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EXISTENCE AND LOCAL ATTRACTIVITY RESULTS FOR A PERTURBED FRACTIONAL INTEGRAL EQUATION

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ABSTRACT. In this paper two existence results concerning local attractivity and local asymptotic attractivity for a perturbed fractional integral equation are proved. In our considerations we apply the technique of measures of noncompactness and the Schauder fixed point principle. The mentioned equation is considered in the Banach space of real functions defined, continuous and bounded on $\mathbb{R}_+$.

1. Introduction

Let $\alpha \in (0,1)$ be fixed number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\Gamma(\cdot)$ denote the Gamma function. This paper is to investigate the following perturbed fractional integral equation

$$x(t) = g(t, x(t)) + \frac{(Ax)(t)}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds, \qquad t \in \mathbb{R}_+, \tag{1}$$

where $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $u: \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ are given functions and $A : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ is an operator. The equation (1) is more general than equations studied in [1]-[3].

The nonlinear integral equations of fractional order play an important part in solving problems of physics, mechanics, engineering and other fields [4]-[6]. Numerous research papers devoted to nonlinear integral equations of fractional order have appeared[1] -[3],[7]-[12]. These papers contain various qualitative properties such as existence, stability, attractivity and positivity behavior for nonlinear integral equations of fractional order.

The goal of this paper is to prove two existence results concerning local attractivity and local asymptotic attractivity of equation (1) in the space $BC(\mathbb{R}_+)$ of real functions defined, continuous and bounded on $\mathbb{R}_+$. The technique of measures of noncompactness and the Schauder fixed point principle are used in our considerations.

It is worthwhile mentioning that the novelty of our approach mainly in obtaining the local attractivity and asymptotic attractivity of solutions for equation (1).

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2. Preliminaries

First we accept a few facts concerning fractional calculus [13]. For $x \in L^1(a,b)$ and a fixed number $\alpha > 0$, the Riemann-Liouville fractional integral of order α of the function $x(t)$ is defined by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a,b).$$

Next, suppose $(E, \|\cdot\|)$ is a real Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x with the radius r. B_r stands for the ball $B(\theta, r)$. If $X \subset E$, then $\bar{X}$ and co(X) stand for the closure and convex closure of X, respectively. Let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ denote the family of all relatively compact sets.

Definition 2.1[14] A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:
(1) The family ker$\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and ker$\mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(\bar{X}) = \mu(X) = \mu(\mathrm{co}(X))$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(5) If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n (n = 1, 2, \cdots)$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \cap_{n=1}^\infty X_n$ is nonempty.

In what follows, We will work in the space $BC(\mathbb{R}_+)$ mentioned in Introduction. This space is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$. For any nonempty and bounded subset $X \subset BC(\mathbb{R}_+)$, $x \in X$, $t \in \mathbb{R}_+$, $T > 0$ and $\varepsilon \geq 0$ define
$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,T], |t-s| \leq \varepsilon\}$,
$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}$,
$\omega_0^T(X) = \lim\limits_{\varepsilon\to 0} \omega^T(X,\varepsilon)$, $\omega_0(X) = \lim\limits_{T\to\infty} \omega_0^T(X)$,
$X(t) = \{x(t) : x \in X\}$, diam$X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}$ and

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \mathrm{diam} X(t). \tag{2}$$

The function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ [14]. The kernel kerμ of this measure consists of nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity.

Now, let Ω be a nonempty subset of $BC(\mathbb{R}_+)$ and $Q : \Omega \to BC(\mathbb{R}_+)$ be an operator. Consider the following operator equation

$$x(t) = (Qx)(t), \qquad t \in \mathbb{R}_+. \tag{3}$$

Definition 2.2[15] Solutions of equation (3) are locally attractive if there exists an $x_0 \in BC(\mathbb{R}_+)$ and an $r > 0$ such that for arbitrary solutions $x(t)$ and $y(t)$ of equation (3) belonging to $B(x_0, r) \cap \Omega$ satisfy

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{4}$$

If for each $\varepsilon > 0$ there exists a $T > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon \tag{5}$$

for all $x(t), y(t) \in B(x_0, r) \cap \Omega$ being solutions of equation (3) and for $t \geq T$, then solutions of equation (3) are uniformly locally attractive on $\mathbb{R}_+$.

Definition 2.3[15] A line $y(t) = at + b(a, b \in \mathbb{R})$, is called an attractor for the solution $x(t) \in BC(\mathbb{R}_+)$ of equation (3) if $\lim\limits_{t\to\infty} [x(t) - (at + b)] = 0$.

Definition 2.4[15] Solutions of equation (3) are locally asymptotically attractive if there exists an $x_0 \in BC(\mathbb{R}_+)$ and an $r > 0$ such that for arbitrary solutions $x(t)$ and $y(t)$ of equation (3) belonging to $B(x_0, r) \cap \Omega$ the condition (4) is satisfied and there is line which is a common attractor to them on $\mathbb{R}_+$. If for each $\varepsilon > 0$ there exists a $T > 0$ such that the inequality (5) is satisfied for $t \geq T$ and for all $x(t), y(t) \in B(x_0, r) \cap \Omega$ being the solutions of equation (3) having a line as a common attractor, then solutions of equation (3) are uniformly locally asymptotically attractive on $\mathbb{R}_+$.

3. Main results

In this section, the equation (1) will be considered under the following assumptions.

(C_1) The function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $p \in \mathbb{R}_+$ such that

$$|g(t, x) - g(t, y)| \leq p|x - y|, \quad \forall t \in \mathbb{R}_+, \quad x, y \in \mathbb{R}.$$

(C_2) The operator $A : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ is continuous and there exist constants $a, b \in \mathbb{R}_+$ such that

$$|(Ax)(t)| \leq a + b|x(t)|, \quad \forall t \in \mathbb{R}_+, \quad x \in BC(\mathbb{R}_+).$$

(C_3) The function $u : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exist a continuous function $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t, s, x)| \leq c(t, s)\varphi(|x|), \quad \forall t, s \in \mathbb{R}_+, \quad x \in BC(\mathbb{R}_+).$$

Moreover,

$$\lim_{t\to\infty} \int_0^t \frac{c(t, s)}{(t - s)^{1-\alpha}} ds = 0.$$

Remark 3.1. The function $\int_0^t \frac{c(t,s)}{(t-s)^{1-\alpha}} ds$ is continuous on $\mathbb{R}_+$ (see [8]). Denote the function $d : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$d(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{c(t, s)}{(t - s)^{1-\alpha}} ds.$$

Then $d(t)$ is continuous on $\mathbb{R}_+$ and $\lim\limits_{t\to\infty} d(t) = 0$. Thus $\bar{d} = \sup\{d(t) : t \in \mathbb{R}_+\}$ is finite.

Now, we denote g_0 by

$$g_0 = \sup\{|g(t, 0)| : t \in \mathbb{R}_+\}. \tag{6}$$

Then $g_0 < \infty$ in view of the assumption (C_1).

Next, we present the last assumption:

(C_4) The inequality

$$pr + g_0 + (a + br)\varphi(r)\bar{d} \leq r$$

has a positive solution r_0 such that $(p + b\bar{d}\varphi(r_0)) < 1$.

Theorem 3.1 Under the assumptions (C_1)-(C_4), equation (1) has at least one solution $x(t)$ in the space $BC(\mathbb{R}_+)$. Moreover, solutions of equation (1) are uniformly locally attractive on $\mathbb{R}_+$.

Proof. Consider the operator V defined on the space $BC(\mathbb{R}_+)$ by the formula

$$(Vx)(t) = g(t, x(t)) + \frac{(Ax)(t)}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds, \qquad t \in \mathbb{R}_+.$$

In order to simplify our considerations, we represent the operator V in the form

$$(Vx)(t) = g(t, x(t)) + (Ax)(t)(Ux)(t), \tag{7}$$

where

$$(Ux)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds.$$

Notice our assumptions, for any function $x \in BC(\mathbb{R}_+)$ the function $g(t, x(t))$ and the operator Ax are continuous on $\mathbb{R}_+$. We show that the same assertion holds also for Ux. To do this fix $T > 0, \varepsilon > 0$, $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leq \varepsilon$. Without loss of generality, we can assume that $t_1 < t_2$. For fixed $t > 0$, the function $s \to c(t, s)$ is continuous on the interval $[0, t]$. Hence $c_t = \sup\{c(t, s) : s \in [0, t]\}$ is finite. Then in view of assumption (C_3), we have

$$\begin{aligned}
&|(Ux)(t_2) - (Ux)(t_1)| \\
=&\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} ds \right| \\
\leq&\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \left| \frac{u(t_2, s, x(s)) - u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} \right| ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \left| \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} - \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \right| ds \\
\leq&\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{\omega^T(u, \varepsilon, \|x\|)}{(t_2 - s)^{1-\alpha}} ds + \frac{c_{t_1}}{\Gamma(\alpha)} \int_0^{t_1} \varphi(|x(s)|) \left[\frac{1}{(t_1 - s)^{1-\alpha}} - \frac{1}{(t_2 - s)^{1-\alpha}} \right] ds \\
&+ \frac{c_{t_2}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\varphi(|x(s)|)}{(t_2 - s)^{1-\alpha}} ds \\
\leq&\frac{\omega^T(u, \varepsilon, \|x\|)}{\Gamma(\alpha)} \frac{t_2^\alpha - (t_2 - t_1)^\alpha}{\alpha} + \frac{c_T \varphi(\|x\|)}{\Gamma(\alpha + 1)} [t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha] + \frac{c_T \varphi(\|x\|)}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha \\
\leq&\frac{1}{\Gamma(\alpha + 1)} [T^\alpha \omega^T(u, \varepsilon, \|x\|) + 2\varepsilon^\alpha c_T \varphi(\|x\|)],
\end{aligned} \tag{8}$$

where $\omega^T(u, \varepsilon, \|x\|) = \sup\{|u(t_2, s, y) - u(t_1, s, y)| : t_1, t_2, s \in [0, T], s \leq t_1, s \leq t_2, |t_1 - t_2| \leq \varepsilon, |y| \leq \|x\|\}$. Using the uniform continuity of the function $u(t, s, y)$ on $[0, T]^2 \times [-\|x\|, \|x\|]$, we have that $\omega^T(u, \varepsilon, \|x\|) \to 0$ as $\varepsilon \to 0$.

By estimate (8), we obtain

$$\omega^T(Ux, \varepsilon) \leq \frac{1}{\Gamma(\alpha + 1)} [T^\alpha \omega^T(u, \varepsilon, \|x\|) + 2\varepsilon^\alpha c_T \varphi(\|x\|)]. \tag{9}$$

From estimate (9), we infer that the function Ux is continuous on $[0, T]$ for any $T > 0$. This yields the continuity of Ux on $\mathbb{R}_+$.

Therefore, we conclude that the function Vx is continuous on $\mathbb{R}_+$.

Now, taking an arbitrary function $x\in BC(\mathbb{R}_+)$, using assumptions (C_1)-(C_3) and Remark 3.1, for a fixed $t\in\mathbb{R}_+$ we get

$$\begin{aligned}|(Vx)(t)| \leq&|g(t,x(t))-g(t,0)|+|g(t,0)|+\frac{a+b\|x\|}{\Gamma(\alpha)}\int_0^t\frac{c(t,s)\varphi(|x(s)|)}{(t-s)^{1-\alpha}}ds\\ \leq& p\|x\|+g_0+(a+b\|x\|)\varphi(\|x\|)\bar{d}.\end{aligned} \tag{10}$$

This shows that Vx is bounded on $\mathbb{R}_+$. Linking this assertion with the earlier proved continuity of the function Vx on $\mathbb{R}_+$, we conclude that Vx is a member of the space $BC(\mathbb{R}_+)$. This shows that the operator V transforms the space $BC(\mathbb{R}_+)$ into itself.

Moreover, estimate (10) yields

$$\|Vx\| \leq p\|x\|+g_0+(a+b\|x\|)\varphi(\|x\|)\bar{d}.$$

Combining this estimate with assumption (C_4), we deduce that there exists a number $r_0>0$ such that the operator $V: BC(\mathbb{R}_+)\to B_{r_0}$, especially $V: B_{r_0}\to B_{r_0}$.

Now, let us take a nonempty subset $X\subset B_{r_0}$. Then, for $x,y\in X$ and for an arbitrarily fixed $t\in\mathbb{R}_+$, using assumptions (C_1)-(C_3) and Remark 3.1 we have

$$\begin{aligned}&|(Vx)(t)-(Vy)(t)|\\ \leq&|g(t,x(t))-g(t,y(t))|+\frac{1}{\Gamma(\alpha)}\left[\left|(Ax)(t)\int_0^t\frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds-(Ay)(t)\int_0^t\frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}ds\right|\right]\\ \leq& p|x(t)-y(t)|+\frac{1}{\Gamma(\alpha)}\left[\left|(Ax)(t)\int_0^t\frac{c(t,s)\varphi(|x(s)|)ds}{(t-s)^{1-\alpha}}\right|+\left|(Ay)(t)\int_0^t\frac{c(t,s)\varphi(|y(s)|)ds}{(t-s)^{1-\alpha}}\right|\right]\\ \leq& p\operatorname{diam}X(t)+2(a+br_0)\varphi(r_0)d(t).\end{aligned} \tag{11}$$

From estimate (11), we derive the following inequality

$$\operatorname{diam}(VX)(t)\leq p\operatorname{diam}X(t)+2(a+br_0)\varphi(r_0)d(t).$$

Then

$$\limsup_{t\to\infty}\operatorname{diam}(VX)(t)\leq p\limsup_{t\to\infty}\operatorname{diam}X(t). \tag{12}$$

Further, For fixed $T>0$, $\varepsilon>0$ and $x\in X$, take $t_1,t_2\in[0,T]$ such that $|t_1-t_2|\leq\varepsilon$. Without loss of generality, we can assume that $t_1<t_2$. Then by representation (7), estimate (8) and taking into account our assumptions, we obtain

$$\begin{aligned}&|(Vx)(t_2)-(Vx)(t_1)|\\ \leq&|g(t_2,x(t_2))-g(t_2,x(t_1))|+|g(t_2,x(t_1))-g(t_1,x(t_1))|\\ &+|(Ax)(t_2)(Ux)(t_2)-(Ax)(t_2)(Ux)(t_1)|+|(Ax)(t_2)(Ux)(t_1)-(Ax)(t_1)(Ux)(t_1)|\\ \leq& p|x(t_2)-x(t_1)|+\omega^T(g,\varepsilon)+\frac{a+br_0}{\Gamma(\alpha+1)}[T^\alpha\omega^T(u,\varepsilon,r_0)+2\varepsilon^\alpha c_T\varphi(r_0)]\\ &+b\bar{d}\varphi(r_0)|x(t_2)-x(t_1)|,\end{aligned} \tag{13}$$

where $\omega^T(g,\varepsilon)=\sup\{|g(t_2,x)-g(t_1,x)|: t_1,t_2\in[0,T],|t_1-t_2|\leq\varepsilon,x\in[-r_0,r_0]\}$. Since the uniform continuity of the functions of $g(t,x)$ on $[0,T]\times[-r_0,r_0]$ and $u(t,s,x)$ on $[0,T]^2\times[-r_0,r_0]$, we infer that $\omega^T(g,\varepsilon)\to 0$ and $\omega^T(u,\varepsilon,r_0)\to 0$ as $\varepsilon\to 0$. Hence by estimate (13) we have

$$\omega_0^T(VX)\leq(p+b\bar{d}\varphi(r_0))\omega_0^T(X).$$

This yields
$$\omega_0(VX) \le (p + b\bar{d}\varphi(r_0))\omega_0(X). \tag{14}$$
Now, linking (12) with (14), keeping in mind formula (2) we obtain
$$\mu(VX) \le (p + b\bar{d}\varphi(r_0))\mu(X). \tag{15}$$

In the following, put $B^1_{r_0} = \mathrm{co}V(B_{r_0})$, $B^2_{r_0} = \mathrm{co}V(B^1_{r_0})$ and so on, then the sequence is decreasing i.e. $B^{n+1}_{r_0} \subset B^n_{r_0} \subset B_{r_0}$ for $n = 1, 2, \cdots$. Moreover, all sets of this sequence are nonempty, bounded, convex and closed. Apart from this, in view of (15) we get $\mu(B^n_{r_0}) \le (p + b\bar{d}\varphi(r_0))^n\mu(B_{r_0})$ for $n = 1, 2, \cdots$. Then by the second inequality of assumption (C_4), we deduce that $\lim\limits_{n\to\infty}\mu(B^n_{r_0}) = 0$. Hence by Definition 2.1, we infer that the set $Y = \cap^{\infty}_{n=1}B^n_{r_0}$ is nonempty, bounded, convex and closed. Moreover, the set Y belongs to $\ker\mu$. In particular, we have
$$\limsup_{t\to\infty} \mathrm{diam}Y(t) = \lim_{t\to\infty} \mathrm{diam}Y(t) = 0. \tag{16}$$
Let us also observe that the operator V maps the set Y into itself.

Next, we show that V is continuous on the set Y.

Let us fix $\varepsilon > 0$ and take arbitrary functions $x, y \in Y$ such that $\|x - y\| \le \varepsilon$. Taking into account the fact that $VY \subset Y$ and using (16), we deduce that there exists a $T > 0$ for an arbitrary $t \ge T$ such that
$$|(Vx)(t) - (Vy)(t)| \le \varepsilon. \tag{17}$$
Further, take $t \in [0, T]$, keeping in mind our assumptions and evaluating similarly as above, we obtain
$$\begin{aligned}
&|x(t) - y(t)| = |(Vx)(t) - (Vy)(t)|\\
\le& p|x(t) - y(t)| + \frac{1}{\Gamma(\alpha)}\left[\left|(Ax)(t)\int_0^t \frac{c(t,s)\varphi(|x(s)|)ds}{(t-s)^{1-\alpha}} - (Ay)(t)\int_0^t \frac{c(t,s)\varphi(|x(s)|)ds}{(t-s)^{1-\alpha}}\right|\right.\\
&+ \left.\frac{1}{\Gamma(\alpha)}|(Ay)(t)|\left[\left|\int_0^t \frac{u(t,s,x(s))ds}{(t-s)^{1-\alpha}} - \int_0^t \frac{u(t,s,y(s))ds}{(t-s)^{1-\alpha}}\right|\right]\right.\\
\le& p\|x - y\| + b\bar{d}\varphi(r_0)\|x - y\| + \frac{a + br_0}{\Gamma(\alpha + 1)}\omega^T(u, \varepsilon)T^\alpha\\
\le& (p + b\bar{d}\varphi(r_0))\varepsilon + \frac{a + br_0}{\Gamma(\alpha + 1)}\omega^T(u, \varepsilon)T^\alpha,
\end{aligned} \tag{18}$$
where $\omega^T(u, \varepsilon) = \sup\{|u(t, s, x) - u(t, s, y)| : t, s \in [0, T], x, y \in Y, |x - y| \le \varepsilon\}$. In virtue of the uniform continuity of the function $u(t, s, x)$, we have $\omega^T(u, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Now by estimates (17) and (18), it follows that the operator V is continuous on Y.

Finally, taking into account all the facts concerning the set Y, the operator $V : Y \to Y$ and using the classical Schauder fixed point principle we infer that the operator V has at least one fixed point $x(t)$ in Y. Obviously, the function $x(t)$ is a solution of equation (1). Moreover, by the fact that the set Y belongs to $\ker\mu$ and the characterization of sets belonging to $\ker\mu$ (see descriptions made after formula (2)), we deduce that all solutions of equation (1) are uniformly locally attractive on $\mathbb{R}_+$ (Definition 2.2). The proof is now completed. □

Next, We introduce the following another two assumptions.
(H_1) The function $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a function $p:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|g(t,x)-g(t,y)|\leq p(t)|x-y|$$

for all $t\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$. Moreover, $\lim_{t\to\infty}p(t)=0$.
(H_2) $\lim_{t\to\infty}g(t,0)=0$.
Theorem 3.2 Under the assumptions (H_1), $(C_2)-(C_4)$ and (H_2), equation (1) has at least one solution $x(t)$ in the space $BC(\mathbb{R}_+)$. Moreover, solutions of equation (1) are uniformly locally asymptotically attractive on $\mathbb{R}_+$.
Proof. By assumption (H_1), it follows that there exists a constant $q\in\mathbb{R}_+$ such that $|p(t)|\leq q$ and

$$|g(t,x)-g(t,y)|\leq q|x-y|. \tag{19}$$

Then linking (19) with assumptions (C_2)-(C_4), by Theorem 3.1, equation (1) has at least one solution $x(t)$ in the space $BC(\mathbb{R}_+)$. Moreover, solutions of equation (1) are uniformly locally attractive on $\mathbb{R}_+$. Let $x(t)$ be any solution of equation (1). In view of assumptions (H_1),(C_2)-(C_4) and Remark 3.1, we have

$$\begin{aligned}|x(t)|\leq&|g(t,x(t))|-|g(t,0)|+|g(t,0)|+\frac{a+b|x(t)|}{\Gamma(\alpha)}\int_0^t\frac{c(t,s)\varphi(|x(s)|)}{(t-s)^{1-\alpha}}ds\\ \leq&p(t)\|x\|+|g(t,0)|+(a+b\|x\|)\varphi(\|x\|)d(t).\end{aligned}$$

Then we obtain $\lim_{t\to\infty}x(t)=0$. Thus, according to Definition 2.3 and Definition 2.4, all solutions of equation (1) are uniformly locally asymptotically attractive on $\mathbb{R}_+$ with the line $y(t)=0$ as a common attractor for them. This completes the proof. $\square$

4. An example

Consider the following nonlinear integral equation of fractional order

$$x(t)=\frac{t\arctan(t+tx(t))}{2+3t^2}+\frac{\frac{|x(t)|}{3(1+|x(t)|)}}{\Gamma(\frac{2}{3})}\int_0^t\frac{(1+t^2+s)^{-\frac{2}{3}}\arctan(|x(s)|)}{(t-s)^{\frac{1}{3}}}ds,\quad t\in\mathbb{R}_+. \tag{20}$$

Observe that equation (20) is a special case of equation (1). If we put

$$\alpha=\frac{2}{3},\ \ g(t,x)=\frac{t\arctan(t+tx)}{2+3t^2},\ \ Ax=\frac{|x|}{3(1+|x|)},\ \ u(t,s,x)=\frac{\arctan(|x|)}{(1+t^2+s)^{\frac{2}{3}}}. \tag{21}$$

In what follows, we show that the assumptions of Theorem 3.1, Theorem 3.2 are satisfied. First, by differential Mean value theorem, there exists a $\xi\in(x,y)$ satisfies

$$\begin{aligned}|g(t,x)-g(t,y)|=&\frac{t}{2+3t^2}|\arctan(t+tx)-\arctan(t+ty)|\\ =&\frac{t}{2+3t^2}|\arctan(t+t\xi)'||x-y|\leq\frac{t}{2+3t^2}\frac{t}{2t+2t^2|\xi|}|x-y|\\ \leq&p(t)|x-y|,\end{aligned}$$

where $p(t)=\frac{t}{2(2+3t^2)}$. Then $\lim_{t\to\infty}p(t)=0$ and $p=\frac{\sqrt{6}}{24}$. Therefore assumptions (C_1) and (H_1) hold.

Next, $|Ax| \leq \frac{1}{3}|x|$, then $a = 0$ and $b = \frac{1}{3}$. Thus the assumption (C_2) is satisfied.

Moreover, $u(t,s,x) \leq c(t,s)\varphi(|x|)$, with $\varphi(|x|) = |x|$ and $c(t,s) = \frac{1}{(1+t^2+s)^{\frac{2}{3}}}$. Furthermore, $\int_0^t \frac{c(t,s)}{(t-s)^{\frac{1}{3}}} ds \leq (1+t^2)^{-\frac{2}{3}} \int_0^t \frac{1}{(t-s)^{\frac{1}{3}}} ds \leq \frac{\frac{3}{2}t^{\frac{2}{3}}}{(1+t^2)^{\frac{2}{3}}}$, then $\lim\limits_{t\to\infty} \int_0^t \frac{c(t,s)}{(t-s)^{1-\alpha}} ds = 0$. Therefore assumption ($\mathrm{C}_3$) holds.

What is more, $d(t) \leq \frac{1}{\Gamma(\frac{2}{3})} \frac{\frac{3}{2}t^{\frac{2}{3}}}{(1+t^2)^{\frac{2}{3}}} \leq \frac{1}{\sqrt[3]{4}\Gamma(\frac{5}{3})}$, Hence $\bar{d} = \frac{1}{\sqrt[3]{4}\Gamma(\frac{5}{3})}$.

Next, by using (6) and (21), we have $g_0 = \frac{\sqrt{6}}{24}\pi$. Then in view of the above estimates of constants p, g_0, a, b and $\bar{d}$, the first inequality of assumption (C_4) has the form

$$\frac{\sqrt{6}}{24} r + \frac{\sqrt{6}}{24}\pi + \frac{1}{3} \frac{1}{\sqrt[3]{4}\Gamma(\frac{5}{3})} r^2 \leq r. \tag{22}$$

It is easy to check that $r_0 = 1$ is a solution of inequality (22). Obviously, the second inequality from assumption (C_4) is satisfied in our situation. Thus assumption (C_4) holds. Then, in light of Theorem 3.1, equation (20) has at least one solution $x(t)$ belongs to $BC(\mathbb{R}_+)$. Moreover, solutions of this equation are uniformly locally attractive on $\mathbb{R}_+$.

Finally, by (21) the assumption (H_2) is satisfied. Thus by Theorem 3.2, equation (20) has at least one solution $x(t)$ belongs to $BC(\mathbb{R}_+)$. Moreover, solutions of this equation are uniformly locally asymptotically attractive on $\mathbb{R}_+$.

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