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EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study existence of positive solutions to the three-point fractional boundary value problem (FBVP for short)

$$\begin{cases} D_{0^+}^{\alpha}u(t)+\lambda f(t,u(t))=0,\ 0<t<1,\ 2<\alpha\leq 3 \\ u(0)=u'(0)=0 \\ u'(1)-\mu u'(\eta)=\displaystyle\int_0^1 g(s)u'(s)ds \end{cases}$$

where λ is a positive parameter, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differential operator of order $\alpha\in(2,3]$, $\eta\in(0,1)$, $\mu\geq 0$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is a continuous function and $g:(0,1)\to(0,+\infty)$ is a continuous increasing function and $\int_0^1 s^{\alpha-2}g(s)ds<+\infty$. Existence results are obtained by means of Krasnosel'skii 's fixed point theorem.

1. Introduction

We investigate in this paper, existence of positive solutions to the three-point fractional boundary value problem (FBVP for short),

$$D_{0^+}^{\alpha}u(t)+\lambda f(t,u(t))=0,\ t\in(0,1)\,, \tag{1}$$

$$u(0)=u'(0)=0,\ u'(1)-\mu u'(\eta)=\int_0^1 g(s)u'(s)ds, \tag{2}$$

where$D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differential operator of order $\alpha\in(2,3]$, λ,μ,η are real parameters with $\lambda>0$, $\mu\geq 0$, $\eta\in(0,1)$, $f:[0,1]\times[0,+\infty)\to(0,+\infty)$ is a continuous function and $g:(0,1)\to(0,+\infty)$ is a continuous function.

By a positive solution to FBVP (1)-(2), we mean a function $u\in C[0,1]$ such that u is positive in $(0,1)$ having the fractional derivative $D_{0^+}^{\alpha}u$ in $C[0,1]$ and u satisfies all equations in (1)-(2).

Throughout this paper, we assume that the following condition hold.

$$\int_0^1 s^{\alpha-2}g(s)ds<+\infty \text{ and } d=1-\mu\eta^{\alpha-2}-\int_0^1 s^{\alpha-2}g(s)ds>0. \tag{3}$$

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It is well known that fractional boundary value problems play a very important role in both theories and applications. Recently, existence of positive solutions for nonlinear three-point fractional boundary value problems has been studied by many authors by using a nonlinear alternative of the Leray-Schauder, coincidence degree theory, fixed point index theory, fixed point theorems in cones and so on. We refer the reader to ([1, 2, 3, 7, 8, 19, 20, 21, 23]) and references therein. However, all of these papers are concerned with problems with three-point boundary conditions, for example

$$
\begin{aligned}
&u(0) = 0,\ u(1) = \beta u(\eta),\\
&u(0) = u(1) = 0,\ u'(1) = \mu u'(\eta),\\
&u(0) = u'(0) = 0,\ u'(1) - \mu u'(\eta) = \lambda,\\
&u(0) - \beta u'(0) = 0,\ \alpha u(\eta) = u(1)\\
&\alpha u(0) - \beta u'(0) = 0,\ u'(\eta) + u'(1) = 0\\
&u'(0) = 0,\ u'(1) = \lambda u'(\eta)\\
&u(0) = 0, u'(0) = u'(1) = \alpha u'(\eta), \text{ etc...}
\end{aligned}
$$

Since many of physical systems can better be described by integral boundary conditions, integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Hence, boundary value problems with integral boundary conditions constitute a very important class of problems. Also, note that integral boundary conditions include multi-point and nonlocal boundary value problems as special cases, see ([4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 22]) and references therein. The aim of this paper is to establish simple criteria for existence of at least one positive solution for (1)-(2). The paper is organized as follows. Essentially in section 2, we present the framework in which FBVP (1)-(2), is formulated in a fixed point equation. Section 3 is devoted to the main results and their proofs and is ended by illustrative examples. The main tool of this paper is the cone compression and expansion principale in a Banach space of norm type, known also by the Krasnosels'kii's fixed point theorem in a cone.

2. Preliminaries

We begin this section by reminding the reader the following compression and expansion of a cone principale in a Banach space, known also to be the Krasnosels'kii's Theorem.

Theorem 2.1. [17] *Let X be a Banach space and $P \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \longrightarrow P$ be a completely continuous operator such that, either*

a) *$\|Tu\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_2$; or*

b) *$\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Now, let us recall some basic facts related to the theory of fractional differential equations. Let β be a positive real number, the Riemann-Liouville fractional integral of order β of a function $f : (0, +\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \tag{4}$$

where $\Gamma(\beta)$ is the gamma function, provided that the right side is pointwise defined on $(0,+\infty)$. For example, we have for any real $\sigma>-1$, $I_{0^+}^{\beta}t^{\sigma}=\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)}t^{\sigma+\beta}$.

The Riemann-Liouville fractional derivative of order β, of a continuous function $f:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0^+}^{\beta}f(t)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{f(s)}{(t-s)^{\beta-n+1}}ds, \tag{5}$$

where $n=[\beta]+1$, $[\beta]$ denotes the integer part of the number β, provided that the right side is pointwise defined on $(0,\infty)$. As a basic example, we quote for $\sigma>-1$, $D_{0^+}^{\beta}t^{\sigma}=\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\beta+1)}t^{\sigma-\beta}$. Thus, if $u\in C(0,1)\cap L^1(0,1)$, then the fractional differential equation $D_{0^+}^{\beta}u(t)=0$ has $u(t)=\sum_{i=1}^{i=[\beta]+1}c_it^{\beta-i}$, $c_i\in\mathbb{R}$, as unique solution and if u has a fractional derivative of order β in $C(0,1)\cap L^1(0,1)$, then

$$I_{0^+}^{\beta}D_{0^+}^{\beta}u(t)=u(t)+\sum_{i=1}^{i=[\beta]+1}c_it^{\beta-i},\ c_i\in\mathbb{R}. \tag{6}$$

For a detailled presentation on fractional differential equations see [16, 18]

Now, let us introduce some functions needeed for the fixed point formulation of FBVP (1)-(2). Set for $(t,s)\in[0,1]\times[0,1]$

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1},\ 0\le s\le t\le 1\\ t^{\alpha-1}(1-s)^{\alpha-2},\ 0\le t\le s\le 1\end{cases} \tag{7}$$

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-2},\ 0\le s\le t\le 1\\ t^{\alpha-2}(1-s)^{\alpha-2},\ 0\le t\le s\le 1\end{cases} \tag{8}$$

and

$$H(t,s)=G(t,s)+\frac{\mu t^{\alpha-1}}{d}G_1(\eta,s). \tag{9}$$

Lemma 2.2. *For all $(t,s)\in[0,1]\times[0,1]$, we have*

P1) $0\le G_1(t,s)\le\frac{1}{\Gamma(\alpha)}t^{\alpha-2}(1-s)^{\alpha-2}$,

P2) $t^{\alpha-1}G(1,s)\le G(t,s)\le G(1,s),\ (t,s)\in[0,1]\times[0,1]$,

P3) $t^{\alpha-1}H(1,s)\le H(t,s)\le H(1,s),\ (t,s)\in[0,1]\times[0,1]$,

Proof.P1) is obvious and P3) is easily obtained from P2). Thus, we have to prove P2). Let $(t,s)\in[0,1]\times[0,1]$, we distinguish two cases.

Case 1: $0\le s\le t\le 1$. In this case, set

$$\theta_s(t)=t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}$$

and note that

$$\begin{aligned}\frac{d}{dt}\theta_s(t) &= (\alpha-1)\,t^{\alpha-2}(1-s)^{\alpha-2}-(\alpha-1)\,(t-s)^{\alpha-2}\\ &= (\alpha-1)\left((t-ts)^{\alpha-2}-(t-s)^{\alpha-2}\right)\ge 0.\end{aligned}$$

Thus, we have

$$\frac{G(t,s)}{G(1,s)}=\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{s(1-s)^{\alpha-2}}=\frac{\theta_s(t)}{\theta_s(1)}\le 1$$

and

$$\frac{G(t,s)}{G(1,s)}\ge\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-ts)^{\alpha-1}}{s(1-s)^{\alpha-2}}=t^{\alpha-1}.$$

Case 2: $0 \leq t \leq s \leq 1$. In this case we have

$$t^{\alpha-1} \leq \frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{s(1-s)^{\alpha-2}} = \frac{t^{\alpha-1}}{s} \leq \frac{s^{\alpha-1}}{s} = s^{\alpha-2} \leq 1.$$

This complete the proof of Lemma 2.2. ■

Let us introduce now, some spaces, cones and operators. In all this paper E is the Banach space of all continuous function defined on $[0,1]$ endowed with its sup-norm, designed by $\|\cdot\|$. E^+ is the cone of E, constituted of all nonnegative function and K is the cone defined by

$$K = \left\{u \in E : u(t) \geq t^{\alpha-1} \|u\| \text{ for all } t \in [0,1]\right\}. \tag{10}$$

Let $L : E \to E$ be the linear operator defined for $u \in E$ by

$$Lu(t) = \int_0^1 H(t,s)u(s)ds + \frac{t^{\alpha-1}}{d} \int_0^1 g(s) \left(\int_0^1 G_1(s,\tau)\, u(\tau) d\tau \right) ds$$

and $F : E^+ \to E^+$ the Nymetski associated with the nonlinearity f. Clearly, we have that L is continuous, $L(E^+) \subset E^+$ and F is bounded (maps bounded sets into bounded sets).

Lemma 2.3. *Assume that Hypothesis (3) holds and let $h \in C[0,1]$ be a given function. Then $\phi = Lh$ is the unique solution to the FBVP*

$$D_{0^+}^{\alpha} u(t) + h(t) = 0,\ \alpha \in (2,3],\ t \in (0,1), \tag{11}$$

$$u(0) = u'(0) = 0,\ u'(1) - \mu u'(\eta) = \int_0^1 g(s)u'(s)ds. \tag{12}$$

Proof. Using (6), we obtain from Equation (11) that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. We have then from (12), $c_2 = c_3 = 0$ and

$$c_1 = \frac{1}{d\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-2} h(s)ds - \mu \int_0^{\eta} (\eta-s)^{\alpha-2} h(s)ds - \psi \right],$$

where

$$\psi = \int_0^1 g(s) \left(\int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau \right) ds.$$

Therefore, the unique solution of (11)-(12) is

$$\begin{aligned}
&u(t) = -\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1}h(s)ds + \tfrac{t^{\alpha-1}}{d\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
&-\tfrac{\mu t^{\alpha-1}}{d\Gamma(\alpha)}\textstyle\int_0^\eta (\eta-s)^{\alpha-2}h(s)ds - \tfrac{t^{\alpha-1}}{d\Gamma(\alpha)}\int_0^1 g(s)\left(\int_0^s (s-\tau)^{\alpha-2}h(\tau)d\tau\right)ds \\
&= -\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1}h(s)ds + \tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
&+\tfrac{(1-d)t^{\alpha-1}}{d\Gamma(\alpha)}\textstyle\int_0^1 (1-s)^{\alpha-2}h(s)ds - \tfrac{\mu t^{\alpha-1}}{d\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-2}h(s)ds \\
&-\tfrac{t^{\alpha-1}}{d\Gamma(\alpha)}\textstyle\int_0^1 g(s)\left(\int_0^s (s-\tau)^{\alpha-2}h(\tau)d\tau\right)ds \\
&= -\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1}h(s)ds + \tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
&+\tfrac{\mu\eta^{\alpha-2}t^{\alpha-1}}{d\Gamma(\alpha)}\textstyle\int_0^1 (1-s)^{\alpha-2}h(s)ds - \tfrac{\mu t^{\alpha-1}}{d\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-2}h(s)ds \\
&+\tfrac{t^{\alpha-1}}{d\Gamma(\alpha)}\left[\textstyle\int_0^1 s^{\alpha-2}g(s)ds\left(\int_0^1 (1-s)^{\alpha-2}h(s)ds - \int_0^s (s-\tau)^{\alpha-2}h(\tau)d\tau\right)ds\right] \\
&= \tfrac{1}{\Gamma(\alpha)}\left[\textstyle\int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds - \int_0^t (t-s)^{\alpha-1}h(s)ds\right] \\
&+\tfrac{\mu t^{\alpha-1}}{d\Gamma(\alpha)}\left[\textstyle\int_0^1 \eta^{\alpha-2}(1-s)^{\alpha-2}h(s)ds - \int_0^\eta (\eta-s)^{\alpha-2}h(s)ds\right] \\
&+\tfrac{t^{\alpha-1}}{d\Gamma(\alpha)}\left[\textstyle\int_0^1 g(s)\left(\int_0^1 s^{\alpha-2}(1-\tau)^{\alpha-2}h(\tau)d\tau - \int_0^s (s-\tau)^{\alpha-2}h(\tau)d\tau\right)ds\right] \\
&= \textstyle\int_0^1 G(t,s)h(s)ds + \tfrac{\mu t^{\alpha-1}}{d}\int_0^1 G_1(\eta,s)h(s)ds + \tfrac{t^{\alpha-1}}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)h(\tau)d\tau\right)ds \\
&= \textstyle\int_0^1 H(t,s)h(s)ds + \tfrac{t^{\alpha-1}}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)h(\tau)d\tau\right)ds.
\end{aligned}$$

The proof is complete. ■

Lemma 2.4. *Assume that Hypothesis (3) holds and let $T_\lambda = \lambda LF$. We have then*
i) *T_λ is completely continuous,*
ii) *$T(E^+) \subset K$ and*
iii) *$u \in K$ is a solution to FBVP (1)-(2) if and only if u is a fixed point of T.*

Proof.i) This is due to the fact that the operator $u \to \frac{t^{\alpha-1}}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)u(\tau)d\tau\right)ds$ is one dimensional and continuous and the function H is uniformly continuous on $[0,1]\times[0,1]$.

ii) Let $u \in E^+$, we have from P3) in Lemma 2.2 that

$$\begin{aligned}
T_\lambda u(t) &= \lambda\int_0^1 H(t,s)f(s,u(s))ds + \frac{\lambda t^{\alpha-1}}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\geq \lambda t^{\alpha-1}\left(\int_0^1 H(1,s)f(s,u(s))ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right) \\
&\geq t^{\alpha-1}\|T_\lambda u\|
\end{aligned}$$

showing that $T_\lambda u \in K$.

iii) This is due to Lemma 2.3. ■

3. Main results

The statement of main results and their proofs need to introduce the following notations. Set for $\nu = 0$ or $+\infty$

$$f_\nu = \lim_{u\to\nu}\inf\left(\min_{t\in[0,1]}\tfrac{f(t,u)}{u}\right) \quad f^\nu = \lim_{u\to\nu}\sup\left(\max_{t\in[0,1]}\tfrac{f(t,u)}{u}\right)$$

$$\Lambda_\nu = \begin{cases} (\Gamma_0 f_\nu)^{-1} & \text{if } 0 < f_\nu < +\infty \\ 0 & \text{if } f_\nu = +\infty \\ +\infty & \text{if } f_\nu = 0 \end{cases} \qquad \Lambda^\nu = \begin{cases} (\Gamma_\infty f^\nu)^{-1} & \text{if } 0 < f^\nu < +\infty \\ +\infty & \text{if } f^\nu = 0 \\ 0 & \text{if } f^\nu = +\infty \end{cases}$$

where

$$\Gamma_\infty = \int_0^1 H(1,s)ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)\,d\tau\right)ds \text{ and}$$
$$\Gamma_0 = \int_0^1 s^{\alpha-1}H(1,s)ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 \tau^{\alpha-1}G_1(s,\tau)\,d\tau\right)ds$$

Theorem 3.1. *Assume that Hypothesis (3) holds and $\Lambda_{+\infty} < \Lambda^0$. Then the FBVP (1)-(2) has at least one positive solution for all $\lambda \in \left(\Lambda_{+\infty}, \Lambda^0\right)$.*

Proof. Let $\lambda \in \left(\Lambda_{+\infty}, \Lambda^0\right)$ and let $\varepsilon > 0$ be such that $\lambda < \left(\left(f^0+\varepsilon\right)\Gamma_1\right)^{-1}$. There exists $r_0 > 0$ such that $f(.,x) \le \left(f^0+\varepsilon\right)x$ for all $x \in [0, r_0]$ and $t \in [0,1]$. Thus, if $u \in K$ is such that $\|u\| = r_0$, we have

$$\begin{aligned}\|T_\lambda u\| &\le \lambda\left(f^0+\varepsilon\right)r_0\left[\int_0^1 H(1,s)ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)\,d\tau\right)ds\right]\\ &= \lambda\left(f^0+\varepsilon\right)r_0\Gamma_\infty \le r_0 = \|u\|.\end{aligned}$$

Now, let $M \in (0, f_\infty)$ be such that $\lambda > (M\Gamma_0)^{-1}$, there exists $c > 0$ such that $f(.,x) \ge Mx - c$ for all $x \ge 0$ and $t \in [0,1]$. At this stage, we claim that there exists r_∞ large such that $\|T_\lambda u\| \ge \|u\|$, for all $u \in K \cap \partial B(0, r_\infty)$. Indeed, if this is not true and for all $n \in \mathbb{N}$, there exists $u_n \in K \cap \partial B(0,n)$ such that $\|T_\lambda u_n\| \le \|u_n\|$, we have then,

$$\begin{aligned}&\|T_\lambda u_n\| + \|\lambda L(c)\| = \|T_\lambda u_n + \lambda L(c)\| \ge T_\lambda u_n(1) + \lambda L(c)(1)\\ &\ge \lambda\left[\int_0^1 H(1,s)Mu_n(s)ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)\,Mu_n(\tau)d\tau\right)ds\right]\\ &\ge \lambda M\left[\int_0^1 H(1,s)s^{\alpha-1}ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)\,\tau^{\alpha-1}d\tau\right)ds\right]\|u_n\|\\ &= \lambda M\|u_n\|\Gamma_0\end{aligned}$$

leading to

$$1 + \frac{\|T_\lambda u_n\|}{\|u_n\|} \ge \frac{\|T_\lambda u_n\|}{\|u_n\|} + \frac{\|\lambda L(c)\|}{\|u_n\|} \ge \lambda M\Gamma_0.$$

in which letting $n \to \infty$, yields the contradiction $1 \ge \lambda M\Gamma_0 > 1$.

At the end, choosing $\Omega_\nu = \{u \in E : \|u\| < r_\nu\}$ for $\nu = 0$ or ∞, we obtain from a) of Theorem 2.1 that T admits a fixed point $u \in K$ with $r_0 \le \|u\| \le r_\infty$, then from iii) in Lemma 2.4, u is a positive solution to FBVP (1)-(2). The proof is complete. ■

Theorem 3.2. *Assume that Hypothesis (3) holds and $\Lambda_0 < \Lambda^{+\infty}$. Then the FBVP (1)-(2) has at least one positive solution for all $\lambda \in (\Lambda_0, \Lambda^{+\infty})$.*

Proof. Let $\lambda \in (\Lambda_0, \Lambda^{+\infty})$ and let $m \in (0, f_0)$ be such that $\lambda > (m\Gamma_0)^{-1}$. There exists $r_0 > 0$ such that $f(.,x) \ge mx$ for all $x \in [0, r_0]$ and $t \in [0,1]$. Thus, if $u \in K$ is such that $\|u\| = r_0$, we have

$$\begin{aligned}\|T_\lambda u\| \ge T_\lambda u(1) &\ge \lambda m r_0\left[\int_0^1 H(1,s)s^{\alpha-1}ds + \frac{1}{d}\int_0^1 g(s)\left(\int_0^1 G_1(s,\tau)\,\tau^{\alpha-1}d\tau\right)ds\right]\\ &= \lambda m r_0\Gamma_0 \ge r_0 = \|u\|.\end{aligned}$$

Now, let $\varepsilon > 0$ be such that $\lambda < ((f^\infty+\varepsilon)\Gamma_\infty)^{-1}$, there exists $c > 0$ such that $f(.,x) \le (f^\infty+\varepsilon)x + c$ for all $x \ge 0$ and $t \in [0,1]$. At this stage, we prove that there exists r_∞ large such that $\|T_\lambda u\| \le \|u\|$ for all $u \in K \cap \partial B(0, r_\infty)$. By the contrary, suppose that for all $n \in \mathbb{N}$, there exists $u_n \in K \cap \partial B(0,n)$ such that $\|T_\lambda u_n\| \geq \|u_n\|$, we have then for all $t \in [0,1]$

$$\begin{array}{l} T_\lambda u_n\,(t) \leq \lambda \int_0^1 H(1,s)\left((f^\infty + \varepsilon)\, u_n(s) + c\right) ds \\ + \frac{\lambda}{d} \int_0^1 g(s) \left(\int_0^1 G_1\,(s,\tau)\left((f^\infty + \varepsilon)\, u_n(\tau) + c\right) d\tau \right) ds \\ \leq \lambda\,(f^\infty + \varepsilon)\,\Gamma_\infty \,\|u_n\| + c\Gamma_\infty \end{array}$$

leading to

$$1 \leq \frac{\|T_\lambda u_n\|}{\|u_n\|} \leq \lambda\,(f^\infty + \varepsilon)\,\Gamma_\infty + \frac{c\Gamma_\infty}{\|u_n\|},$$

in which letting $n \to \infty$, yields the contradiction $1 \leq \lambda\,(f^\infty + \varepsilon)\,\Gamma_\infty < 1$.

At the end, choosing $\Omega_\nu = \{u \in E : \|u\| < r_\nu\}$ for $\nu = 0$ or ∞, we obtain from b) of Theorem 2.1 that T admits a fixed point $u \in K$ with $r_0 \leq \|u\| \leq r_\infty$, then from iii) in Lemma 2.4, u is a positive solution to FBVP (1)-(2). The proof is complete. ■

Remark 3.3. *Clearly Theorems 3.1 and 3.2 covers respectively the cases,* $f^0 = 0$, $f_\infty = +\infty$ *and* $f^\infty = 0$, $f_0 = +\infty$. *In particular, if* $f\,(t,u) = u^\gamma$ *with* $\gamma \in (0,1) \cup (1,+\infty)$ *then FBVP (1)-(2) admits a positive solution for all* $\lambda > 0$.

Example 3.4. *Consider FBVP (1)-(2) with* $f(t,u) = \frac{Au}{1+u} + \frac{Bu^3}{1+u^2}$. *In this case we have,* $f^0 = f_0 = A$ *and* $f^\infty = f_\infty = B$. *Applying Theorems 3.1 and 3.2, we obtain that for such a nonlinearity, FBVP (1)-(2) admits a positive solution for all* $\lambda \in (\Lambda^-, \Lambda^+)$ *where*

$$\Lambda^- = \begin{cases} (A\Gamma_0)^{-1} & \text{if } A\Gamma_0 > B\Gamma_\infty \\ (B\Gamma_\infty)^{-1} & \text{if } A\Gamma_0 < B\Gamma_\infty \end{cases} \qquad \Lambda^+ = \begin{cases} (B\Gamma_\infty)^{-1} & \text{if } A\Gamma_0 > B\Gamma_\infty \\ (A\Gamma_0)^{-1} & \text{if } A\Gamma_0 < B\Gamma_\infty. \end{cases}$$

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