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THE EXISTENCE OF DISTRIBUTIONAL CHAOS IN ABSTRACT DEGENERATE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Fractional calculus is a rapidly growing field of research, with numerous applications in the areas of engineering, physics, chemistry, biology and other sciences. In this paper, we analyze distributionally chaotic properties of abstract degenerate (multi-term) fractional differential equations with Caputo derivatives, providing also several illustrative examples and possible applications. Our results are formulated in the setting of infinite-dimensional complex Fréchet spaces.

1. Introduction and preliminaries

Distributional chaos is very popular field of research in the theory of topological dynamics of linear operators. Let us recall that the notion of distributional chaos for interval maps was introduced by Schweizer and Smítal [1994]; for some other relevant references on distributional chaos, one may refer e.g. to [4, 10-11, 35, 40]. A linear continuous operator T acting on a Fréchet space X is said to be distributionally chaotic iff there exist an uncountable set $S \subseteq X$ (scrambled set) and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{dens}\Big(\big\{k \in \mathbb{N} : d\big(T^k x, T^k y\big) \geq \sigma\big\}\Big) = 1 \text{ and}$$
$$\overline{dens}\Big(\big\{k \in \mathbb{N} : d\big(T^k x, T^k y\big) < \epsilon\big\}\Big) = 1,$$

where $d(\cdot,\cdot)$ denotes the metric on X and the upper density of a set $D \subseteq \mathbb{N}$ is defined by

$$\overline{dens}(D) := \limsup_{n \to +\infty} \frac{\text{card}(D \cap [1,n])}{n}.$$

If we can choose S to be dense in X, then we say that T is densely distributionally chaotic; see [21] for more details about linear dynamics of single operators.

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The notion of a (densely) distributionally chaotic strongly continuous semigroup on Fréchet space has been recently introduced in [15] (joint work with Conejero, Miana and Murillo-Arcila; cf. also [1, 6-8, 13] for further information concerning distributionally chaotic strongly continuous semigroups on Banach spaces) as follows: A strongly continuous semigroup $(T(t))_{t\geq 0} \subseteq L(X)$ is said to be distributionally chaotic iff there are an uncountable set $S \subseteq X$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{Dens}\big(\{t \geq 0 : d(T(t)x, T(t)y) \geq \sigma\}\big) = 1 \text{ and}$$
$$\overline{Dens}\big(\{t \geq 0 : d(T(t)x, T(t)y) < \epsilon\}\big) = 1,$$

where the upper density of a set $D \subseteq [0, \infty)$ is defined now by

$$\overline{Dens}(D) := \limsup_{t\to+\infty} \frac{m(D \cap [0, t])}{t},$$

with $m(\cdot)$ being the Lebesgue's measure on $[0, \infty)$. If, moreover, we can choose S to be dense in X, then $(T(t))_{t\geq 0}$ is said to be densely distributionally chaotic. The question whether an operator $T \in L(X)$ or a strongly continuous semigroup $(T(t))_{t\geq 0} \subseteq L(X)$ is distributionally chaotic or not is closely connected with the existence of distributionally irregular vectors, i.e., those elements $x \in X$ such that for each $\sigma > 0$

$$\overline{dens}\Big(\big\{k \in \mathbb{N} : d\big(T^k x, 0\big) > \sigma\big\}\Big) = 1 \text{ and}$$
$$\overline{dens}\Big(\big\{k \in \mathbb{N} : d\big(T^k x, 0\big) < \sigma\big\}\Big) = 1,$$

respectively,

$$\overline{Dens}\big(\{t \geq 0 : d(T(t)x, 0) > \sigma\}\big) = 1 \text{ and}$$
$$\overline{Dens}\big(\{t \geq 0 : d(T(t)x, 0) < \sigma\}\big) = 1.$$

Distributionally chaotic properties of abstract non-degenerate fractional differential equations in Banach spaces has been analyzed in [29], where it has been pointed out that the notion of distributional chaos is much more appropriate for dealing with fractional equations than that of the usually considered Devaney chaos. On the other hand, the most intriguing hypercyclic and topologically mixing properties of abstract degenerate (multi-term) fractional differential equations has been recently considered in [27]-[28]. The main aim of this paper is continue these research studies by enquiring into the basic distributionally chaotic properties of solutions to abstract degenerate (multi-term) fractional differential equations with Caputo derivatives. The notion of subspace distributional chaoticity plays an important role in our analysis (cf. [5] and [25] for more details about subspace hypercyclicity and subspace topologically mixing properties of abstract differential equations).

Throughout this paper, we assume that X is an infinite-dimensional Fréchet space over the field od complex numbers, and that the topology of X is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms (our results admit very simply reformulations in the setting of real Fréchet spaces and, because of that, we will omit all related details for the sake of brevity and better exposition). The translation invariant metric $d : X \times X \to [0, \infty)$, defined by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)},\ x,\ y \in X, \tag{1}$$

satisfies the following properties:

$$d(x+u,y+v) \leq d(x,y)+d(u,v), \quad x,\ y,\ u,\ v \in X, \tag{2}$$

$$d(cx,cy) \leq (|c|+1)d(x,y),\ c \in \mathbb{C},\ x,\ y \in X, \tag{3}$$

and

$$d(\alpha x,\beta x) \geq \frac{|\alpha-\beta|}{1+|\alpha-\beta|}d(0,x), \quad x \in X,\ \alpha,\ \beta \in \mathbb{C}. \tag{4}$$

Given $x \in X$ and $\epsilon > 0$ in advance, set $L(x,\epsilon) := \{y \in X : d(x,y) < \epsilon\}$ (if $(Z, \|\cdot\|_Z)$ is a Banach space under consideration, then it will be assumed that the metric on Z is given by $d_Z(x,y) := \|x-y\|_Z,\ x,\ y \in Z$; the norm on X will be abbreviated to $\|\cdot\|$). Let Y be another Fréchet space over the field of complex numbers, let the topology of Y be induced by the fundamental system $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms, and let $d_Y(\cdot,\cdot)$ denote the induced metric on Y (cf. (1)). By $L(X,Y)$ we denote the space which consists of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. Let $\mathcal{B}$ be the family of bounded subsets of X and let $p_{n,B}(T) := \sup_{x\in B} p_n^Y(Tx)$, $n \in \mathbb{N}$, $B \in \mathcal{B}$, $T \in L(X,Y)$. Then $p_{n,B}(\cdot)$ is a seminorm on $L(X,Y)$ and the system $(p_{n,B})_{(n,B)\in\mathbb{N}\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(X,Y)$. Henceforth A and B denote two closed linear operator acting on X, $C \in L(X)$ denotes an injective operator satisfying $CA \subseteq AC$, $CB \subseteq BC$, and the convolution like mapping $*$ is given by $f*g(t) := \int_0^t f(t-s)g(s)\,ds$. The domain, range and kernel space of A are denoted by $D(A)$, $R(A)$ and $N(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Suppose now that F is a linear subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Set $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$, $p_{m,n}(x) := \sum_{i=0}^n p_m(A^i x)$, $x \in D_\infty(A)$, $m,\ n \in \mathbb{N}$, and $A_\infty := A_{|D_\infty(A)}$. Then the system $(p_{m,n})_{m,n\in\mathbb{N}}$ induces a Fréchet topology on $D_\infty(A)$. We will denote this space by $[D_\infty(A)]$; then it is clear that $A_\infty \in L([D_\infty(A)])$. If $\tilde{X}$ is a closed linear subspace of X, then $\tilde{X}$ is a Fréchet space itself and the fundamental system of seminorms which induces the topology on $\tilde{X}$ is $(p_{n|\tilde{X}})_{n\in\mathbb{N}}$. By I we denote the identity operator on X; if Z is a general topological space and $Z_0 \subseteq Z$, then by $\overline{Z_0}^Z$ we denote the adherence of Z_0 in Z (we will use the abbreviation $\overline{Z_0}$, if there is no risk for confusion).

Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{C}_- := \{z \in \mathbb{C} : \Re z < 0\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta > 0,\ t > 0)$, $\mathbb{N}_l := \{1,\cdots,l\}$, $\mathbb{N}_l^0 := \{0,1,\cdots,l\}$ $(l \in \mathbb{N})$ and $g_0(t) :=$ the Dirac δ-distribution.

Assume $\alpha > 0$, $m = \lceil \alpha \rceil$, $\beta > 0$ and $\gamma \in (0,1)$. Recall that the Caputo fractional derivative $\mathbf{D}_t^\alpha u$ ([9], [25]) is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : X)$; if this is the case, then we have

$$\mathbf{D}_t^\alpha u(t) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

Denote by $E_{\alpha,\beta}(z)$ the Mittag-Leffler function $E_{\alpha,\beta}(z) := \sum_{n=0}^\infty z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$ ([9]). Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. We shall use the following asymptotic formulae ([9], [25]): If $0 < \alpha < 2$ and $\beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \varepsilon_{\alpha,\beta}(z),\ |\arg(z)| < \alpha\pi/2, \tag{5}$$

and

$$E_{\alpha,\beta}(z)=\varepsilon_{\alpha,\beta}(z),\ |\arg(-z)|<\pi-\alpha\pi/2, \tag{6}$$

where

$$\varepsilon_{\alpha,\beta}(z)=\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}+O\big(|z|^{-N}\big),\ |z|\to\infty. \tag{7}$$

The reader may consult [25] for further information concerning the Laplace transform and analytical properties of functions with values in sequentially complete locally convex spaces (cf. [2] for the Banach space case). By $\mathcal{L}$ and $\mathcal{L}^{-1}$ we denote the Laplace transform and its inverse transform, respectively. We say that a function $h:(a,\infty)\to X$ belongs to the class $LT-X$ iff there exists a function $f\in C([0,\infty):X)$ such that for each $n\in\mathbb{N}$ there exists $M_n>0$ satisfying $p_n(f(t))\leq M_ne^{at}$, $t\geq 0$ and $h(\lambda)=\int_0^\infty e^{-\lambda t}f(t)\,dt,\ \lambda>a\ (a\in\mathbb{R})$.

In the theory of non-degenerate equations, of concern is the following multi-term problem:

$$\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=0,\quad t\geq 0,$$
$$u^{(k)}(0)=u_k,\quad k=0,\cdot\cdot\cdot,\lceil\alpha_n\rceil-1,$$

where $n\in\mathbb{N}\setminus\{1\}$, $A_1,\cdot\cdot\cdot,A_{n-1}$ are closed linear operators on X and $0\leq\alpha_1<\cdot\cdot\cdot<\alpha_n$. The reader may consult [25, Section 2.10] for an extensive survey of recent results on abstract multi-term fractional differential equations with Caputo fractional derivatives (cf. [9, 19, 23, 25, 33, 36-37] for further information concerning fractional calculus and fractional differential equations). Set $m_i:=\lceil\alpha_i\rceil$, $i\in\mathbb{N}_n$, $T_{i,L}u(t):=A_i\mathbf{D}_t^{\alpha_i}u(t)$, if $t\geq 0$, $i\in\mathbb{N}_n$ and $\alpha_i>0$, and $T_{i,R}u(t):=\mathbf{D}_t^{\alpha_i}A_iu(t)$, if $t\geq 0$ and $i\in\mathbb{N}_n$; here, $A_n:=B$ is also a closed linear operator on X. Assume that, for every $t\geq 0$ and $i\in\mathbb{N}_n$, $T_iu(t)$ denotes exactly one of the terms $T_{i,L}u(t)$ or $T_{i,R}u(t)$. In [27], we have considered the following degenerate multi-term fractional Cauchy problem:

$$\sum_{i=1}^{n}T_iu(t)=0,\quad t\geq 0, \tag{8}$$

accompanied with the following initial conditions:

$$u^{(k)}(0)=u_k,\ 0\leq k\leq m_Q-1\text{ and }\big(A_iu\big)^{(k)}(0)=u_{i,k}\text{ if }m_i-1\geq k\geq m_Q, \tag{9}$$

where
$\mathcal{I}=\{i\in\mathbb{N}_n:\alpha_i>0\text{ and }T_{i,L}u(t)\text{ appears on the left hand side of (8)}\}$, $Q=\max\mathcal{I}$, if $\mathcal{I}\neq\emptyset$ and $Q=m_Q=0$, if $\mathcal{I}=\emptyset$. In order to simplify the notation, we will use the shorthand (ACP) to denote the problem [(8)-(9)].

The most important subcases of problem (ACP) are Sobolev linear degenerate equations:

$$B\frac{d}{dt}u(t)=Au(t),\ u(0)=x\text{ and }\frac{d}{dt}Bu(t)=Au(t),\ Bu(0)=Bx\ \ (t\geq 0). \tag{10}$$

The reader may consult the monographs by Favini, Yagi [20], Carroll, Showalter [12], Demidenko, Uspenskii [17], Melnikova, Filinkov [34] and Sviridyuk, Fedorov [41] for further information concerning the wellposedness of problems stated in (10). For the basic information about abstract degenerate Volterra equations and abstract degenerate fractional differential equations, the reader may consult the forthcoming monograph [26].

We need the following definition from [27].

Definition 1 A function $u \in C([0,\infty) : X)$ is said to be a strong solution of problem (ACP) iff the term $T_i u(t)$ is well defined and continuous for any $t \geq 0$, $i \in \mathbb{N}_n$, and (ACP) holds identically on $[0,\infty)$.

We focus special attention on distributionally chaotic solutions of the following fractional Sobolev equations:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^{\alpha} Bu(t) = Au(t), & t \geq 0, \\ (Bu)^{(j)}(0) = Bx_j, \ 0 \leq j \leq \lceil \alpha \rceil - 1 \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B\mathbf{D}_t^{\alpha} u(t) = Au(t), & t \geq 0, \\ u^{(j)}(0) = x_j, \ 0 \leq j \leq \lceil \alpha \rceil - 1. \end{cases}$$

Along with the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, we consider the associated abstract integral equation:

$$Bu(t) = f(t) + \int_0^t g_{\alpha}(t-s)Au(s)\,ds, \ t \geq 0, \tag{11}$$

where $f \in C([0,\infty) : X)$. Henceforth (DFP) denotes either $(\mathrm{DFP})_R$ or $(\mathrm{DFP})_L$. By a mild solution of the problem $(\mathrm{DFP})_R$ we mean any continuous function $t \mapsto u(t)$, $t \geq 0$ such that the mapping $t \mapsto Bu(t)$, $t \geq 0$ is continuous and $A(g_\alpha * u)(t) = Bu(t) - \sum_{k=0}^{\lceil \alpha \rceil -1} g_{k+1}(t)Bx_k$, $t \geq 0$. The set of all vectors $\vec{x} = (Bx_0, Bx_1, \cdot\cdot\cdot, Bx_{\lceil \alpha \rceil -1})$ for which there exists a mild solution of problem $(\mathrm{DFP})_R$ will be denoted by $Z_{\alpha,R}^{mild}(A,B)$.

2. Distributionally chaotic properties of abstract degenerate fractional differential equations

We start this section by repeating some known facts about the existence and uniqueness of strong solutions of problem (ACP); cf. [28] for more details. Recall that $n \in \mathbb{N} \setminus \{1\}$, as well as that $A_1, \cdot\cdot\cdot, A_n$ are closed linear operators on X and $0 \leq \alpha_1 < \cdot\cdot\cdot < \alpha_n$; $m_i = \lceil \alpha_i \rceil$ $(i \in \mathbb{N}_n)$, $T_{i,L}u(t) = A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geq 0$, $i \in \mathbb{N}_n$, $\alpha_i > 0$, $T_{i,R}u(t) = \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geq 0$, $i \in \mathbb{N}_n$, and for every $t \geq 0$, $i \in \mathbb{N}_n$, $T_i u(t)$ denotes exactly one of the terms $T_{i,L}u(t)$ or $T_{i,R}u(t)$. Denote by $\mathfrak{T}$ the exact number of initial values subjected to the problem (ACP); that is, $\mathfrak{T}$ is the sum of number m_Q and the cardinality of set consisting of those pairs $(i,j) \in \mathbb{N}_n \times \mathbb{N}_{m_n-1}^0$ for which $m_i - 1 \geq j \geq m_Q$. To make this more precise, suppose that $\{i_1, \cdot\cdot\cdot, i_s\} = \{i \in \mathbb{N}_n : m_i - 1 \geq m_Q\}$ and $i_1 < \cdot\cdot\cdot < i_s$. Then the set of all initial values appearing in (9) is given by $\{u_0, \cdot\cdot\cdot, u_{m_Q-1}; u_{i_1,m_Q}, \cdot\cdot\cdot, u_{i_1,m_{i_1}-1}; \cdots; u_{i_s,m_Q}, \cdots, u_{i_s,m_{i_s}-1}\} = \{(u_j)_{0 \leq j \leq m_Q-1}; (u_{i_{s'},j})_{1 \leq s' \leq s, m_Q \leq j \leq m_{i_{s'}}-1}\}$ so that $\mathfrak{T} = m_{i_1} + \cdot\cdot\cdot + m_{i_s} + (1-s)m_Q$. Denote by $\mathfrak{Z}$ $(\mathfrak{Z}_{uniq})$ the set of all tuples of initial values $\vec{x} = ((u_j)_{0 \leq j \leq m_Q-1}; (u_{i_{s'},j})_{1 \leq s' \leq s, m_Q \leq j \leq m_{i_{s'}}-1}) \in X^{\mathfrak{T}}$ for which there exists a (unique) strong solution of problem (ACP). Then $\mathfrak{Z}$ is a linear subspace of $X^{\mathfrak{T}}$ and $\mathfrak{Z}_{uniq} \subseteq \mathfrak{Z}$, with equality iff the zero function is a unique strong solution of the problem (ACP) with the initial value $\vec{x} = \vec{0}$.

The notion of (subspace) distributional chaoticity of problem (ACP) is introduced in the following definition.

Definition 2 Let $\tilde{X}$ be a closed linear subspace of $X^{\mathfrak{T}}$. Then it is said that the abstract Cauchy problem (ACP) is $\tilde{X}$-distributionally chaotic iff there are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $\vec{x}, \ \vec{y} \in S$ of distinct tuples we have that there exist strong solutions $t \mapsto u(t;\vec{x})$, $t \geq 0$ and $t \mapsto u(t;\vec{y})$, $t \geq 0$ of problem (ACP) with the property that

$$\overline{Dens}\Big(\big\{t \geq 0 : d\big(u(t;\vec{x}), u(t;\vec{y})\big) \geq \sigma\big\}\Big) = 1 \text{ and}$$
$$\overline{Dens}\Big(\big\{t \geq 0 : d\big(u(t;\vec{x}), u(t;\vec{y})\big) < \epsilon\big\}\Big) = 1.$$

If we can choose S to be dense in $\tilde{X}$, then we also say that the problem (ACP) is densely $\tilde{X}$-distributionally chaotic (S is called a $\sigma_{\tilde{X}}$-scrambled set). In the case that $\tilde{X} = X$, it is also said that the problem (ACP) is (densely) distributionally chaotic; S is then called a σ-scrambled set.

Observe that any $\tilde{X}$-distributionally chaotic problem (ACP) is automatically $\tilde{\tilde{X}}$-distributionally chaotic for any closed linear subspace $\tilde{\tilde{X}}$ of $X^{\mathfrak{T}}$ containing $\tilde{X}$. This implies that it is very important to know the minimal linear subspace $\tilde{X}$ of $X^{\mathfrak{T}}$ such that the problem (ACP) is $\tilde{X}$-distributionally chaotic. On the contrary, we are always trying to find the maximal possible linear subspace $\tilde{X}$ of $X^{\mathfrak{T}}$ for which the problem (ACP) is densely $\tilde{\tilde{X}}$-distributionally chaotic.

As mentioned in [27], it is very difficult to create a general theoretical concept which would enable us to investigate distributionally chaotic properties of abstract degenerate differential equations in a safe and sound way. For example, in Definition 2, we work only with strong solutions. Without any doubt, this is inevitable for the problem $(\text{DFP})_L$, and a large number of similar problems, because then we cannot define the notion of a mild solution so easily. On the other hand, the notion introduced in Definition 2 can be slightly modified for the problem $(\text{DFP})_R$ by requiring that, for every two distinct tuples vectors of $\sigma_{\tilde{X}}$-scrambled set $S \subseteq \tilde{X} \cap Z^{mild}_{\alpha,R}(A,B)$, there exist mild solutions $t \mapsto u(t;\vec{x})$, $t \geq 0$ and $t \mapsto u(t;\vec{y})$, $t \geq 0$ of problem $(\text{DFP})_R$ obeying the properties prescribed. We will not follow this approach henceforth.

Definition 3 Let $n \in \mathbb{N}$, let $\tilde{X}$ be a closed linear subspace of $X^{\mathfrak{T}}$, and let $\vec{x} \in \tilde{X} \cap \mathfrak{Z}$. Then it is said that the vector $\vec{x}$ is:

(i) $\tilde{X}$-(ACP)-distributionally near to 0 iff there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geq 0$ of problem (ACP) such that

$$\overline{Dens}(Z) = 1 \text{ and } \lim_{t \in Z, t \to +\infty} u\big(t;\vec{x}\big) = 0;$$

(ii) $\tilde{X}$-(ACP)-distributionally n-unbounded iff there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geq 0$ of problem (ACP) such that

$$\overline{Dens}(Z) = 1 \text{ and } \lim_{t \in Z, t \to +\infty} p_n\big(u(t;\vec{x})\big) = +\infty;$$

$\vec{x}$ is said to be $\tilde{X}$-(ACP)-distributionally unbounded iff there exists $q \in \mathbb{N}$ such that $\vec{x}$ is $\tilde{X}$-(ACP)-distributionally q-unbounded (if $(X, \|\cdot\|)$ is a Banach space, this simply means that there exist a set $Z \subseteq [0,\infty)$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geq 0$ of problem (ACP) such that $\overline{Dens}(Z) = 1$ and $\lim_{t \in Z, t \to +\infty} \|u(t;\vec{x})\| = +\infty$);

(iii) a $\tilde{X}$-(ACP)-distributionally irregular vector iff there exist an integer $q \in \mathbb{N}$, two subsets $B_0,\ B_\infty$ of $[0,\infty)$ with $\overline{Dens}(B_0) = \overline{Dens}(B_\infty) = 1$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geq 0$ of problem (ACP) such that

$$\lim_{t\in B_0, t\to+\infty} u\big(t;\vec{x}\big) = 0 \text{ and } \lim_{t\in B_\infty, t\to+\infty} p_q\big(u(t;\vec{x})\big) = +\infty. \tag{12}$$

In the case that $\tilde{X} = X^{\mathfrak{T}}$, then we also say that $\vec{x}$ is (ACP)-distributionally near to 0, resp., (ACP)-distributionally n-unbounded, (ACP)-distributionally unbounded; a $\tilde{X}$-distributionally irregular vector for (ACP) is then called a distributionally irregular vector for (ACP).

Suppose that $X' \subseteq \tilde{X} \cap \mathfrak{Z}$ is a linear manifold. Then we say that X' is a $\tilde{X}$-distributionally irregular manifold for (ACP) (distributionally irregular manifold for (ACP), in the case that $\tilde{X} = X^{\mathfrak{T}}$) iff any element $x \in X' \setminus \{0\}$ is $\tilde{X}$-distributionally irregular vector for (ACP). Further on, we say that X' is a uniformly $\tilde{X}$-distributionally irregular manifold for (ACP) (uniformly distributionally irregular manifold for (ACP), in the case that $\tilde{X} = X^{\mathfrak{T}}$) iff there exists $q \in \mathbb{N}$ such that, for every $\vec{x} \in X' \setminus \{0\}$, there exist two subsets $B_0,\ B_\infty$ of $[0,\infty)$ with $\overline{Dens}(B_0) = \overline{Dens}(B_\infty) = 1$ and a strong solution $t \mapsto u(t;\vec{x})$, $t \geq 0$ of problem (ACP) such that (12) holds. It can be simply verified with the help of translation invariance of metric $d(\cdot,\cdot)$ and inequalities (2)-(4) that the following holds: If $0 \neq \vec{x} \in \tilde{X}\cap\mathfrak{Z}$ is a $\tilde{X}$-distributionally irregular vector for (ACP), then $X' \equiv span\{\vec{x}\}$ is a uniformly $\tilde{X}$-distributionally irregular manifold for (ACP).

Remark 1:

(i) If $\vec{x}$ is a $\tilde{X}$-distributionally irregular vector for (ACP), then $\vec{x}$ is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded. The converse statement holds provided that strong solutions of problem (ACP) are unique. If this is not the case and $\vec{x} \neq 0$ is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded, then we can prove the following (cf. Definition 2): There are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ (S is, in fact, equal to $span\{\vec{x}\}$) and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $\vec{x},\ \vec{y} \in S$ of distinct vectors we have that there exist strong solutions $t \mapsto u_i(t;\vec{x})$, $t \geq 0$ and $t \mapsto u_i(t;\vec{y})$, $t \geq 0$ $(i = 1,2)$ of problem (ACP) with the property that

$$\overline{Dens}\Big(\big\{t \geq 0 : d\big(u_1(t;\vec{x}), u_1(t;\vec{y})\big) \geq \sigma\big\}\Big) = 1 \text{ and}$$
$$\overline{Dens}\Big(\big\{t \geq 0 : d\big(u_2(t;\vec{x}), u_2(t;\vec{y})\big) < \epsilon\big\}\Big) = 1.$$

If this is the case, we say that the problem (ACP) is quasi $\tilde{X}$-distributionally chaotic (quasi distributionally chaotic, provided that $\tilde{X} = X^{\mathfrak{T}}$). The set S is called quasi $\sigma_{\tilde{X}}$-scrambled set (quasi σ-scrambled set, provided that $\tilde{X} = X^{\mathfrak{T}}$).

(ii) Suppose that (ACP) is $\tilde{X}$-distributionally chaotic and S is the corresponding $\sigma_{\tilde{X}}$-scrambled set. Then, for every two distinct vectors $\vec{x},\ \vec{y} \in S$, $\vec{x} - \vec{y}$ is a $\tilde{X}$-distributionally vector for (ACP).

(iii) Suppose that (ACP) is quasi $\tilde{X}$-distributionally chaotic and S is the corresponding quasi $\sigma_{\tilde{X}}$-scrambled set. Then, for every two distinct vectors $\vec{x},\ \vec{y} \in S$, $\vec{x} - \vec{y}$ is a quasi $\tilde{X}$-distributionally vector for (ACP), i.e., $\vec{x} - \vec{y}$

is both $\tilde{X}$-(ACP)-distributionally near to 0 and $\tilde{X}$-(ACP)-distributionally unbounded.

It is worth noting that the non-triviality of subspace $\bigcap_{i=1}^{n} N(A_i)$ in X immediately implies that the problem (ACP) is distributionally chaotic:

Example 1 Suppose that $0 \neq x \in \bigcap_{i=1}^{n} N(A_i)$. We can always find a sequence $(a_n)_{n\in\mathbb{N}_0}$ of non-negative real numbers and a scalar-valued function $f \in C^{\infty}([0,\infty))$ such that $a_0 = 0$, $a_n > a_{n-1} + 2$, $n \in \mathbb{N}$, $\lim_{n\to+\infty}(a_{n-1} - 1)(a_n - 1)^{-1} = 0$, $f(t) = 0$ for $t \in \bigcup_{n\in\mathbb{N}}[a_{2n-1}, a_{2n}]$ and $f(t) = 2n$ for $t \in \bigcup_{n\in\mathbb{N}_0}[a_{2n} + 1, a_{2n+1} - 1]$. Since the sets $B_0 := \bigcup_{n\in\mathbb{N}}[a_{2n-1}, a_{2n}]$ and $B_\infty := \bigcup_{n\in\mathbb{N}_0}[a_{2n} + 1, a_{2n+1} - 1]$ have the upper densities equal to 1, it is very simple to verify that the function $u(t;\vec{x}) := f(t)x$, $t \geq 0$ is a strong solution of problem (ACP) with the initial value $\vec{x} = ((u_j \equiv f^{(j)}(0)x)_{0\leq j\leq m_Q-1}; (u_{i_{s'},j} \equiv 0)_{1\leq s'\leq s, m_Q\leq j\leq m_{i_{s'}}-1}) \in X^{\mathfrak{T}}$, as well as that $\vec{x}$ is a distributionally irregular vector for (ACP). In particular, $\vec{x} = \vec{0}$ can be a distributionally irregular vector for (ACP).

We leave to the interested reader problem of finding a quasi distributionally chaotic problem (ACP) that is not distributionally chaotic. In the sequel, we will consider only the classical notion of (subspace) distributional chaoticity of problem (ACP).

Lemma 1 ([15, Theorem 4.1]) Let Y be another Fréchet space over the field of complex numbers, let the topology of Y be induced by the fundamental system $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms, and let $d_Y(\cdot,\cdot)$ denote the induced metric on Y. Suppose that X is separable, X_0 is a dense linear subspace of X, $(T(t))_{t\geq 0} \subseteq L(X,Y)$ is a strongly continuous operator family, as well as:

(i) $\lim_{t\to+\infty} T(t)x = 0$, $x \in X_0$,
(ii) there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq [0,\infty)$ such that $\overline{Dens}(B) = 1$, and $\lim_{t\to+\infty,t\in B} p_m^Y(T(t)x) = \infty$, resp. $\lim_{t\to+\infty,t\in B} \|T(t)x\|_Y = \infty$ if $(Y, \|\cdot\|_Y)$ is a Banach space.

Then there exist a dense linear subspace S of X and a number $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{Dens}\Bigl(\Bigl\{t \geq 0 : d_Y\bigl(T(t)x, T(t)y\bigr) \geq \sigma\Bigr\}\Bigr) = 1 \text{ and}$$
$$\overline{Dens}\Bigl(\Bigl\{t \geq 0 : d_Y\bigl(T(t)x, T(t)y\bigr) < \epsilon\Bigr\}\Bigr) = 1.$$

In the analysis of existence and uniqueness of abstract degenerate fractional Cauchy problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$, the notions of exponentially equicontinuous (g_α, C)-regularized resolvent family for (11) and exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B are crucially important (cf. [20] and [41] for some other approaches):

Definition 4 Suppose that $\alpha > 0$, $C \in L(X)$ is injective, $CA \subseteq AC$ and $CB \subseteq BC$.

(i) [30, Definition 2.2] Suppose that $R(t) : D(B) \to E$ is a linear mapping $(t \geq 0)$. Then the operator family $(R(t))_{t\geq 0}$ is said to be an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11) iff there exists $\omega \geq 0$ such that the following holds:
  (a) The mapping $t \mapsto R(t)x$, $t \geq 0$ is continuous for every fixed element $x \in D(B)$.

(b) The family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous, i.e., for every $n \in \mathbb{N}$, there exist $c > 0$ and $m \in \mathbb{N}$ such that

$$p_n\big(e^{-\omega t}R(t)x\big) \leq cp_m(x), \quad x \in D(B),\ t \geq 0.$$

(c) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, the operator $\lambda^\alpha B - A$ is injective, $C(R(B)) \subseteq R(\lambda^\alpha B - A)$ and

$$\lambda^{\alpha-1}\big(\lambda^\alpha B - A\big)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in D(B). \tag{13}$$

(ii) [31, Definition 2.3] An operator family $(R(t))_{t\geq 0} \subseteq L(X, [D(B)])$ is said to be an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by $A,\ B$ iff there exists $\omega \geq 0$ such that the following holds:

(a) The mappings $t \mapsto R(t)x$, $t \geq 0$ and $t \mapsto BR(t)x$, $t \geq 0$ are continuous for every fixed element $x \in X$.

(b) The family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X, [D(B)])$ is equicontinuous, i.e., for every $n \in \mathbb{N}$, there exist $c > 0$ and $m \in \mathbb{N}$ such that

$$p_n\big(e^{-\omega t}R(t)x\big) + p_n\big(e^{-\omega t}BR(t)x\big) \leq cp_m(x), \quad x \in X,\ t \geq 0. \tag{14}$$

(c) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, the operator $\lambda^\alpha B - A$ is injective, $R(C) \subseteq R(\lambda^\alpha B - A)$ and

$$\lambda^{\alpha-1}\big(\lambda^\alpha B - A\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in X.$$

From [30-31], we know the following facts about exponentially equicontinuous (g_α, C)-regularized resolvent families introduced above (cf. Remark 2.4 of [31] for their mutual relationship, and [25, Subsection 2.1.1] for non-degenerate case).

Lemma 2

(i) Let $(R(t))_{t\geq 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11). Suppose that the following condition holds:

(P) There exists a number $\omega_1 > \omega$ such that, for every $x \in X$, there exists a function $h(\lambda; x) \in LT - X$ such that $h(\lambda; x) = \lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx$, provided $\Re\lambda > \omega_1$.

Let $x_0, \cdots, x_{\lceil\alpha\rceil-1} \in D(A) \cap D(B)$. Then the function $u(t; (BCx_0, \cdots, BCx_{\lceil\alpha\rceil-1})) := \sum_{j=0}^{\lceil\alpha\rceil} \int_0^t g_j(t-s)R(s)x_j\,ds$, $t \geq 0$ is a unique strong solution of $(\mathrm{DFP})_R$, with the initial values Bx_j replaced by BCx_j $(0 \leq j \leq \lceil\alpha\rceil - 1)$.

(ii) Let $(R(t))_{t\geq 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by $A,\ B$. Then, for every $x_0, \cdots, x_{\lceil\alpha\rceil-1} \in D(A) \cap D(B)$, the function $u(t; (Cx_0, \cdots, Cx_{\lceil\alpha\rceil-1})) := \sum_{j=0}^{\lceil\alpha\rceil} \int_0^t g_j(t-s)R(s)Bx_j\,ds$, $t \geq 0$ is a unique strong solution of problem $(\mathrm{DFP})_L$, with the initial values x_j replaced by Cx_j $(0 \leq j \leq \lceil\alpha\rceil - 1)$.

In the following theorem, we will consider the subspace distributionally chaotic properties of problem $(\mathrm{DFP})_R$.

Theorem 1 Suppose that $\alpha > 0$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geq 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11), (P) holds and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Let F_i be a separable complex Fréchet space, let $F_i \subseteq D(A) \cap D(B)$, and let F_i be continuously embedded in X $(i \in \mathcal{V})$. Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(CBf_i) \leq c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$. Suppose, further, that for each $i \in \mathcal{V}$ there exists a dense subset F_i^0 of F_i satisfying that $\lim_{t\to+\infty}(g_i * R(\cdot)f_i)(t) = 0$, $f_i \in F_i^0$. Let there exist $\vec{f_\infty} = (f_{0,\infty}, \cdots, f_{\lceil\alpha\rceil-1,\infty}) \in F$, $m \in \mathbb{N}$ and a set $D \subseteq [0,\infty)$ such that $\overline{Dens}(D) = 1$, and $\lim_{t\to+\infty, t\in D} p_m(\sum_{i\in\mathcal{V}}(g_i * R(\cdot)f_{i,\infty})(t)) = +\infty$, resp. $\lim_{t\to+\infty,t\in D} \|\sum_{i\in\mathcal{V}}(g_i * R(\cdot)f_{i,\infty})(t)\| = +\infty$ if $(X, \|\cdot\|)$ is a Banach space. Then we have that the problem $(\mathrm{DFP})_R$ is densely $\overline{\{(CBf_0, \cdots, CBf_{\lceil\alpha\rceil-1}) : \vec{f} = (f_0, \cdots, f_{\lceil\alpha\rceil-1}) \in F\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.

Proof. It is clear that F is an infinite-dimensional separable complex Fréchet space. Define $V(t)\vec{f} := \sum_{i=0}^{\lceil\alpha\rceil-1}(g_i * R(\cdot)f_i)(t)$, $t \geq 0$ ($\vec{f} = (f_0, \cdots, f_{\lceil\alpha\rceil-1}) \in F$) and $F_0 := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i^0$, where $G_i^0 := F_i^0$, if $i \in \mathcal{V}$, and $G_i^0 := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$. Then F_0 is dense in F and $(V(t))_{t\geq0} \subseteq L(F, X)$ is a strongly continuous operator family. An application of Lemma 1 yields that there exist a dense linear subspace S of F and a number $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $\vec{f'}$, $\vec{f''} \in S$ of distinct vectors we have that

$$\overline{Dens}\Big(\Big\{t \geq 0 : d\big(V(t)\vec{f'}, V(t)\vec{f''}\big) \geq \sigma\Big\}\Big) = 1 \text{ and}$$
$$\overline{Dens}\Big(\Big\{t \geq 0 : d\big(V(t)\vec{f'}, V(t)\vec{f''}\big) < \epsilon\Big\}\Big) = 1.$$

Suppose that $CBf_i = 0$ for all $i \in \mathcal{V}$ and $f_i \in F_i$. Then (13) and the uniqueness theorem for the Laplace transform together imply that $R(t)f_i = 0$ for all $i \in \mathcal{V}$ and $f_i \in F_i$, which contradicts the existence of m-distributionally unbounded vector $\vec{f_\infty}$ from F. Hence, there exist $i \in \mathcal{V}$ and $f_i \in F_i$ such that $CBf_i \neq 0$. Using this fact and the continuity of mapping $CB : F_i \to X$ for each $i \in \mathcal{V}$, we can simply verify that $\{(CBf_0, \cdots, CBf_{\lceil\alpha\rceil-1}) : \vec{f} = (f_0, \cdots, f_{\lceil\alpha\rceil-1}) \in S\}$ is a non-trivial subspace of $X^{\lceil\alpha\rceil}$. Now the final conclusion simply follows by using the continuity of mappings $CB : F_i \to X$ ($i \in \mathcal{V}$) once more, and Lemma 2(i).

Similarly, by using Lemma 1 and Lemma 2(ii), we can prove the following theorem on subspace distributional chaoticity of problem $(\mathrm{DFP})_L$.

Theorem 2 Suppose that $\alpha > 0$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geq0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B, and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Let F_i be a separable complex Fréchet space, and let $F_i \subseteq D(A) \cap D(B)$ ($i \in \mathcal{V}$). Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(Bf_i) + p_n(Cf_i) \leq c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$. Suppose, further, that for each $i \in \mathcal{V}$ there exists a dense subset F_i^0 of F_i satisfying that $\lim_{t\to+\infty}(g_i * R(\cdot)Bf_i)(t) = 0$, $f_i \in F_i^0$. Let there exist $\vec{f} = (f_0, \cdots, f_{\lceil\alpha\rceil-1}) \in F$, $m \in \mathbb{N}$ and a set $D \subseteq [0,\infty)$ such that $\overline{Dens}(D) = 1$, and $\lim_{t\to+\infty,t\in D} p_m(\sum_{i\in\mathcal{V}}(g_i * R(\cdot)Bf_i)(t)) = +\infty$, resp. $\lim_{t\to+\infty,t\in D} \|\sum_{i\in\mathcal{V}}(g_i * R(\cdot)Bf_i)(t)\| = +\infty$ if $(X, \|\cdot\|)$ is a Banach space. Then we have that the problem $(\mathrm{DFP})_L$ is densely $\overline{\{(Cf_0, \cdots, Cf_{\lceil\alpha\rceil-1}) : \vec{f} = (f_0, \cdots, f_{\lceil\alpha\rceil-1}) \in F\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.

Remark 2:

(i) Suppose that $l \in \mathbb{N}_0$. Then the increasing family of seminorms $p_{n,l}^{A,B,C}(\cdot) := p_n(C^{-l}\cdot) + p_n(C^{-l}A\cdot) + p_n(C^{-l}B\cdot)$ $(n \in \mathbb{N})$ turns $C^l(D(A) \cap D(B))$ into a Fréchet space, which will be denoted by $[D(A) \cap D(B)]_C^l$ in the sequel. In the concrete situation of Theorem 1 or Theorem 2, F_i can be chosen to be some of closed linear subspaces of $[D(A) \cap D(B)]_C^l$ that is separable for the topology induced from $[D(A) \cap D(B)]_C^l$. If X is separable and $C = I$, then for each number $\lambda > \omega$ the mapping $(\lambda^\alpha B - A)^{-1} : X \to [D(A) \cap D(B)]$ $(\equiv [D(A) \cap D(B)]_I^0)$ is a linear topological isomorphism and, in this case, $[D(A) \cap D(B)]$ and F_i will be separable $(i \in \mathcal{V})$.

(ii) If we suppose additionally that $R(t)B \subseteq BR(t)$, $t \geq 0$ in the formulation of Theorem 2, then we do not need to assume that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(Bf_i) \leq cq_{n,i}(f_i)$, $f_i \in F_i$ (because, in this case, the operator $V_L(t)\vec{\cdot} = \sum_{i=0}^{\lceil\alpha\rceil-1}(g_i * R(\cdot)B\cdot_i)(t)$, $t \geq 0$ $(\vec{\cdot} = (\cdot_0, \cdots, \cdot_{\lceil\alpha\rceil-1}) \in F)$ belongs to the space $L(F, X)$ and $(V_L(t))_{t\geq 0}$ is a strongly continuous operator family in $L(F, X)$ on account of (14)).

In [27, Theorem 5] and [28, Remark 1(ii)], we have recently reconsidered the Desch-Schappacher-Webb and Banasiak-Moszyński criteria for chaos of strongly continuous semigroups ([18], [5]). The following important theorem holds good (cf. [27]-[28] for the notion):

Theorem 3 Suppose that $\alpha \in (0, 2)$ and Ω is an open connected subset of $\mathbb{C}$ which satisfies $\Omega \cap (-\infty, 0] = \emptyset$ and $\Omega \cap i\mathbb{R} \neq \emptyset$. Let $f : \Omega^\alpha \to X$ be an analytic mapping such that $f(\lambda^\alpha) \in N(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$, and let $\tilde{X} := \overline{span\{f(\lambda^\alpha) : \lambda \in \Omega\}}$. Then the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are $\tilde{X}$-topologically mixing; furthermore, if $Af(\lambda^\alpha) \in \tilde{X}$ for all $\lambda \in \Omega$, then the problems $(\mathrm{DFP})_R^{\tilde{X}}$ and $(\mathrm{DFP})_L^{\tilde{X}}$, obtained by replacing the operators A and B in $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ with the operators $A_{|\tilde{X}}$ and $B_{|\tilde{X}}$, respectively, are topologically mixing in the Fréchet space $\tilde{X}$.

Keeping Theorem 3 in mind, it is very natural to raise the following issue:

Problem 1. Suppose that $\alpha \in (0, 2)$ and Ω is an open connected subset of $\mathbb{C}$ which satisfies $\Omega \cap (-\infty, 0] = \emptyset$ and $\Omega \cap i\mathbb{R} \neq \emptyset$. Let $f : \Omega^\alpha \to X$ be an analytic mapping such that $f(\lambda^\alpha) \in N(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$. Does there exist a closed linear subspace X' of $X^{\lceil\alpha\rceil}$ such that the problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ are (densely) X'-distributionally chaotic?

The method proposed in the proofs of [15, Theorem 4.1] and its discrete precursor [11, Theorem 15] cannot be applied here and, because of that, we will have to follow some other paths capable of moving us towards a solution of this problem. Unfortunately, we will present only some partial answers to Problem 1 by assuming that the strong solutions of problem (DFP) are goverened by an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11) (in the case of problem $(\mathrm{DFP})_R$) or an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B (in the case of problem $(\mathrm{DFP})_L$); this has not been the case in our previous research studies [27-28]. We start by stating the following result.

Theorem 4

(i) Suppose that $0 < \alpha < 2$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geq 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11), (P) holds and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Let F_i be a separable complex Fréchet space, let $F_i \subseteq D(A) \cap D(B)$, and let F_i be continuously embedded in X $(i \in \mathcal{V})$. Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(CBf_i) \leq c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$. Let $H_i : \Omega^\alpha \to F_i$ be an analytic mapping such that $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$ $(i \in \mathcal{V})$. Set $F_i' := \overline{span\{H_i(\lambda^\alpha) : \lambda \in \Omega\}}^{F_i}$ $(i \in \mathcal{V})$, $F_i' := \{0\}$ $(i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V})$ and $F' := \prod_{i=0}^{\lceil\alpha\rceil-1} F_i'$. Then the problem $(\mathrm{DFP})_R$ is densely $\overline{\{(CBf_0', \cdots, CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \cdots, f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.

(ii) Suppose that $0 < \alpha < 2$, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, $(R(t))_{t\geq 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B, and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$. Let F_i be a separable complex Fréchet space, and let $F_i \subseteq D(A) \cap D(B)$ $(i \in \mathcal{V})$. Suppose that for each $n \in \mathbb{N}$ and $i \in \mathcal{V}$ there exist a number $c_{n,i} > 0$ and a continuous seminorm $q_{n,i}(\cdot)$ on F_i so that $p_n(Bf_i) + p_n(Cf_i) \leq c_{n,i}q_{n,i}(f_i)$, $f_i \in F_i$. Set $G_i := F_i$, if $i \in \mathcal{V}$, $G_i := \{0\}$, if $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $F := \prod_{i=0}^{\lceil\alpha\rceil-1} G_i$. Let $H_i : \Omega^\alpha \to F_i$ be an analytic mapping such that $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$ $(i \in \mathcal{V})$. Set $F_i' := \overline{span\{H_i(\lambda^\alpha) : \lambda \in \Omega\}}^{F_i}$ $(i \in \mathcal{V})$, $F_i' := \{0\}$ $(i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V})$ and $F' := \prod_{i=0}^{\lceil\alpha\rceil-1} F_i'$. Then the problem $(\mathrm{DFP})_L$ is densely $\overline{\{(Cf_0', \cdots, Cf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0', \cdots, f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$-distributionally chaotic.

Proof. Suppose that Ω_0 is an arbitrary open connected subset of Ω which admits a cluster point in Ω. Then the (weak) analyticity of mapping $\lambda \mapsto H_i(\lambda^\alpha) \in F_i$, $\lambda \in \Omega$ implies that $\Psi(\Omega_0, i) := \mathrm{span}\{H_i(\lambda^\alpha) : \lambda \in \Omega_0\}$ is dense in the Fréchet space F_i'; in particular, $(F_i')_0 := \Psi(\Omega \cap \mathbb{C}_-, i)$ is dense in F_i' $(i \in \mathcal{V})$. The remaining part of proof is almost the same in cases (i) and (ii), so that we will consider only (i). Since $H_i(\lambda^\alpha) \in N(A - \lambda^\alpha B) \setminus \{0\}$, $\lambda \in \Omega$, we can apply the uniqueness theorem for Laplace transform, (13) and the well known identity

$$\int_0^\infty e^{-zt} t^{\beta-1} E_{\alpha,\beta}\big(t^\alpha \lambda^\alpha\big)\, dt = \frac{z^{\alpha-\beta}}{z^\alpha - \lambda^\alpha}, \quad \Re z > |\lambda|,\ \lambda \in \Omega,$$

see e.g. [9, (1.26)], in order to see that $R(t)H_i(\lambda^\alpha) = E_\alpha(t^\alpha\lambda^\alpha)CH_i(\lambda^\alpha)$, $t \geq 0$, $\lambda \in \Omega$ and that $(g_i * E_\alpha(\cdot^\alpha\lambda^\alpha))(t) = t^i E_{\alpha,i+1}(t^\alpha\lambda^\alpha)$, $t \geq 0$, $i \in \mathbb{N}_0$ $(i \in \mathcal{V})$. Now the claimed assertion follows from an application of Theorem 1 and the asymptotic expansion formulae (5)-(7).

Remark 3: Suppose that the requirements of Problem 1 hold, $C \in L(X)$ is injective, $CA \subseteq AC$, $CB \subseteq BC$, X is separable, $l \in \mathbb{N}_0$ and $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$.

(i) Let $(R(t))_{t\geq 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family for (11), and let (P) hold. Then the closed graph theorem implies that $(\lambda^\alpha B - A)^{-1}C \in L(X)$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_1$. Suppose that $\Re\lambda_0 > \omega_1$, $(\lambda^\alpha B - A)^{-1}CA \subseteq A(\lambda^\alpha B - A)^{-1}C$ and $(\lambda^\alpha B - A)^{-1}CB \subseteq$

$B(\lambda^\alpha B-A)^{-1}C$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_1$. Set

$$X_l := \overline{span\Big\{C^l\big(\lambda_0^\alpha B-A\big)^{-1}Cf\big(\lambda^\alpha\big) : \lambda \in \Omega\Big\}}^{[D(A)\cap D(B)]_C^l}.$$

Then the mapping $G : X \to [D(A)\cap D(B)]_C^l$, given by $G(x) := C^l(\lambda_0^\alpha B - A)^{-1}Cx$, $x \in X$, is continuous and the space $\overline{G(X)}^{[D(A)\cap D(B)]_C^l}$ is separable. Set $F_i := [D(A)\cap D(B)]_C^l$ $(i \in \mathcal{V})$. Define G_i and the space F as in the formulation of Theorem 4(i). Then, for every $i \in \mathcal{V}$, the mapping $H_i : \Omega^\alpha \to F_i$, given by $H_i(\lambda^\alpha) := C^l(\lambda_0^\alpha B - A)^{-1}Cf(\lambda^\alpha)$, $\lambda \in \Omega$, is analytic $(i \in \mathcal{V})$. Define now F_i' and F' as in the formulation of Theorem 4(i). Applying Theorem 4(i), we get that $(\mathrm{DFP})_R$ is densely $\overline{\{(CBf_0',\cdots,CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0',\cdots,f_{\lceil\alpha\rceil-1}') \in F'\}}^{X^{\lceil\alpha\rceil}}$ -distributionally chaotic (observe that $C^l(X_m) = X_{m+l}$ and $C(X_l) \subseteq X_l$ for all $l,\ m \in \mathbb{N}_0$, as well as that $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_l \supseteq \cdots$ and $\{(CBf_0',\cdots,CBf_{\lceil\alpha\rceil-1}') : \vec{f'} = (f_0',\cdots,f_{\lceil\alpha\rceil-1}') \in F'\} = \{(x_0,\cdots,x_{\lceil\alpha\rceil-1}) \in X^{\lceil\alpha\rceil} : x_i = 0$ for $i \in \mathbb{N}^0_{\lceil\alpha\rceil-1} \setminus \mathcal{V}$, and $x_i \in B(X_{l+1})$ for $i \in \mathcal{V}\}$; in the sequel, we will use the abbreviation $X^{\lceil\alpha\rceil}_{B,l,\mathcal{V}}$ to denote the above set).

(ii) Let $(R(t))_{t\geq 0}$ be an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by A, B, obeying additionally that $(\lambda^\alpha B - A)^{-1}C$ commutes with A and B for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$. Then Theorem 4(ii) and a similar argumentation imply that the problem $(\mathrm{DFP})_L$ is densely $\overline{\{(x_0,\cdots,x_{\lceil\alpha\rceil-1}) \in X^{\lceil\alpha\rceil} : x_i = 0,\ i \in \mathbb{N}^0_{\lceil\alpha\rceil-1}\setminus\mathcal{V};\ x_i \in X_{l+1},\ i \in \mathcal{V}\}}^{X^{\lceil\alpha\rceil}}$ -distributionally chaotic. We will denote the above set simply by $X^{\lceil\alpha\rceil}_{l,\mathcal{V}}$.

Now we will present an illustrative example of application of obtained theoretical results.

Example 2. Suppose that $0 < \alpha < 2$, $\cos(\pi/\alpha) \leq 0$, $l \in \mathbb{N}_0$, $\emptyset \neq \mathcal{V} \subseteq \mathbb{N}^0_{\lceil\alpha\rceil-1}$, $1 \leq p < \infty$, $\omega \geq 0$, $P_1(z)$ and $P_2(z)$ are non-zero complex polynomials, $N_1 = dg(P_1(z))$, $N_2 = dg(P_2(z))$, $P_2(x) \neq 0$ for all $x \in \mathbb{R}$, $\beta > \frac{1}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ and

$$\sup_{x\in\mathbb{R}} \Re\left(\left(\frac{P_1(x)}{P_2(x)}\right)^{1/\alpha}\right) \leq \omega.$$

Then there exist numbers $z_0 \in \mathbb{C}$ and $r \geq 0$ such that:

$$P_1\big(-iz_0\big) = re^{\pm i\alpha\pi/2}P_2\big(-iz_0\big),\ P_2\big(z_0\big) \neq 0$$

and

$$P_1(-i\cdot)'\big(z_0\big)P_2\big(-iz_0\big) - P_1\big(z_0\big)P_2(-i\cdot)'\big(z_0\big) \neq 0.$$

Let $a > 0$ be such that $|\Re\big(z_0\big)| < a/p$. Set $\rho(x) := e^{-a|x|}$, $x \in \mathbb{R}$,

$$L^p_\rho(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f(\cdot) \text{ is measurable, } \int_{\mathbb{R}} |f(x)|^p \rho(x)\,dx < \infty \right\}$$

and $\|f\| := (\int_{\mathbb{R}} |f(x)|^p \rho(x)\,dx)^{1/p}$; equipped with this norm, $X := L^p_\rho(\mathbb{R})$ becomes an infinite-dimensional separable complex Banach space. It is well known that the operator $-iA_0$, defined by

$$D\big(-iA_0\big) := \big\{f \in X \mid f(\cdot) \text{ is loc. abs. continuous, } f' \in X\big\},\ \big(-iA_0\big)f := f',$$

is the generator of a C_0-group on X (cf. [18, Theorem 4.9]). Therefore, we can define the closed linear operators $A := \overline{P_1(A_0)}$ and $B := \overline{P_2(A_0)}$ on X by using the functional calculus for bounded commuting C_0-groups (cf. [16] and [25, Section 2.5] for more details); these operators are densely defined, and B is injective ([30]). Arguing as in [27, Example 7], we obtain that there exist an open connected subset Ω of $\mathbb{C} \setminus (-\infty, 0]$ intersecting the imaginary axis and an open connected neighborhood W of point z_0, contained in the vertical strip $\{z \in \mathbb{C} : |\Re(z)| < a/p\}$, such that the mapping $(P_1(-i\cdot)/P_2(-i\cdot))^{-1} : \Omega^\alpha \to W$ is well defined, analytic and bijective. Set

$$f\big(\lambda^\alpha\big) := e^{(P_1(-i\cdot)/P_2(-i\cdot))^{-1}(\lambda^\alpha)\cdot}, \quad \lambda \in \Omega$$

and, for every $t \geq 0$,

$$R_\alpha(t) := \left(E_\alpha\left(t^\alpha \frac{P_1(x)}{P_2(x)}\right)\big(1 + |x|^2\big)^{-\beta/2}\right)\big(A_0\big), \quad G_\alpha(t) := \overline{P_2\big(A_0\big)}^{-1} R_\alpha(t).$$

Then $(R_\alpha(t))_{t\geq 0} \subseteq L(X)$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family for (11), (P) holds, $(G_\alpha(t))_{t\geq 0} \subseteq L(X, [D(B)])$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family generated by A, B, the mapping $f : \Omega^\alpha \to X$ is analytic and $Af(\lambda^\alpha) = \lambda^\alpha Bf(\lambda^\alpha)$, $\lambda \in \Omega$ [30-31]. Furthermore, there exists $\omega_1 > \omega$ such that $(\lambda^\alpha B - A)^{-1}C$ commutes with A and B for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_1$; here $C = R_\alpha(0)$. By the considerations from Remark 3, it readily follows that the problem $(\mathrm{DFP})_R$ ($(\mathrm{DFP})_L$) is densely $X^{\lceil\alpha\rceil}_{B,l,\mathcal{V}}$-distributionally chaotic (densely $X^{\lceil\alpha\rceil}_{l,\mathcal{V}}$-distributionally chaotic); unfortunately, in the present situation, we do not know to say anything about the optimality of this result. Before quoting some concrete examples where the established conclusions can be applied, it should be noticed that we can prove a similar result provided that the state space is chosen to be the Banach space $C_{0,\rho}(\mathbb{R})$ (cf. [18, Definition 4.3]) or the Fréchet space $X' := \{f \in C^\infty(\mathbb{R}) : f^{(n)} \in L^p_\rho(\mathbb{R})$ for all $n \in \mathbb{N}_0\}$, equipped with the following family of seminorms $p_n(f) := \sum_{j=0}^n \|f^{(j)}\|_{L^p_\rho(\mathbb{R})}$, $n \in \mathbb{N}_0$; in this case, the operators $A_{|X'}$ and $B_{|X'}$ are linear and continuous on X', $(C_{|X'})^{-1} \in L(X')$, as well as $((C_{|X'})^{-1}R_\alpha(t)_{|X'})_{t\geq 0} \subseteq L(X')$ is a global exponentially equicontinuous $(g_\alpha, I_{X'})$-regularized resolvent family for (11), (P) holds in our concrete situation, and $((C_{|X'})^{-1}G_\alpha(t)_{|X'})_{t\geq 0} \subseteq L(X', [D(B_{|X'})])$ is a global exponentially equicontinuous $(g_\alpha, I_{X'})$-regularized resolvent family generated by $A_{|X'}$, $B_{|X'}$):

(i) Assuming that $P_1(z) = -\alpha_0 z^2 - \beta_0 z^4$ and $P_2(z) = \gamma_0 + z^2$, where α_0, β_0, γ_0 are positive real numbers, we are in a position to clarify some results on subspace distributionally chaoticity of equation

$$\big(\gamma_0 - \Delta\big)\mathbf{D}^\alpha_t u = \alpha_0 \Delta u - \beta_0 \Delta^2 u,$$

for which it is well known that plays an important role in evolution modelling of some problems appearing in the theory of liquid filtration ([41]), on $L^p_\rho(\mathbb{R})$ or $C_{0,\rho}(\mathbb{R})$; see also [27, Example 8], where we have considered topologically mixing properties of this equation on symmetric spaces of non-compact type, Damek-Ricci, Riemannian symmetric or Heckman-Opdam root spaces ([23], [3], [38]).

(ii) Assuming that $P_1(z) = z^2$ and $P_2(z) = -\eta z^2 - 1$, where $\eta > 0$, we are in a position to clarify some results on subspace distributionally chaoticity of the Barenblatt-Zheltov-Kochina equation (cf. [17, Example 1.6, p. 50]):

$$(\eta\Delta - 1)\mathbf{D}_t^{\alpha}u(t) + \Delta u = 0 \quad (\eta > 0).$$

This equation is very important in the study of fluid filtration in fissured rocks.

In the remaining part of paper we will always assume that $T_n u(t) = B\mathbf{D}_t^{\alpha_n} u(t) = T_{n,L}u(t)$. Then it is evident that the abstract degenerate Cauchy problem

$$B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} T_i u(t) = 0; \quad u^{(k)}(0) = u_k,\ 0 \le k \le m_n - 1 \tag{15}$$

is a special subcase of problem (ACP). The Caputo fractional derivative $\mathbf{D}_t^{\alpha_n} u(t)$ is defined for any strong solution $t \mapsto u(t)$, $t \ge 0$ of problem (15) and this, in turn, implies that we can define the Caputo fractional derivative $\mathbf{D}_t^{\zeta} u(t)$ for any number $\zeta \in [0, \alpha_n]$ ([25]). Motivated by our recent research study [28], we introduce the following notion:

Definition 5. Let $\tilde{X}$ be a closed linear subspace of X^{m_n}, let $k \in \mathbb{N}$, and let $\vec{\beta} = (\beta_1, \beta_2, \cdot \cdot \cdot, \beta_k) \in [0, \alpha_n]^k$. Then it is said that the abstract Cauchy problem (15) is $(\tilde{X}, \vec{\beta})$-distributionally chaotic iff there are an uncountable set $S \subseteq \tilde{X} \cap \mathfrak{Z}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $\vec{x},\ \vec{y} \in S$ of distinct tuples we have that there exist strong solutions $t \mapsto u(t; \vec{x})$, $t \ge 0$ and $t \mapsto u(t; \vec{y})$, $t \ge 0$ of problem (15) with the property that

$$\overline{Dens}\Biggl(\Biggl\{ t \ge 0 : \sum_{i=1}^{k} d\Bigl(\mathbf{D}_t^{\beta_i} u(t; \vec{x}), \mathbf{D}_t^{\beta_i} u(t; \vec{y})\Bigr) \ge \sigma \Biggr\}\Biggr) = 1 \text{ and}$$

$$\overline{Dens}\Biggl(\Biggl\{ t \ge 0 : \sum_{i=1}^{k} d\Bigl(\mathbf{D}_t^{\beta_i} u(t; \vec{x}), \mathbf{D}_t^{\beta_i} u(t; \vec{y})\Bigr) < \epsilon \Biggr\}\Biggr) = 1.$$

As before, if we can choose S to be dense in $\tilde{X}$, then we say that the problem (15) is densely $(\tilde{X}, \vec{\beta})$-distributionally chaotic (S is called a $(\sigma_{\tilde{X}}, \vec{\beta})$-scrambled set). In the case $\tilde{X} = X^{m_n}$, it is also said that the problem (15) is (densely) $\vec{\beta}$-distributionally chaotic; S is then called a $(\sigma, \vec{\beta})$-scrambled set.

Before proceeding further, it would be worthwhile to mention that the classical definitions of $\tilde{X}$-distributional chaos of problem (15) follows by plugging $\vec{\beta} = (0, 0, \cdot \cdot \cdot, 0) \in [0, \alpha_n]^k$ in Definition 5 and that we can also define some other notions of $\tilde{X}$-distributional chaos of problem (15) by replacing, optionally, some of terms $\mathbf{D}_t^{\beta_i} u(t; \vec{x})$ and $\mathbf{D}_t^{\beta_i} u(t; \vec{y})$ in Definition 5 with $\mathbf{D}_t^{\beta_i} A_i' u(t; \vec{x})$ or $A_i'' \mathbf{D}_t^{\beta_i} u(t; \vec{x})$, and $\mathbf{D}_t^{\beta_i} B_i' u(t; \vec{y})$ or $B_i'' \mathbf{D}_t^{\beta_i} u(t; \vec{y})$, respectively, where A_i', A_i'', B_i', B_i'' are closed linear operators on X $(1 \le i \le k)$. We leave details to the reader.

Now we would like to present an illustrative example from [14].

Example 3. The study of various hypercyclic, chaotic and topologically mixing properties of the viscous van Wijngaarden-Eringen equation:

$$\bigl(1 - a_0^2 u_{xx}\bigr)u_{tt} = (\mathrm{Re}_b)^{-1} u_{xxt} + u_{xx}, \tag{16}$$

which corresponds to the linearized version of the equation that models the acoustic planar propagation in bubbly liquids, has been recently carried out by Conejero, Lizama and Murillo-Arcila in [14]; here $a_0 > 0$ denotes the dimensionless bubble radius and $\mathrm{Re}_b > 0$ is a Reynolds number. The state space in their analysis is chosen to be the space X_ρ of real analytic functions of Herzog type

$$X_\rho := \left\{ f : \mathbb{R} \to \mathbb{C} \ ; \ f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, \ x \in \mathbb{R} \text{ for some } (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\},$$

which is an isomorphic copy of the sequence space $c_0(\mathbb{N}_0)$. More precisely, it has been proved that the bounded matricial operator

$$A := \begin{bmatrix} O & I \\ -(1-a_0^2 u_{xx})^{-1} u_{xx} & (\mathrm{Re}_b)^{-1}(1-a_0^2 u_{xx})^{-1} u_{xx} \end{bmatrix}$$

generates a strongly continuous semigroup on X_ρ^2 satisfying the assumptions of Desch-Schappacher-Webb criterion, provided $a_0 < 1$, $\sqrt{5}/6 < a_0 \mathrm{Re}_b < 1/2$ and $\rho > r_0 a_0^{-1}/(2^{-1}a_0^{-2}(\mathrm{Re}_b)^{-1} - 3r_0)$ $(r_0 := 4^{-1}a_0^{-2}(\mathrm{Re}_b)^{-1}(1-4a_0^2\mathrm{Re}_b^2)^{1/2})$. This immediately implies that the abstract degenerate second order Cauchy problem (16) is densely $(0,1)$-distributionally chaotic (cf. Definition 5, as well as [27, Remark 6] and Theorem 5 below).

From the point of view of linear dynamics, the abstract degenerate equations with integer order derivatives have certain peculiarities compared with abstract degenerate fractional differential equations. For example, the following theorem cannot be so simply reformulated for fractional multi-term problem (15) (cf. [28, Remark 2(iii)], and [28, Section 2] for the notion used):

Theorem 5. ([28, Theorem 3]) Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying that

$$P_\lambda f(\lambda) = \left(\lambda^{\alpha_n} B + \sum_{i=0}^{n-1} \lambda^{\alpha_i} A_i \right) f(\lambda) = 0, \quad \lambda \in \Omega. \tag{17}$$

Set $\vec{x_\lambda} := [f(\lambda)\ \lambda f(\lambda)\ \cdots\ \lambda^{n-1} f(\lambda)]^T$ $(\lambda \in \Omega)$, $E_0 := \mathit{span}\{\vec{x_\lambda} : \lambda \in \Omega\}$, $\tilde{E} := \check{E} := \overline{E_0}$, $\vec{\beta} := (0, 1, \cdots, n-1)$, $W := \mathbb{N}_n$ and $\hat{E}_i := \mathit{span}\{f(\lambda) : \lambda \in \Omega\}$, $i \in W$. Let $\emptyset \neq S \subseteq E^n$ be such that $E_0 \subseteq \mathrm{Orb}(S; (D_i)_{1\leq i \leq l})$. Then $\vec{x_\lambda} \in \mathfrak{M}_{\mathfrak{D}}$, $\lambda \in \Omega$ and the abstract Cauchy problem

$$(ACP)_{B,n} \ : \ Bu^{(n)}(t) + \sum_{i=0}^{n-1} T_i u(t) = 0, \ t \geq 0; \quad u^{(i)}(0) = x_i, \ 0 \leq i \leq n-1$$

is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing provided that $\sum_{j=1}^q e^{\lambda_j \cdot} f(\lambda_j) \in \mathfrak{P}(\sum_{j=1}^q \vec{x_{\lambda_j}})$ for any $\sum_{j=1}^q \vec{x_{\lambda_j}} \in E_0$ $(q \in \mathbb{N};\ \lambda_j \in \Omega,\ 1 \leq j \leq q)$; here, $T_0 u(t) = A_0 u(t)$.

The following problem arises immediately:

Problem 2. Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{C}$ intersecting the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying (17). Does there exist a tuple $\vec{\beta} \in [0, \alpha_n]^k$ and a closed linear subspace X' of X^n such that the problem $(\mathrm{ACP})_{B,n}$ is (densely) $(X', \vec{\beta})$-distributionally chaotic?

This problem can be also rephrased for certain classes of abstract degenerate multi-term fractional differential equations that are not of general type (ACP) considered here; because of that, we shall skip all details (see also [28, Remark 2(ii)]).

Now we are going to enquire into the basic distributionally chaotic properties of the following special subcase of problem (15):

$$B\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=0;\quad u^{(k)}(0)=u_k,\ 0\leq k\leq m_n-1. \tag{18}$$

In our analysis, we are primarily concerned with exploiting Lemma 1 and, because of that, we need to assume that strong solutions of (18) are goverened by some known degenerate resolvent families for (18); besides of that, it is very important to know whether strong solutions of (18) are unique or not. In this paper, we will focus our attention on the use of so-called (C_1, C_2)-existence and uniqueness families for (18) (cf. [32, Definition 3.1] for a slightly general notion):

Definition 6. Suppose that the operators $C_1 \in L(X)$ and $C_2 \in L(X)$ are injective.

(i) A strongly continuous operator family $(E(t))_{t\geq 0} \subseteq L(X)$ is said to be a C_1-existence family for (18) iff, for every $x \in X$, the following holds: $E(\cdot)x \in C^{m_n-1}([0,\infty) : [D(B)])$, $E^{(i)}(0)x = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)x \in C([0,\infty) : X)$ for $0 \leq j \leq n$, and

$$BE^{(m_n-1)}(t)x+\sum_{j=1}^{n-1}A_j\big(g_{\alpha_n-\alpha_j}*E^{(m_n-1)}\big)(t)x=C_1x,\quad t\geq 0.$$

(ii) A strongly continuous operator family $(U(t))_{t\geq 0} \subseteq L(X)$ is said to be a C_2-uniqueness family for (18) iff, for every $t \geq 0$ and $x \in \bigcap_{0\leq j\leq n} D(A_j)$, the following holds:

$$U(t)Bx+\sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j}*U(\cdot)A_jx\big)(t)=g_{m_n}(t)C_2x.$$

Suppose $0 \leq i \leq m_n - 1$. Then we define $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$, $D_i' := \mathbb{N}_{n-1} \setminus D_i$ and

$$\mathbf{D}_i:=\Bigg\{u_i\in\bigcap_{j\in D_i'}D(A_j):A_ju_i\in R(C_1),\ j\in D_i'\Bigg\}.$$

The existence of a C_2-uniqueness family for (18) implies the uniqueness of strong solutions of this problem, while the existence of a C_1-existence family for (18) implies the following:

Lemma 3. ([32, Theorem 3.4(i)]) Suppose that $(E(t))_{t\geq 0}$ is a C_1-existence family for (18), and $u_i \in \mathbf{D}_i$ for $0 \leq i \leq m_n - 1$. Define, for every $t \geq 0$,

$$u(t):=\sum_{i=0}^{m_n-1}u_ig_{i+1}(t)-\sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big(g_{\alpha_n-\alpha_j}*E^{(m_n-1-i)}\Big)(t)C_1^{-1}A_ju_i. \tag{19}$$

Then the function $t \mapsto u(t)$, $t \geq 0$ is a strong solution of (18).

The Laplace transform techniques can be used to prove the following:

Lemma 5. ([32, Theorem 3.5]) Suppose that $(E(t))_{t\geq 0} \subseteq L(X)$, $(U(t))_{t\geq 0} \subseteq$

$L(X)$, $\omega \geq 0$, $C_1 \in L(X)$ and $C_2 \in L(X)$ are injective. Set

$$\mathbf{P}_\lambda := B + \sum_{j=1}^{n-1} \lambda^{\alpha_j - \alpha_n} A_j, \quad \Re\lambda > 0.$$

(i) Suppose that the operator $\mathbf{P}_\lambda$ is injective for every $\lambda > \omega$, as well as that there exist strongly continuous operator families $(W(t))_{t\geq 0} \subseteq L(X)$ and $(W_j(t))_{t\geq 0} \subseteq L(X)$ such that $\{e^{-\omega t}W(t) : t \geq 0\}$ and $\{e^{-\omega t}W_j(t) : t \geq 0\}$ are equicontinuous $(1 \leq j \leq n)$ as well as that:

$$\int_0^\infty e^{-\lambda t} W(t)x\,dt = \lambda^{-1}\mathbf{P}_\lambda^{-1}C_1 x \text{ and } \int_0^\infty e^{-\lambda t} W_j(t)x\,dt = \lambda^{\alpha_j - \alpha_n - 1}A_j \mathbf{P}_\lambda^{-1} C_1 x,$$

for every $\lambda > \omega$, $x \in X$ and $j \in \mathbb{N}_n$. Then there exists a C_1-existence family for (18), denoted by $(E(t))_{t\geq 0}$. Furthermore, $E^{(m_n-1)}(t)x = W(t)x$, $t \geq 0$, $x \in X$ and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(t)x = W_j(t)x$, $t \geq 0$, $x \in X$, $j \in \mathbb{N}_n$.

(ii) Suppose $(U(t))_{t\geq 0}$ is strongly continuous and the operator family $\{e^{-\omega t}U(t) : t \geq 0\}$ is equicontinuous. Then $(U(t))_{t\geq 0}$ is a C_2-uniqueness family for (18) iff, for every $x \in \bigcap_{j=0}^n D(A_j)$, the following holds:

$$\int_0^\infty e^{-\lambda t} U(t)\mathbf{P}_\lambda x\,dt = \lambda^{-m_n} C_2 x, \quad \Re\lambda > \omega.$$

The following theorem can be proved by using Lemma 1 and Lemma 3.

Theorem 6. Suppose that $C_1 \in L(X)$ is injective and $(E(t))_{t\geq 0}$ is a C_1-existence family for (18). Let F be a separable complex Fréchet space, let $F \subseteq \mathbf{D}_0 \times \mathbf{D}_1 \times \cdots \times \mathbf{D}_{m_n-1}$, and let F be continuously embedded in X^{m_n}. Define $V(t) : F \to X$ by $V(t)\vec{u} := \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)})(t)C_1^{-1}A_j u_i$ $(t \geq 0$, $\vec{u} = (u_0, u_1, \cdots, u_{m_n-1}) \in F$; cf. (19)). Suppose that $V(t) \in L(F, X)$ for all $t \geq 0$, as well as that F_0 is a dense subset of F satisfying that $\lim_{t\to+\infty} V(t)\vec{x} = 0$, $\vec{x} \in F_0$. Let there exist $\vec{x} \in F$, $m \in \mathbb{N}$ and a set $B \subseteq [0, \infty)$ such that $\overline{Dens}(B) = 1$, and $\lim_{t\to+\infty, t\in B} p_m(V(t)\vec{x}) = +\infty$, resp. $\lim_{t\to+\infty,t\in B} \|V(t)\vec{x}\|_X = +\infty$ if $(X, \|\cdot\|_X)$ is a Banach space. Then the problem (18) is densely $\overline{F}^{X^{m_n}}$-distributionally chaotic.

Now we would like to present an illustrative application of Theorem 6 in the study of distributionally chaotic properties of fractional analogons of the viscous van Wijngaarden-Eringen equation.

Example 4. Suppose $1/2 < \alpha \leq 1$ and $p > 2$. We are considering the following fractional degenerate multi-term problem:

$$\begin{aligned}\big(1 + a_0^2\Delta_{X,p}^\natural\big)\mathbf{D}_t^{2\alpha}u(t,x) + (\mathrm{Re}_b)^{-1}\Delta_{X,p}^\natural \mathbf{D}_t^\alpha u(t,x) + \Delta_{X,p}^\natural u(t,x) = 0, \quad t \geq 0;\\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \qquad (20)\end{aligned}$$

on a symmetric space X of non-compact type and rank one. Let P_p be the parabolic domain defined in [23]; then we know that $int(P_p) \subseteq \sigma_p(\Delta_{X,p}^\natural)$. In our concrete situation, we have that $n = 3$, $\alpha_3 = 2\alpha$, $\alpha_2 = \alpha$, $\alpha_1 = 0$, $B = (1 + a_0^2\Delta_{X,p}^\natural)$, $A_2 = (\mathrm{Re}_b)^{-1}\Delta_{X,p}^\natural$, $A_1 = \Delta_{X,p}^\natural$ and $\mathbf{P}_\lambda = 1 + (a_0^2 + \lambda^{-\alpha} + \lambda^{-2\alpha})\Delta_{X,p}^\natural$ for $\Re\lambda > 0$. Then it is clear that $z(\lambda) := (a_0^2 + \lambda^{-\alpha} + \lambda^{-2\alpha})^{-1} \to a_0^{-2}$ for $|\lambda| \to \infty$, as well as that

$$\lambda^{-1}\mathbf{P}_\lambda^{-1} = \lambda^{-1}z(\lambda)\big(z(\lambda) + \Delta_{X,p}^\natural\big)^{-1}, \quad \Re\lambda > 0 \text{ suff. large.}$$

Taking into account [25, Theorem 1.2.5] and the fact that the operator $-\Delta^{\natural}_{X,p}$ generates an analytic strongly continuous semigroup on X, we may conclude from the above that $\lambda^{-1}\mathbf{P}_{\lambda}^{-1} \in LT - L(X)$. Since $\lambda^{-1}z(\lambda)I \in LT - L(X)$ (cf. the proof of [27, Theorem 11]), we can apply the resolvent equation and [25, Theorem 1.2.5] in order to see that

$$\lambda^{-2\alpha-1}\Delta^{\natural}_{X,p}z(\lambda)\big(z(\lambda)+\Delta^{\natural}_{X,p}\big)^{-1} \in LT - L(X),$$

$$\lambda^{-\alpha-1}(\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}z(\lambda)\big(z(\lambda)+\Delta^{\natural}_{X,p}\big)^{-1} \in LT - L(X)$$

and

$$\lambda^{-1}\big(1+a_0^2\Delta^{\natural}_{X,p}\big)z(\lambda)\big(z(\lambda)+\Delta^{\natural}_{X,p}\big)^{-1} \in LT - L(X).$$

By Lemma 4(i), we have that there exists an exponentially bounded I-existence family $(E(t))_{t\geq 0}$ for (20). It is not difficult to see with the help of Lemma 4(ii) that $(E(t))_{t\geq 0}$ is likewise an exponentially bounded I-uniqueness family for (20), so that the strong solutions of (20) are unique. Furthermore, we have that $\mathbf{D}_i = D(\Delta^{\natural}_{X,p})$ for $i = 0, 1$. Let $f : int(P_p) \rightarrow X \setminus \{0\}$ be an analytic mapping satisfying that $\Delta^{\natural}_{X,p}f(\lambda) = \lambda f(\lambda)$, $\lambda \in int(P_p)$. Using the proof of [27, Theorem 11], we get that the function $t \mapsto u(t; (f(\lambda), f(\lambda')))$, $t \geq 0$, given by

$$u\Big(t; \big(f(\lambda), f(\lambda')\big)\Big) := H_0(\lambda,t)f(\lambda) + H_1(\lambda',t)f(\lambda'), \quad t \geq 0 \quad \big(\lambda,\ \lambda' \in int(P_p)\big),$$

where

$$H_0(\lambda,t) := \mathcal{L}^{-1}\left(\frac{z^{2\alpha-1} - (\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1}z^{\alpha-1}}{z^{2\alpha} - (\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1}z^{\alpha} - \big(\lambda - a_0^2\big)^{-1}}\right)(t), \quad t \geq 0,$$

and

$$H_1(\lambda',t) := \mathcal{L}^{-1}\left(\frac{z^{2\alpha-1}}{z^{2\alpha} - (\mathrm{Re}_b)^{-1}\big(\lambda' - a_0^2\big)^{-1}z^{\alpha} - \big(\lambda' - a_0^2\big)^{-1}}\right)(t), \quad t \geq 0,$$

is a unique strong solution of (20) with $u(0,\cdot) = f(\lambda)$ and $u_t(0,\cdot) = f(\lambda')$. Direct computations, similar to those already established in [27, Example 13], show that

$$H_0(\lambda,t) = \frac{r_1(\lambda) - (\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1}}{\sqrt{D_\lambda}}e^{r_1(\lambda)t} - \frac{r_2(\lambda) - (\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1}}{\sqrt{D_\lambda}}e^{r_2(\lambda)t}$$

if $\alpha = 1$,

$$\begin{aligned} H_0(\lambda,t) =& \frac{t^{-\alpha}}{\sqrt{D_\lambda}}\Big[E_{\alpha,1-\alpha}\big(r_1(\lambda)t^{\alpha}\big) - E_{\alpha,1-\alpha}\big(r_2(\lambda)t^{\alpha}\big)\Big] \\ & - \frac{(\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1}}{\sqrt{D_\lambda}}\Big[E_{\alpha}\big(r_1(\lambda)t^{\alpha}\big) - E_{\alpha}\big(r_2(\lambda)t^{\alpha}\big)\Big], \quad t > 0, \end{aligned}$$

if $0 < \alpha < 1$, and

$$H_1(\lambda',t) = \frac{t^{1-\alpha}}{\sqrt{D_\lambda}}\Big[E_{\alpha,2-\alpha}\big(r_1(\lambda')t^{\alpha}\big) - E_{\alpha,2-\alpha}\big(r_2(\lambda')t^{\alpha}\big)\Big], \quad t > 0,$$

where

$$r_{1,2}(\lambda) := \frac{(\mathrm{Re}_b)^{-1}\big(\lambda - a_0^2\big)^{-1} \pm \sqrt{(\mathrm{Re}_b)^{-2}\big(\lambda - a_0^2\big)^{-2} + 4\big(\lambda - a_0^2\big)^{-1}}}{2}$$

and

$$D_\lambda := (\mathrm{Re}_b)^{-2}\big(\lambda - a_0^2\big)^{-2} + 4\big(\lambda - a_0^2\big)^{-1}.$$

Set $F := [D(\Delta^{\natural}_{X,p})] \times [D(\Delta^{\natural}_{X,p})]$. Then F is a separable infinite-dimensional complex Banach space. Define $V(t)$ as in the formulation of Theorem 6, with $C_1 = I$; then it is clear that $V(t) \in L(F, X)$ for all $t \geq 0$. From the uniqueness of strong solutions of (20), it readily follows that $V(t)(f(\lambda), f(\lambda')) = u(t; (f(\lambda), f(\lambda')))$, $t \geq 0$ $(\lambda,\ \lambda' \in int(P_p))$. Using the asymptotic expansion formulae (5)-(7) and a simple analysis, we obtain that there exist a sufficiently small number $\epsilon > 0$ and two sufficiently large negative numbers $x_- < 0$ and $x_-^{'} < 0$ such that the first requirement in Theorem 6 holds with $F_0 = span\{(1 + \Delta^{\natural}_{X,p})^{-1} f(\lambda) : \lambda \in L(x_-, \epsilon)\} \times span\{(1 + \Delta^{\natural}_{X,p})^{-1} f(\lambda') : \lambda' \in L(x_-^{'}, \epsilon)\}$. It is clear that there exists a great number of concrete situations (consider, for example, the case in which $a_0 \to 0+$ and $\mathrm{Re}_b \to +\infty$) in which there exists a number $\lambda_0 \in int(P_p)$ such that

$$r_1(\lambda_0) \in \Sigma_{\gamma\pi/2}.$$

If this is the case, the vector $(f(\lambda_0), 0)$ is distributionally unbounded and the problem (20) is densely distributionally chaotic; observe also that the problem (20) is densely $(X \times \{0\})$-distributionally chaotic and $(\{0\} \times X)$-distributionally chaotic by Theorem 6. The same holds for the problem (20)' obtained by interchanging the terms $(\mathrm{Re}_b)^{-1}\Delta^{\natural}_{X,p}\mathbf{D}_t^{\alpha}u(t,x)$ and $(\mathrm{Re}_b)^{-1}\mathbf{D}_t^{\alpha}\Delta^{\natural}_{X,p}u(t,x)$ in (20); this follows directly from Definition 2 and the fact that the mapping $t \mapsto u(t; (f(\lambda), f(\lambda')))$, $t \geq 0$, defined above, is still a strong solution of (20)' for $\lambda,\ \lambda' \in int(P_p)$; cf. [27]. Observe, finally, that all established conclusions for the problems (20) and (20)' continue to hold if we replace the operator $\Delta^{\natural}_{X,p}$ and the state space X in our analysis with the operator $(\Delta^{\natural}_{X,p})_\infty$ and the Fréchet space $[D_\infty(\Delta^{\natural}_{X,p})]$, respectively.

In the previous example, we have employed some ideas contained in the proof of [27, Theorem 11]. Assuming that the requirements of this theorem hold, we can pose the problem of existence of a closed linear subspace X' of X^{m_n}, an integer $k \in \mathbb{N}$ and a tuple $\vec{\beta} \in [0, \alpha_n]^k$ such that the problem [27, (8)] is (densely) $(X', \vec{\beta})$-distributionally chaotic; a similar question can be posed for the problem (15), cf. [28, Remark 1(iii)] for more details.

3. Conclusion

In this paper, we have enquired into the most important distributionally chaotic properties of abstract degenerate (multi-term) fractional differential equations with Caputo derivatives. We have proposed three interesting problems. In the present situation, we can give only some partial answers to these problems by using Lemma 1 as an essential tool in the consideration.

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