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N- FRACTIONAL CALCULUS OF GENERALIZED DOUBLE ZETA FUNCTION

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Abstract. Many authors investigated fractional derivative operators on some special functions such as using the extended Riemann - Liouville fractional derivative operator on extended Beta function (cf. [1]), and the theory of fractional integration operators (of Marichev- Saigo- Maeda type) on the Mittag - Leffler function with four parameters (cf. [2]) and the so- called pathway fractional integration operator due to Nair with a finite product of Bessel functions of the first kind (cf. [4]). In this paper ,some new theorems have been established by applying N- fractional Differintegration on the generalized double Zeta function and on its real and imaginary parts. Also some particular cases have been deduced

1. Introductions and definitions

Bin- Saad in [3] defined the generalized double Zeta function by

$$\zeta_{\lambda}^{\mu}\left(x,y;z,a\right)=\sum_{m=0}^{\infty}(\mu)_m\Phi(y,z,a+\lambda m)\frac{x^m}{m!}. \tag{1}$$

For
$|x|<1,\quad |y|<1,\quad \mu\in\ \mathbb{C},\mu\neq 0,-1,-2,...,\quad a\in\mathbb{C},a\neq-(n+\lambda m),\quad \lambda\in\mathbb{C},\lambda\neq 0,\quad z\in\mathbb{C},\quad n,m\in\mathbb{N}.$
where $\Phi(y,z,a)$ is Hurwitz - Lerch Zeta function(cf. [5],p.27, eq.1.11(1)) , defined by

$$\Phi(y,z,a)=\sum_{n=0}^{\infty}\frac{y^n}{(a+n)^z},\quad |y|<1,\quad a\in\mathbb{C},\quad a\neq 0,-1,-2,.... \tag{2}$$

A further generalization of the Hurwitz - Lerch Zeta function (cf. [6], p. 100, eq.(1.5)), which is called the generalized Hurwitz - Lerch Zeta function $\Phi_{\mu}^{*}\left(x,z,a\right)$ is defined by

$$\Phi_{\mu}^{*}\left(x,z,a\right)=\sum_{n=0}^{\infty}\frac{(\mu)_n x^n}{(a+n)^z n!},\quad |x|<1,\quad z,\quad \mu\in\mathbb{C},\quad a\neq 0,-1,-2,.... \tag{3}$$

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where $(\mu)_n$ is the Pochhammers symbol defined by
$(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \mu(\mu+1)(\mu+2)...(\mu+n-1)$
Also in [3], the following relationships are deduced :

$$\zeta_\lambda^\mu\left(0,1;z,a\right)=\zeta_1^1\left(1,0;z,a\right)=\zeta\left(z,a\right), \tag{4}$$

$$\zeta_\lambda^\mu\left(0,y;z,a\right)=\Phi\left(y,z,a\right), \tag{5}$$

$$\zeta_1^\mu\left(x,0;z,a\right)=\Phi_\mu^*\left(x,z,a\right), \tag{6}$$

Where $\zeta\left(z,a\right)$ is the generalized Zeta function.
In [8] Nishomoto defined the fractional Differintegration of arbitrary orderν(derivatives of order ν for $\nu>0$, and integrals of order$-\nu$ for $\nu<0$, for the function f with respect to z , by the following :Let

$$D=\left\{D_-,D_+\right\},C=\left\{C_-,C_+\right\}$$

C_- be a curve along the cut joining two points z and $-\infty+iIm(z)$
C_+ be a curve along the cut joining two points z and $\infty+iIm(z)$.

where D contains the point over the curve C. Moreover , let $f = f(z)$ be a regular function in $D\left(z\in D\right)$

$$(f)_\nu = {}_c\left(f\right)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_c \frac{f(\zeta)}{\left(\zeta-z\right)^{\nu+1}}\,d\zeta\,,(\nu\notin\mathbb{Z}^-), \tag{7}$$

$$(f)_{-m} = \lim_{\nu\to-m}\left(f\right)_\nu\,,(m\in\mathbb{Z}^+), \tag{8}$$

where

$$-\pi\le arg(\zeta-z)\le\pi, \qquad \text{for} \quad C_-$$

$$0\le arg(\zeta-z)\le 2\pi, \qquad \text{for} \quad C_+$$

$$\zeta\ne z\,,\ z\in\mathbb{C},\ \nu\in\mathbb{R}, \quad \Gamma(.) \qquad \text{is gamma function.}$$

The fractional calculus operator (Nishiomoto's operator) is defined as

$$N^\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_c \frac{d\zeta}{\left(\zeta-z\right)^{\nu+1}}\ ,(\nu\notin\mathbb{Z}^-), \tag{9}$$

with

$$N^{-m} = \lim_{\nu\to-m} N^\nu,\ (m\in\mathbb{Z}^+), \tag{10}$$

Here some useful results (cf. [7] and [8]),which are used in our investigations :
for $a\ne 0$, $(z,\,\nu\in\mathbb{C})$,

$$\left(e^{az}\right)_\nu = a^\nu e^{az}, \tag{11}$$

$$\left(\cos az\right)_\nu = a^\nu\cos\left(az+\frac{\pi}{2}\nu\right), \tag{12}$$

$$\left(\sin az\right)_\nu = a^\nu\sin\left(az+\frac{\pi}{2}\nu\right), \tag{13}$$

2. Main Results

Theorem 1:

$$\left[\zeta_\lambda^\mu\left(x,y;z,a\right)\right]_{\alpha(z)}=e^{i\pi\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)\right]^\alpha\frac{1}{(a+\lambda m+n)^z},\tag{14}$$

$\alpha\in\mathbb{R},\ a>a_1$ (a_1 is the one such that $a_1 log\, a_1=1$)

Proof:

Using (1.2)in (1.1) , we get

$$\begin{aligned}\zeta_\lambda^\mu\left(x,y;z,a\right)&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}(a+\lambda m+n)^{-z}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}e^{log(a+\lambda m+n)^{-z}}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}e^{zlog(a+\lambda m+n)^{-1}},\quad(a)\end{aligned}$$

Now applying Nishomoto's operator N^α defined by (9) to both sides of (a),and using (11) ,we get

$$\begin{aligned}\left[\zeta_\lambda^\mu\left(x,y;z,a\right)\right]_{\alpha(z)}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[e^{zlog(a+\lambda m+n)^{-1}}\right]_{\alpha(z)}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)^{-1}\right]^\alpha e^{zlog(a+\lambda m+n)^{-1}}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[-log(a+\lambda m+n)\right]^\alpha\frac{1}{(a+\lambda m+n)^z}\\&=e^{i\pi\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)\right]^\alpha\frac{1}{(a+\lambda m+n)^z}\end{aligned}$$

Which ends the proof.

Theorem 2:

$$\{\left[\zeta_\lambda^\mu\left(x,y;z,a\right)\right]_{\alpha(z)}\}_{\beta(z)}=\{\left[\zeta_\lambda^\mu\left(x,y;z,a\right)\right]_{\beta(z)}\}_{\alpha(z)}=\{\left[\zeta_\lambda^\mu\left(x,y;z,a\right)\right]\}_{(\alpha+\beta)(z)}\tag{15}$$

$\alpha\ \beta\in\mathbb{R}$, $a>a_1$ (a_1 is the one such that $a_1 log\, a_1=1$)

Proof: This theorem easily can be proved by using theorem (2)

.

Theorem 3:

$$\left[\zeta_\lambda^\mu\left(x,y;u+iv,a\right)\right]_{\alpha(u)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)^{-1}\right]^\alpha\frac{1}{(a+\lambda m+n)^z},\tag{16}$$

$$\left[\zeta_{\lambda}^{\mu}\left(x,y;u+iv,a\right)\right]_{\beta(v)}$$
$$= e^{i\pi\frac{\beta}{2}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)^{-1}\right]^{\beta} \frac{1}{(a+\lambda m+n)^{z}}\,, \tag{17}$$

$\alpha\ ,\beta \in \mathbb{R}\ ,\ a > a_1$ (a_1 is the one such that $a_1 log\, a_1 = 1$)

Proof:
Equation(16)can be proved as follows:

$$\begin{aligned}\left[\zeta_{\lambda}^{\mu}\left(x,y;z,a\right)\right]_{\alpha(z)} &= \left[\zeta_{\lambda}^{\mu}\left(x,y;u+iv,a\right)\right]_{\alpha(u)}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[(a+\lambda m+n)^{-(u+iv)}\right]_{\alpha(u)}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{log(a+\lambda m+n)^{-iv}} \left[e^{log(a+\lambda m+n)^{-u}}\right]_{\alpha(u)}\end{aligned}$$

Now using (11) , we get:

$$\begin{aligned}\left[\zeta_{\lambda}^{\mu}\left(x,y;z,a\right)\right]_{\alpha(u)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{iv log(a+\lambda m+n)^{-1}} \left[log(a+\lambda m+n)^{-1}\right]^{\alpha} e^{u log(a+\lambda m+n)^{-1}}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)^{-1}\right]^{\alpha} e^{log(a+\lambda m+n)^{-(u+iv)}}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)^{-1}\right]^{\alpha} (a+\lambda m+n)^{-(u+iv)}\end{aligned}$$

put $u+iv=z$, then we get the required result.
The proof of (17) can be done in the same wayas in (16).

Theorem 4:
Let $\mathrm{Re}\{\zeta_{\lambda}^{\mu}\left(x,y;z,a\right)\} = w(u,v) = w$ and $\mathrm{Im}\{\zeta_{\lambda}^{\mu}\left(x,y;z,a\right)\} = t(u,v) = t$
we have

$$\begin{aligned}w_{\alpha(u)} = e^{i\pi\alpha} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)\right]^{\alpha} \frac{1}{(a+\lambda m+n)^{u}}\\ \times \cos\left[v log(a+\lambda m+n)^{-1}\right],\end{aligned} \tag{18}$$

$$\begin{aligned}w_{\beta(v)} = e^{i\pi\beta} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)\right]^{\beta} \frac{1}{(a+\lambda m+n)^{u}}\\ \times \cos\left[v log(a+\lambda m+n)^{-1} + \frac{\pi}{2}\beta\right],\end{aligned} \tag{19}$$

$$\begin{aligned}t_{\alpha(u)} = e^{i\pi\alpha} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)\right]^{\alpha} \frac{1}{(a+\lambda m+n)^{u}}\\ \times \sin\left[v log(a+\lambda m+n)^{-1}\right],\end{aligned} \tag{20}$$

$$t_{\beta(v)} = e^{i\pi\beta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[log(a+\lambda m+n)\right]^{\beta} \frac{1}{(a+\lambda m+n)^u} \times \sin\left[vlog(a+\lambda m+n)^{-1} + \frac{\pi}{2}\beta\right], \tag{21}$$

$\alpha\ ,\beta \in \mathbb{R}\ ,\ a > a_1$ (a_1 is the one such that $a_1 log\, a_1 = 1$)

Proof:
first we have to prove (18)

$$\begin{aligned}
\zeta_{\lambda}^{\mu}(x,y;z,a) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} (a+\lambda m+n)^{-z} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} (a+\lambda m+n)^{-(u+iv)} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{log(a+\lambda m+n)^{-u}} e^{log(a+\lambda m+n)^{-iv}} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{ulog(a+\lambda m+n)^{-1}} \{\cos\left[vlog(a+\lambda m+n)^{-1}\right] \\
&+ i\sin\left[vlog(a+\lambda m+n)^{-1}\right]\},
\end{aligned} \tag{22}$$

From (22) it is clear that

$$\mathrm{Re}\{\zeta_{\lambda}^{\mu}(x,y;z,a)\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{ulog(a+\lambda m+n)^{-1}} \cos\left[vlog(a+\lambda m+n)^{-1}\right] = w, \tag{23}$$

and

$$\mathrm{Im}\{\zeta_{\lambda}^{\mu}(x,y;z,a)\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{ulog(a+\lambda m+n)^{-1}} \sin\left[vlog(a+\lambda m+n)^{-1}\right] = t, \tag{24}$$

Using (11), from (23), we have

$$w_{\alpha(u)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} \left[e^{ulog(a+\lambda m+n)^{-1}}\right]_{\alpha(u)} \cos\left[vlog(a+\lambda m+n)^{-1}\right], \tag{25}$$

Now using (11) on (25), we get the result.
To prove (19),from (23), we have

$$w_{\beta(v)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_m x^m y^n}{m!} e^{ulog(a+\lambda m+n)^{-1}} \{\cos\left[vlog(a+\lambda m+n)^{-1}\right]\}_{\beta(v)}, \tag{26}$$

Now using (12) on (26), we get the result.
The proof of (20) and (21) can be deduced from (24) by using (25) and (13), in the same manner as in the proof of (18)and (19) .

Theorem 5:

$$\left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]_{\alpha(u)} - (-i)^{\alpha}\left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]_{\alpha(v)} = 0, \tag{27}$$

where $z = u + iv$, $\mathrm{Re}(z) > 1$, α , u , $v \in \mathbb{R}$, $a > a_1$ (a_1 is the one such that $a_1 log\, a_1 = 1$)

Proof:
we have for $z = u + iv$, $\mathrm{Re}(z) > 1$, and by using theorem 3,

$$i^{\alpha}\left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]_{\alpha(u)} - \left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]_{\alpha(v)}$$

$$= i^{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)^{-1}\right]^{\alpha}\frac{1}{(a+\lambda m+n)^z}$$

$$-i^{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m y^n}{m!}\left[log(a+\lambda m+n)^{-1}\right]^{\alpha}\frac{1}{(a+\lambda m+n)^z}$$

$$= 0, \quad (b)$$

. multiplying both sides of (b) by $(-i)^{\alpha}$ we get the theorem.
Corollary:

$$\frac{\partial^n\left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]}{\partial u^n} - (-i)^n\frac{\partial^n\left[\zeta_{\lambda}^{\mu}(x,y;z,a)\right]}{\partial v^n} = 0, \tag{28}$$

Proof:
In theorem 5, put $\alpha = n \in \mathbb{Z}^+$, , we get the result.

3. Special cases

In theorem 1, put $x = 0$, $y = 1$, $\mu = \lambda = 1$, and using (4) , we get

$$\zeta_{\alpha}(z,a) = e^{i\pi\alpha}\sum_{n=0}^{\infty}\left[log(a+n)\right]^{\alpha}\frac{1}{(a+n)^z}, \tag{29}$$

Which is the result given by Nishimoto in [9].
Also in theorem 1, put $x = 0$, , and using (5) , we get the result

$$\left[\Phi(y;z,a)\right]_{\alpha(z)} = e^{i\pi\alpha}\sum_{n=0}^{\infty}y^n\left[log(a+n)\right]^{\alpha}\frac{1}{(a+n)^z}, \tag{30}$$

where $\Phi(y;z,a)$ is the function defined by (2).
In theorem 1, put $y = 0$, $\lambda = 1$, and using (6) , we get

$$\left[\Phi_{\mu}^{*}(x;z,a)\right]_{\alpha(z)} = e^{i\pi\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_m x^m}{m!}\left[log(a+m)\right]^{\alpha}\frac{1}{(a+m)^z}, \tag{31}$$

In theorem 3 put $x = 0$,using (5) and $y = 0$, $\lambda = 1$,using (6), respectively ,we get the results

$$[\Phi\,(y;z,a)]_{\alpha(u)} = \sum_{n=0}^{\infty} \left[log(a+n)^{-1}\right]^{\alpha} \frac{y^n}{(a+n)^z}, \tag{32}$$

$$[\Phi\,(y;z,a)]_{\beta(v)} = e^{i\beta\frac{\pi}{2}} \sum_{n=0}^{\infty} \left[log(a+n)^{-1}\right]^{\beta} \frac{y^n}{(a+n)^z}, \tag{33}$$

$$\left[\Phi_{\mu}^{*}\,(x;z,a)\right]_{\alpha(u)} = \sum_{m=0}^{\infty} \left[log(a+m)^{-1}\right]^{\alpha} \frac{(\mu)_m x^m}{m!(a+m)^z}, \tag{34}$$

$$\left[\Phi_{\mu}^{*}\,(x;z,a)\right]_{\beta(v)} = e^{i\beta\frac{\pi}{2}} \sum_{m=0}^{\infty} \left[log(a+m)^{-1}\right]^{\beta} \frac{(\mu)_m x^m}{m!(a+m)^z}, \tag{35}$$

In theorem 3 put $x = 0$, $y = 1$, $\mu = \lambda = 1$, ,using (4) and then writing $u = x$, $v = y$, respectively ,we get

$$\zeta_{\alpha(x)}\,(z,a) = \sum_{n=0}^{\infty} \left[log(a+n)^{-1}\right]^{\alpha} \frac{1}{(a+n)^z}, \tag{36}$$

$$\zeta_{\beta(y)}\,(z,a) = \sum_{n=0}^{\infty} \left[log(a+n)^{-1}\right]^{\beta} \frac{1}{(a+n)^z}, \tag{37}$$

Which is the result given by Nishimoto in [9].
In theorem 4,if we put $x = 0$,using (5) we get

$$Q_{\alpha(u)} = e^{i\pi\alpha} \sum_{n=0}^{\infty} [log(a+n)]^{\alpha} \frac{y^n}{(a+n)^u} \cos\left[vlog(a+n)^{-1}\right], \tag{38}$$

$$Q_{\beta(v)} = e^{i\pi\beta} \sum_{n=0}^{\infty} [log(a+n)]^{\beta} \frac{y^n}{(a+n)^u} \cos\left[vlog(a+n)^{-1} + \frac{\pi}{2}\beta\right], \tag{39}$$

$$S_{\alpha(u)} = e^{i\pi\alpha} \sum_{n=0}^{\infty} [log(a+n)]^{\alpha} \frac{y^n}{(a+n)^u} \sin\left[vlog(a+n)^{-1}\right], \tag{40}$$

$$S_{\beta(v)} = e^{i\pi\beta} \sum_{n=0}^{\infty} [log(a+n)]^{\beta} \frac{y^n}{(a+n)^u} \sin\left[vlog(a+n)^{-1} + \frac{\pi}{2}\beta\right], \tag{41}$$

where $Q = Q(u,v) = \mathrm{Re}\{\Phi(y;z,a)\}$, $S = S(u,v) = \mathrm{Im}\{\Phi(y;z,a)\}$
Also In theorem 4 , if we put $x = 0, y = 1$, $\mu = \lambda = 1$, using (5), we get the results given by Nishimoto in [9].

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