined on* $\mathbb{N}_{a+\alpha}$.
(*ii*) $\Delta_{a^+}^{\alpha}$ *maps functions defined on* $\mathbb{N}_a$ *to functions defined on* $\mathbb{N}_{a+(n-\alpha)}$,

where $\mathbb{N}_c = \{c, c+1, c+2, ...\}$.

Lemma 2.6. [8] *Assume that* f *is a real-valued function and* $\mu > 0,\ 0 \leq n-1 < \nu \leq n$. *Then*

(Q_1) $\Delta_{a^+}^{-\mu} \Delta_{a^+}^{-\nu} f(t) = \Delta_{a^+}^{-(\mu+\nu)} f(t) = \Delta_{a^+}^{-\nu} \Delta_{a^+}^{-\mu} f(t)$,
(Q_2) $\Delta_{a^+}^{-\nu}\ \Delta_{a^+}^{\nu} f(t) = f(t) + c_1(t-a)^{\underline{\nu-1}} + c_2(t-a)^{\underline{\nu-2}} + ... + c_n(t-a)^{\underline{\nu-n}}$, $c_i \in \mathbb{R},\ i = 1, 2, ..., n$.
(Q_3) $\Delta_{a^+}^{\nu}\ \Delta_{a^+}^{-\nu} f(t) = f(t)$.
(Q_4) $\Delta_{a^+}^{-\nu} (t-a)^{\underline{\nu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\underline{\mu+\nu}},\ \mu+\nu+1 \notin \mathbb{Z}_{\leq 0}$.

In this position we define relevant Banach spaces that will be covered the main purposes as below:

$$E = \mathfrak{B} \times \mathfrak{B}, \quad \text{(6a)}$$

$$\mathfrak{B} = \left(\left\{ x \middle| x : [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}, x(\alpha+a-2) = x(\alpha+b+1) = 0 \right\}, \|.\|_{\mathfrak{B}} \right), \quad \text{(6b)}$$

endowed with the norm

$$\|(x,y)\|_E = \|x\|_{\mathfrak{B}} + \|y\|_{\mathfrak{B}}, \quad \|x\|_{\mathfrak{B}} = \sup_{t \in [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}} |x(t)|. \quad \text{(6c)}$$

As stated above, we use the nonlinear alternative of the Leray-Schauder and Krasnoselskii -Zabreiko fixed point theorems for solvability verification of the coupled system (2). So we state these theorems as follows, respectively.

Theorem 2.7. [14],[10] *Let* C *be a convex subset of a Banach space,* U *be a open subset of* C *with* $0 \in U$. *Then every completely continuous map* $T : \overline{U} \to C$ *has at least one of the two following properties:*

(E_1) *There exist an* $u \in \overline{U}$ *such that* $Tu = u$.
(E_2) *There exist an* $v \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $v = \lambda T v$.

Theorem 2.8. [11],[9] *Let* X *be a Banach space. Assume that* $T : X \to X$ *is a completely continuous mapping. If* $L : X \to X$ *be a linear bounded mapping such that* 1 *is not an eigenvalue of* L *and*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0, \quad (7)$$

then T *has a fixed point in* X.

3. Main Results

Lemma 3.1. *Let* $1 < \alpha \leq 2,\ a, b \in \mathbb{Z},\ a \geq 1, b \geq 2$ *and* $h : [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$. $u(t)$ *solves the fractional* Δ*-difference boundary value problem*

$$\begin{cases} \Delta^{\alpha}_{a^+} u(t) + h(t+\alpha-1) = 0,\ 1 < \alpha \leq 2,\ t = a, a+1, ..., b, b+1, \\ \\ u(\alpha+a-2) = 0,\ u(\alpha+b+1) = 0, \end{cases} \tag{8}$$

if and only if $u(t)$ *be the unique solution of the fractional* Δ*-sum equation*

$$u(t) = \sum_{s=a}^{b+1} \mathcal{G}(t,s) h(s+\alpha-1), \tag{9}$$

where
$\mathcal{G}(t,s)$

$$= \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\underline{\alpha-1}}(\alpha+b-s)^{\underline{\alpha-1}}}{(\alpha+b-a+1)^{\underline{\alpha-1}}} - (t-\sigma(s))^{\underline{\alpha-1}}, & a \leq s+\alpha-1 \leq t \leq b+1, \\ \\ \dfrac{(t-a)^{\underline{\alpha-1}}(\alpha+b-s)^{\underline{\alpha-1}}}{(\alpha+b-a+1)^{\underline{\alpha-1}}}, & a \leq t \leq s+\alpha-1 \leq b+1. \end{cases} \tag{10}$$

Proof. Using property (Q_2) in Lemma 2.6, one can reduce the fractional Δ- difference equation (8) to the fractional Δ-sum equation

$$u(t) = -\Delta^{\alpha}_{a^+} h(t+\alpha-1) + c_1(t-a)^{\underline{\alpha-1}} + c_2(t-a)^{\underline{\alpha-2}}. \tag{11}$$

Boundary condition $u(\alpha+a-2) = 0$ and property (C_3) of Lemma 2.2, ensure that $c_2 = 0$. Afterward, boundary condition $u(\alpha+b+1) = 0$ leads us to the unique coefficient c_1 given by:

$$c_1 = \frac{\Delta^{-\alpha}_{a^+} h(t+\alpha-1)\Big|_{t=\alpha+b+1}}{(\alpha+b-a+1)^{\underline{\alpha-1}}}. \tag{12}$$

Substituting c_1, c_2 into (11), it follows that

$$\begin{aligned} u(t) &= \frac{-1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1) \\ &\quad + \frac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)(\alpha+b-a+1)^{\underline{\alpha-1}}} \sum_{s=a}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} h(s+\alpha-1) \\ &= \frac{-1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \left[(t-\sigma)^{\underline{\alpha-1}} - \frac{(t-a)^{\underline{\alpha-1}}(\alpha+b-s)^{\underline{\alpha-1}}}{(\alpha+b-a+1)^{\underline{\alpha-1}}} \right] h(s+\alpha-1) \\ &\quad + \frac{(t-a)^{\underline{\alpha-1}}}{\Gamma(\alpha)(\alpha+b-a+1)^{\underline{\alpha-1}}} \sum_{s=t-\alpha+1}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} h(s+\alpha-1) \\ &= \sum_{s=a}^{b+1} \mathcal{G}(t,s) h(t+\alpha-1). \end{aligned}$$

The proof is completed. $\square$

Lemma 3.2. *The Green function $\mathcal{G}(t,s)$ defined by (10) possesses the following properties:*

(*i*) $\mathcal{G}(t,s) > 0$ *for* $t \in [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$ *and* $s \in [a, b+1]_{\mathbb{N}}$.

(*ii*)

$$\max_{\substack{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\\ s\in[a,b+1]_{\mathbb{N}}}} \mathcal{G}(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \dfrac{b-a+2\alpha}{b-a+2}\dfrac{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)}, & a+b: even, \\ \\ \dfrac{b-a+2\alpha+1}{b-a+3}\dfrac{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+1\right)}, & a+b: odd. \end{cases} \tag{13}$$

(*iii*) *There exist a positive number* $\gamma \in (0,1)$ *such that for* $s \in [a, b+1]_{\mathbb{N}}$

$$\min_{t\in[\frac{b-a+1+\alpha}{4}, \frac{3(b-a+1+\alpha)}{4}]} \mathcal{G}(t,s) \geq \gamma \max_{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} \mathcal{G}(t,s).$$

Proof. Taking into account that the proof of items (i) and (iii) are same as in Theorem 3.2 [3], we joust prove the item (ii). To this aim using property (C_1) of Lemma 2.2, a simple calculation yields us

$$\begin{aligned} \Delta_t\mathcal{G}(t,s) < 0, &\qquad a \leq s+\alpha-1 \leq t \leq b+1, \\ \Delta_t\mathcal{G}(t,s) > 0, &\qquad a \leq t \leq s+\alpha-1 \leq b+1. \end{aligned} \tag{14}$$

Thus we conclude that for $t \in [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}},\ s \in [a, b+1]_{\mathbb{N}}$,

$$\max_{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} \mathcal{G}(t,s) = \mathcal{G}(s+\alpha-1, s), \quad s \in [a, b+1]_{\mathbb{N}}.$$

Equivalently it follows that

$$\max_{\substack{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\\ s\in[a,b+1]_{\mathbb{N}}}} \mathcal{G}(t,s) = \max_{s\in[a,b+1]_{\mathbb{N}}} \frac{\mathcal{G}_2(s+\alpha-1,s)}{\Gamma(\alpha)}, \tag{15}$$

where

$$\mathcal{G}_2(t,s) = \frac{(t-a)^{\underline{\alpha-1}}(\alpha+b-s)^{\underline{\alpha-1}}}{(\alpha+b-a+1)^{\underline{\alpha-1}}}, \quad a \leq t \leq s+\alpha-1 \leq b+1.$$

On the other hand since

$$\Delta\mathcal{G}_2(s+\alpha-1,s) = \frac{(1-\alpha)\Gamma(s+\alpha-a)\Gamma(\alpha+b-s)}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}(s-a+2)!(b-s+2)!}[2s-(a+b)],$$

we conclude that $\mathcal{G}(s+\alpha-1,s)$ is increasing for $s<\frac{a+b}{2}$ and is decreasing for $s>\frac{a+b}{2}$. Therefore it follows that

$$\begin{aligned}\max_{s\in[a,b+1]_{\mathbb{N}}}\mathcal{G}(s+\alpha-1,s)&=\frac{\mathcal{G}_2\left(\dfrac{a+b}{2}+\alpha-1,\dfrac{a+b}{2}\right)}{\Gamma(\alpha)}\\&=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha}{b-a+2}\frac{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)},\end{aligned}\tag{16}$$

if $a+b$ is even and

$$\begin{aligned}\max_{s\in[a,b+1]_{\mathbb{N}}}\mathcal{G}(s+\alpha-1,s)&=\frac{\mathcal{G}_2\left(\dfrac{a+b+1}{2}+\alpha-1,\dfrac{a+b+1}{2}\right)}{\Gamma(\alpha)}\\&=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha+1}{b-a+3}\frac{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+1\right)},\end{aligned}\tag{17}$$

if $a+b$ is odd. This completes the proof. □

We define the operator $\mathcal{K}:C\subset E\to E$ as

$$\mathcal{K}(u,v)(t)=\left(\left(\mathcal{K}_1v\right)(t),\left(\mathcal{K}_2u\right)(t)\right),\quad t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\tag{18}$$

where

$$\begin{aligned}\left(\mathcal{K}_1\right)v(t)&=\sum_{s=a}^{b+1}\mathcal{G}(t,s)f(s+\alpha-1,v(s+\alpha-1)),\\&\qquad\qquad t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\\\left(\mathcal{K}_2\right)u(t)&=\sum_{s=a}^{b+1}\mathcal{G}(t,s)g(s+\alpha-1,u(s+\alpha-1)),\end{aligned}\tag{19}$$

and

$$\begin{aligned}C=C_1\oplus C_2&=\left\{(u,v)\in E\,\middle|\,w(t)\geq0,\ t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}},\right.\\&\qquad\left.\min_{t\in\Delta}w(t)\geq\gamma\|w\|_{\mathfrak{B}},\ w\in\{u,v\}\right\},\\C_1&=\left\{(0,v)\in E\,\middle|\,v(t)\geq0,\ t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}},\quad\min_{t\in\Delta}v(t)\geq\gamma\|v\|_{\mathfrak{B}}\right\},\\C_2&=\left\{(u,0)\in E\,\middle|\,u(t)\geq0,\ t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}},\quad\min_{t\in\Delta}u(t)\geq\gamma\|u\|_{\mathfrak{B}}\right\},\\\Delta&=\left[\frac{b-a+1+\alpha}{4\alpha},\frac{3(b-a+1+\alpha)}{4\alpha}\right].\end{aligned}\tag{20}$$

Remark 3.3. *The operator $\mathcal{K}$ can be written in the form of the following matrix equation*

$$\mathcal{K}(u,v)(t)=\\ \sum_{s=a}^{b+1}\begin{pmatrix}\mathcal{G}(t,s)\\ \mathcal{G}(t,s)\end{pmatrix}\begin{pmatrix} f(s+\alpha-1,v(s+\alpha-1)) & 0\\ 0 & g(s+\alpha-1,u(s+\alpha-1))\end{pmatrix}. \tag{21}$$

So one can deduce that the coupled system of two-point fractional Δ-difference boundary value problems (2) solves uniquely the matrix equation (21).

Remark 3.4. *We notice that the item (iii) in Lemma 3.2, ensures that $\mathcal{K}(C)\subset C$.*

Relied on the above preliminaries, now we are ready to establish the existence of at least one positive solution for the coupled system of two-point fractional Δ-difference boundary value problems (2) via the nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems stated above.

3.1. **Existence.** Let us turn to the nonlinear alternative of Leray-Schauder fixed point theorem (Theorem (2.7)). To prove the existence at least one solution for coupled system (2), we shall prove that just the first part of the nonlinear alternative of Leray-Schauder fixed point theorem (E_1) is satisfied. To this aim we need to impose some necessary conditions on both nonlinearities f and g of discrete coupled system (2). As stated above $f,g:[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times\mathbb{R}\to\mathbb{R}^+$ are positive continuous functions. So based on these functions we consider the following hypotheses.

Hypotheses 3.5. *There exist positive continuous functions $\phi_i:[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\to\mathbb{R}^+,\ i=1,2$ and $\psi_i:\mathbb{R}\to\mathbb{R}^+,\ i=1,2$ with ψ_i increasing, such that*

(H_1) $f(z,v)\leq\phi_1(z)\psi_1(|v|),\ (z,v)\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times\mathbb{R}$;

(H_2) $g(z,u)\leq\phi_2(z)\psi_2(|u|),\ (z,u)\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times\mathbb{R}$.

Remark 3.6. *Finitely discrete nature of the summation operator $\mathcal{K}$ given by (18)-(19) in combination with the Hypotheses 3.5, indicate that $\mathcal{K}$ is trivially completely continuous.*

Theorem 3.7. *Assume that the hypotheses (H_1) and (H_2) are satisfied. If there exist positive constant ϱ such that*

$$\frac{1}{M}>\frac{1}{\varrho}\left[\psi_1(\varrho)\sum_{a}^{b+1}\phi_1(s+\alpha-1)+\psi_2(\varrho)\sum_{a}^{b+1}\phi_2(s+\alpha-1)\right], \tag{22}$$

where $M\in\{M_{even},M_{odd}\}$, in which

$$M_{even}=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha}{b-a+2}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a}{2}+1\right)},$$

$$M_{odd}=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha+1}{b-a+3}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)},$$

then the coupled system of two-point fractional Δ-difference boundary value problems (2) has at least one positive solution in C.

Proof. Let us consider the following coupled system of fractional λ-parametric Δ-difference boundary value problems

$$\begin{cases} \begin{pmatrix} \Delta^{\alpha}_{a^+}u(t) \\ \Delta^{\beta}_{a^+}v(t) \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} f(t+\alpha-1, v(t+\alpha-1) \\ g(t+\alpha-1, u(t+\alpha-1)) \end{pmatrix} = 0, \\ \\ \begin{pmatrix} u(\alpha+a-2) \\ u(\alpha+b+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(\alpha+a-2) \\ v(\alpha+b+1) \end{pmatrix}, \end{cases} \tag{23}$$

for $\lambda \in (0,1)$. So solving (23) is equivalent to solving the fixed point problem $(u,v) = \lambda\mathcal{K}(u,v)$ where $\mathcal{K}$ is given by (18)-(19). Define

$$\Omega = \Big\{(u,v) \in C|\ \|u\|_{\mathfrak{B}}, \|v\|_{\mathfrak{B}} < \frac{\varrho}{2}\Big\}. \tag{24}$$

We have to prove that $(u,v) \neq \lambda\mathcal{K}(u,v)$ for $(u,v) \in \partial\Omega$ and $\lambda \in (0,1)$. To this aim suppose on contrary that there exists $(u,v) \in \partial\Omega$ such that $(u,v) = \lambda\mathcal{K}(u,v) = \lambda(\mathcal{K}_1 v, \mathcal{K}_2 u)$. So for $\lambda \in (0,1)$ it follows that

$$\begin{aligned} \|u\|_{\mathfrak{B}} &= \lambda \sup_{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} |\mathcal{K}_1 v| \\ &\le \sup_{t\in[\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} \left|\sum_{s=a}^{b+1} \mathcal{G}(t,s) f(s+\alpha-1, v(s+\alpha-1))\right| \\ &\le M \sum_{s=a}^{b+1} f(s+\alpha-1, v(s+\alpha-1)) \\ &\le M\psi_1(\varrho) \sum_{s=a}^{b+1} \phi_1(s+\alpha-1). \end{aligned} \tag{25}$$

Therefore it follows that

$$\varrho \le 2M\psi_1(\varrho) \sum_{s=a}^{b+1} \phi_1(s+\alpha-1). \tag{26}$$

Similarly one has from $v = \lambda\mathcal{K}_2 u$ that

$$\varrho \le 2M\psi_2(\varrho) \sum_{s=a}^{b+1} \phi_2(s+\alpha-1). \tag{27}$$

From inequalities (26) and (27), it follows that

$$\frac{2}{M} \le \frac{2}{\varrho}\left[\psi_1(\varrho)\sum_{a}^{b+1}\phi_1(s+\alpha-1) + \psi_2(\varrho)\sum_{a}^{b+1}\phi_2(s+\alpha-1)\right],$$

which contradicts with (22). This contradiction demonstrates that according to the Theorem 2.7, (E_2) is not satisfied. Therefore we conclude that there exists an $(u,v) \in \overline{\Omega}$ such that $(v,u) = \mathcal{K}(u,v)$. Equivalently the coupled system of two-point fractional Δ-difference boundary value problems (2) has at least one positive solution in C. The proof is completed. □

In this position, we consider the Krasnoselskii-Zabreiko fixed point theorem (Theorem 2.8). According to this fixed point theorem, we shall approximate the fractional Δ-sum operator $\mathcal{K}(u,v)$ defined by (18) with a linear operator that dos not admit 1 as its eigenvalue. Therefore once again considering the nonlinearities f and g defined above, the following hypotheses will help us to reach to the above mentioned linear operators.

Hypotheses 3.8. *Suppose there exit positive continuous functions* $\Theta_i : [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}^+,\ i=1,2$ *such that the following hypotheses hold:*

(S_1) $\displaystyle\lim_{\|v\|_{\mathfrak{B}}\to\infty} \frac{f(z,v)}{|v|} = \Theta_1(z), \quad (z,v) \in [\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R};$

(S_2) $\displaystyle\lim_{\|u\|_{\mathfrak{B}}\to\infty} \frac{g(z,u)}{|u|} = \Theta_2(z), \quad (z,u) \in [\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R}.$

Theorem 3.9. *Assume that the hypotheses* (S_1) *and* (S_2) *hold. Let*

$$\sum_{s=a}^{b+1} \mathcal{G}(t,s) < \max\left\{\|\Theta_1\|_{\mathfrak{B}},\ \|\Theta_2\|_{\mathfrak{B}}\right\}^{-1}, \quad t \in [\alpha+a-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}. \tag{28}$$

Then the coupled system of two-point fractional Δ*-difference boundary value problems (2) has at least one positive solution in* C.

Proof. Consider the linear bounded mappings $L_i : C_i \to E,\ i=1,2$ given by

$$\begin{aligned} L_1 v(t) &= \sum_{s=a}^{b+1} \mathcal{G}(t,s) v(s+\alpha-1)\Theta_1(s+\alpha-1), \\ L_2 u(t) &= \sum_{s=a}^{b+1} \mathcal{G}(t,s) u(s+\alpha-1)\Theta_2(s+\alpha-1). \end{aligned} \tag{29}$$

Obviously one can derive

$$\begin{aligned} \|L_1 v\|_{\mathfrak{B}} &\le \sum_{s=a}^{b+1} \mathcal{G}(t,s)\|v\|_{\mathfrak{B}}\|\Theta_1\|_{\mathfrak{B}} \\ &\le \sum_{s=a}^{b+1} \mathcal{G}(t,s)\|v\|_{\mathfrak{B}} \max\left\{\|\Theta_1\|_{\mathfrak{B}},\ \|\Theta_2\|_{\mathfrak{B}}\right\} < \|v\|_{\mathfrak{B}}, \end{aligned} \tag{30}$$

which illustrates that 1 can not be an eigenvalue of L_1. Similarly L_2 can not admit 1 as its eigenvalue. Therefore if we define $L(u,v) = (L_1 v, L_2 u)$, then $(1,1)$ can not be the eigenvalue of L. Now considering the limit approach of the hypotheses (S_1) and (S_2), for arbitrary $\epsilon > 0$ we have

$$\begin{aligned} \|\mathfrak{T}_1 v - L_1 v\|_{\mathfrak{B}} &\le \sum_{s=a}^{b+1} \mathcal{G}(t,s)\|f(t+\alpha-1,v) - |v|\Theta_1\|_{\mathfrak{B}} \\ &\le \sum_{s=a}^{b+1} \mathcal{G}(t,s)\epsilon\|v\|_{\mathfrak{B}} < (b-a+2)M\epsilon\|v\|_{\mathfrak{B}}. \end{aligned} \tag{31}$$

Similarly we have

$$\|\mathfrak{T}_2 u - L_2 u\|_{\mathfrak{B}} < (b-a+2)M\epsilon\|u\|_{\mathfrak{B}}. \tag{32}$$

Note that M appeared in (31 is defined as Theorem 3.7. Using the inequalities (31) and (32), we conclude that

$$\begin{aligned}\lim_{(u,v)\to(\infty,\infty)} \frac{\|\mathcal{K}(u,v)-L(u,v)\|_E}{\|(u,v)\|_E} &\le \lim_{\substack{\|u\|_{\mathfrak{B}}\to\infty \\ \|v\|_{\mathfrak{B}}\to\infty}} \left\{\frac{\|\mathcal{K}_1 v - L_1 v\|_{\mathfrak{B}}}{\|v\|_{\mathfrak{B}}} + \frac{\|\mathcal{K}_2 u - L_2 u\|_E}{\|u\|_{\mathfrak{B}}}\right\} \\ &< 2\epsilon(b-a+2)M,\end{aligned} \tag{33}$$

for arbitrary $\epsilon > 0$. Thereby Theorem 2.8 ensures that the matrix equation $\mathcal{K}(u,v)$ defined by (18)-(19) has at least one fixed point in C. Equivalently the coupled system of two-point fractional Δ-difference boundary value problems (2) has at least one positive solution in C. The proof is completed. □

After proving the existence at least one positive solution for the coupled system (2), we are going to present the uniqueness result for coupled system (2) as follows.

3.2. **Uniqueness.**

Theorem 3.10. *Assume that the nonlinearities $f(z,v)$ and $g(z,u)$ both are Lipschitzian in v and u, respectively, that is there exist real parameters $l_i > 0,\ i = 1,2$ such that for $(u_i, v_i) \in C,\ i = 1,2$*

$$|f(z,v_1) - f(z,v_2)| \le l_1\|v_1 - v_2\|_{\mathfrak{B}}, \quad |g(z,u_1) - g(z,u_2)| \le l_2\|u_1 - u_2\|_{\mathfrak{B}}. \tag{34}$$

Then the coupled system of two-point fractional Δ-difference boundary value problems (2) has exactly one positive solution in C provided that

$$(b-a+2)l_i M < 1, \quad i = 1,2, \tag{35}$$

where $M \in \{M_{even}, M_{odd}\}$, in which

$$M_{even} = \frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha}{b-a+2}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a}{2}+1\right)},$$

$$M_{odd} = \frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha+1}{b-a+3}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)}.$$

Proof. First, let us recall once again the fractional Δ-sum operator $\mathcal{K}(u,v)$.

$$\mathcal{K}(u,v)(t) = \left((\mathcal{K}_1 v)(t), (\mathcal{K}_2 u)(t)\right), \quad t \in [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \tag{36}$$

where

$$\begin{aligned}(\mathcal{K}_1)\, v(t) &= \sum_{s=a}^{b+1} \mathcal{G}(t,s) f(s+\alpha-1, v(s+\alpha-1)), \\ &\qquad\qquad\qquad t \in [\alpha+a-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \\ (\mathcal{K}_2)\, u(t) &= \sum_{s=a}^{b+1} \mathcal{G}(t,s) g(s+\alpha-1, u(s+\alpha-1)).\end{aligned} \tag{37}$$

Therefore, using the assumption (34), one has

$$\begin{aligned}\|\mathcal{K}_1(v_1)-\mathcal{K}_1(v_2)\|_{\mathfrak{B}} &= \Big\|\sum_{s=a}^{b+1}\mathcal{G}(t,s)\Big\{f(s+\alpha-1,v_1)-f(s+\alpha-1,v_2)\Big\}\Big\|_{\mathfrak{B}} \\ &\le M\sum_{s=a}^{b+1}\|f(s+\alpha-1,v_1)-f(s+\alpha-1,v_2)\|_{\mathfrak{B}} \\ &\le (b-a+2)Ml_1\|v_1-v_2\|_{\mathfrak{B}}.\end{aligned} \tag{38}$$

So we have

$$\|\mathcal{K}_1(v_1)-\mathcal{K}_1(v_2)\|_{\mathfrak{B}} \le (b-a+2)Ml_1\|v_1-v_2\|_{\mathfrak{B}}. \tag{39}$$

Similarly, one may derive the following

$$\|\mathcal{K}_2(u_1)-\mathcal{K}_2(u_2)\|_{\mathfrak{B}} \le (b-a+2)Ml_2\|u_1-u_2\|_{\mathfrak{B}}. \tag{40}$$

Using the inequalities (39) and (40), we conclude that

$$\begin{aligned}\|\mathcal{K}(u_1,v_1)-\mathcal{K}(u_2,v_2)\|_E &= \|\mathcal{K}_1(v_1)-\mathcal{K}_1(v_2)\|_{\mathfrak{B}}+\|\mathcal{K}_2(u_1)-\mathcal{K}_2(u_2)\|_{\mathfrak{B}} \\ &\le (b-a+2)l_1M\|v_1-v_2\|_{\mathfrak{B}}+(b-a+2)l_2M\|u_1-u_2\|_{\mathfrak{B}} \\ &< \|v_1-v_2\|_{\mathfrak{B}}+\|u_1-u_2\|_{\mathfrak{B}} = \|(u_1,v_1)-(u_1,v_1)\|_E.\end{aligned} \tag{41}$$

Thus we have proved that the fractional Δ-sum operator $\mathcal{K}(u,v)$ is a contraction mapping. So the Banach fixed point theorem ensures that the discrete fixed point equation $(v,u)=\mathcal{K}(u,v)$ and equivalently the coupled system of two-point fractional Δ-difference boundary value problems (2) has exactly one positive solution in C. This completes the proof. □

4. Numerical Examples

Example 4.1. *Consider the following coupled system of two-point fractional ∇-difference boundary value problems*

$$\begin{cases}\begin{pmatrix}\Delta_{0^+}^{\frac{3}{2}}u(t)\\ \\ \Delta_{0^+}^{\frac{3}{2}}v(t)\end{pmatrix}+\begin{pmatrix}f(t+\frac{1}{2},v(t+\frac{1}{2})\\ \\ g(t+\frac{1}{2},u(t+\frac{1}{2}))\end{pmatrix}=0,\\ \\ \begin{pmatrix}u(\frac{1}{2})\\ \\ u(\frac{19}{2})\end{pmatrix}=\begin{pmatrix}0\\ \\ 0\end{pmatrix}=\begin{pmatrix}v(\frac{1}{2})\\ \\ v(\frac{19}{2})\end{pmatrix}.\end{cases} \tag{42}$$

Considering the system (42), one can recognize that $\alpha=\frac{3}{2}$ and $a=1,\ b=8$. Taking into account that $a+b$ is odd, we have $M=M_{odd}$. Thereby choosing $\varrho=5$ and taking

$$\begin{aligned} f(t+\alpha-1,v) &= \underbrace{\exp(-(t+\alpha-1))}_{\phi_1}\left(\underbrace{1+\frac{|v|}{3}}_{\psi_1}\right), \\ g(t+\alpha-1,u) &= \underbrace{\exp(-(t+\alpha-1))}_{\phi_2}\underbrace{\ln(1+|u|)}_{\psi_2},\end{aligned} \tag{43}$$

it is easy to check that the conditions (H_1) *and* (H_2) *hold. Consequently a direct calculation demonstrates that*

$$\frac{1}{M} \approx 5.18485 > 0.082042 \approx \frac{1}{\varrho}\left[\psi_1(\varrho)\sum_{a}^{b+1}\phi_1(s+\alpha-1) + \psi_2(\varrho)\sum_{a}^{b+1}\phi_2(s+\alpha-1)\right].$$

Therefore Theorem 3.7 implies that the coupled system (42) admits at least one positive solution in C*. On the other hand, since*

$$\begin{aligned}|f(t+\alpha-1,v_1) - f(t+\alpha-1,v_2)| &\leq 0.027362\|v_1 - v_2\|_{\mathfrak{B}},\\ |g(t+\alpha-1,u_1) - g(t+\alpha-1,u_2)| &\leq 0.082085\|u_1 - u_2\|_{\mathfrak{B}},\end{aligned} \tag{44}$$

so, choosing $l_1 = 0.03$ *and* $l_2 = 0.09$ *we deduce that*

$$(b-a+2)Ml_1 \approx 0.052075 < 1, \quad (b-a+2)Ml_2 \approx 0.150225 < 1.$$

Therefore, Theorem 3.10 ensures the existence of a unique positive solution for coupled system (42) in C*.*

Example 4.2. *Let us consider the coupled system (42). Suppose that*

$$\begin{aligned}f(t+\alpha-1,v) &= \underbrace{(10t+\alpha-1)^{-2}}_{\Theta_1}v,\\ g(t+\alpha-1,u) &= \underbrace{\exp(-(10t+\alpha-1))}_{\Theta_2}u, \quad t \in [1,10]_{\mathbb{N}}.\end{aligned} \tag{45}$$

Thus hypotheses (S_1) *and* (S_2) *are satisfied. Since* $a+b$ *is even, so we have* $M \approx 0.018368$*. On the other hand a simple computation shows that* $\|\Theta_1\|_{\mathfrak{B}} = 0.007561$ *and* $\|\Theta_2\|_{\mathfrak{B}} \approx 0.000027$*. Thus we have*

$$\sum_{s=2}^{9}\mathcal{G}(t,s) \approx 0.165313 < 132.25 \approx \max\left\{\|\Theta_1\|_{\mathfrak{B}},\ \|\Theta_2\|_{\mathfrak{B}}\right\}^{-1}.$$

Thereby Theorem 3.9 implies that the coupled system (42) has at least one positive solution in C*. On the other hand, since*

$$\begin{aligned}|f(t+\alpha-1,v_1) - f(t+\alpha-1,v_2)| &\leq 0.00907\|v_1 - v_2\|_{\mathfrak{B}},\\ |g(t+\alpha-1,u_1) - g(t+\alpha-1,u_2)| &\leq 0.000028\|u_1 - u_2\|_{\mathfrak{B}},\end{aligned} \tag{46}$$

so, choosing $l_1 = 0.01$ *and* $l_2 = 0.00003$*, clearly* $(b-a+2)Ml_i < 1,\ i = 1,2$*. Thus Theorem 3.10 implies the existence of a unique positive solution for coupled system (42) in* C*.*

Acknowledgment

The authors are indebted to the anonymous referees for comments to improve the presentation of the paper and suggesting the uniqueness results for original version of the manuscript.

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