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A NUMERICAL SCHEME FOR SPATIAL FRATIONAL DIFFERENTIAL EQUATIONS BY RBFS

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Abstract. In this paper, we combine Gaussian integration formula and radial basis function(RBF) interpolation to obtain a numerical scheme for spatial given, fractional differential equations. Utilizing the stable conditions of this schemed, we provide the main analytical results, including local truncated error from Taylor expansion and RBF theory, and convergence rate, which at least is same with shifted Grünwald formula. For verification, some numerical examples are constructed to demonstrate the availability and verify the theoretical results.

1. Introduction

In recent years, fractional differential equations, equipped with suitable initial conditions, reads as follow:

$$D_*^{\alpha} y(x) = f(x, y(x)).$$

Where $\alpha > 0$ (but not necessarily $\alpha \in N$).

This series of equations is extensively used in engineering and mathematical fields, such as porous media, anomalous diffusion, viscoelastic mechanics, Hamiltonian chaos systems, bioengineering[1, 2, 3, 4](including references therein). They are involved to model physical processes referring to memory properties, genetic characters and path dependence. However, only a very few fractional differential equations can be solved analytically because of their complicated form. Hence, the recent rapid development of numerical methods for fractional differential equations has attracted more and more attentions from researchers.

Until now, a lot of numerical methods have been obtained. Diethelm and Walz[5] and Diethelm et al.[6, 7]derived extrapolation method, predictor-corrector method and Adams method for fractional differential equations from integer order partial differential equations. Meerschaert and Tadejeran [8, 9] proposed a finite difference approximation of fractional advection-diffusion flow equation and two-sided space-fractional differential equation. Liu, Zhuang et al.[10] and Liu, Zhuang et al.[11]

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obtained the implicit difference approximation for space-time fractional diffusion equation in 1-D and 2-D case. Momani and Shawagfeh[12] and Momani, Odibat and Erturk[13] used decomposition method for fractional Riccati differential equation and variation iteration method(VIM) to solve a space and time fractional diffusion wave equation. Langlands and Henry[14] presented a novel scheme through L1 scheme form[15] and implicit Euler scheme. Lin and Xu[16] proposed a method to solve time-fractional fractional differential equation based on finite difference method in time and spectral method in space. Ervin and Roop[17] gave a theoretic framework of variational formulation for the stationary fractional advection dispersion equation. In their another paper[18], Ervin , Heuer and Roop solved time dependent, nonlinear, spatial fractional diffusion differential equation by finite element method and displayed corresponding prior error estimate. Ford et al.[19] proposed a finite element method for time fractional partial differential equation. Sousa[20] found a high accuracy method based on approximating fractional derivative by spline interpolation. Piret and Hanert[21] involved RBF-FD method in integer partial differential equation into spatial fractional differential equation. Pedas and Tamme[22] presented the convergence behavior of spline collocation approximations for nonlinear fractional differential equations. Rada and Kazemb[23] solved numerical solution of fractional differential equations with a Tau method based on Legendre and Bernstein polynomials. Xianshan[24] investigated the existence and uniqueness of a positive solution to a two-point boundary-value problem of fractional-order switched system with p-Laplacian operator.

In 1990, Kansa[25, 26] proposed a collocation method with radial basis function(RBF), which provide us a new idea to solve partial differential equations numerically. Although most work to date on RBFs relates to scattered data approxiamtion and in general to interpolation theory, there has recently been an increasing interest in their use for solving PDEs. This approach, which approximates the whole solution of PDE by a translates of RBFs, is very attractive due to the fact that it is a truly meshless method and spatial dimension independent, which can be easily applied to solve high order differential equations.

Another merit of RBF-based meshless methods is no explicit connectivity between these nodes required, in contrast to the information required to store volumes, surfaces, and nodes in a conformal hexahedral or tetrahedral mesh-based method. The flexibility in the node distribution allows for conformal and multi-scale modeling. Additionally, adaptive refinement can be performed with a significantly smaller computational effort since nodes can be added, removed, or displaced with an overhead much smaller than conventional re-meshing.

The remainder of this paper is organized as follows. In section 2, we deduce the algorithm for solving spatial fractional differential equations, further on, we present stable conditions, truncated error and convergence rate estimate of the algorithm. In section 3, a numerical example is given to verify the proposed method. Finally, a remark from the numerical results is given in section 4.

2. Discretization and collocation

Spatial fractional differential equations are often used to model anomalous diffusion and dispersion phenomenon, which disagree with classical Brownian motions.

Its basic model can be described as

$$\frac{\partial u(x,t)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d(x)\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(x,t), L < x < R.$$

where $\frac{\partial^\beta}{\partial x^\beta}$ stands for Riemann-Liouville fractional derivative of order β, detail information referred to [27]. Furthermore, for physical consideration, β must fit $1 < \beta \le 2$. When $v(x) \ge 0$ and $d(x) \ge 0$, the particles move from left to right. Additional equations are initial condition $u(x, t = 0) = u_0(x)$ and boundary conditions $u(L, t) = \varpi_1(t)$, $u(R, t) = \varpi_2(t)$. For the sake of simplicity, we consider the follow equation.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^\beta u(x,t)}{\partial x^\beta} + s(x,t). \\ u(x,0) = \theta(x), u(L,t) = \varpi_1(t), u(R,t) = \varpi_2(t). \end{cases} \tag{1}$$

Where $\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_L^x (x-\xi)^{1-\beta}u(\xi,t)d\xi$,$1 < \beta < 2$, diffusion coefficient $d(x,t) \ge 1$, $s(x,t)$ is reaction term.

RBF interpolation is used to discrete (1) and unsymmetric collocation method is engaged. Larsson and Fornberg[28] have investigated several collocation methods in RBF interpolation.

2.1. **Discretization by RBF meshless method.** In (1), let $t_n = n\Delta t$ and $0 \le t \le T$, using explicit Euler scheme, $\frac{\partial u(x,t)}{\partial t}$ can be approximated as

$$\frac{\partial u(x,t)}{\partial t}|_{t=t_n} \approx \frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t}. \tag{2}$$

According to the relation of Riemann-Liouville fractional derivative and Caputo fractional derivative [27],

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \begin{cases} \frac{1}{\Gamma(2-\beta)}\int_L^x \frac{u_{\xi\xi}(\xi,t)}{(x-\xi)^{\beta-1}}d\xi + \frac{u(L,t)(x-L)^{-\beta}}{\Gamma(1-\beta)} + \\ \frac{u_\xi(\xi,t)|_{\xi=L}(x-L)^{1-\beta}}{\Gamma(2-\beta)}. \end{cases} \tag{3}$$

From(2) and (3)we can get an approximation of (1) at $t = t_n$,

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t} = s(x,t_n) + d(x,t_n)\begin{cases} \frac{1}{\Gamma(2-\beta)}\int_L^x \frac{u_{\xi\xi}(\xi,t)}{(x-\xi)^{\beta-1}}d\xi + \\ \frac{u(L,t)(x-L)^{-\beta}}{\Gamma(1-\beta)} + \\ \frac{u_\xi(\xi,t)|_{\xi=L}(x-L)^{1-\beta}}{\Gamma(2-\beta)}. \end{cases} \tag{4}$$

Let $\{x_k\}_{k=1}^M$ distribute in district [L,R], we assume that

$$u(x,t) = \sum_{k=1}^M \lambda_k(t)\phi(\|x - x_k\|). \tag{5}$$

When $x = x_m$, we substitute (5) into (4),

$$\frac{u(x_m,t_{n+1}) - u(x_m,t_n)}{\Delta t} = s_m^n + d_m^n\sum_{k=1}^M \lambda_k(t_n)\hat{g}(x_m,x_k). \tag{6}$$

Where

$$\hat{g}(x_m,x_k) = \begin{cases} \frac{1}{\Gamma(2-\beta)}\int_L^{x_m} \frac{\phi''_{\xi\xi}(\|\xi-x_k\|)}{(x_m-\xi)^{\beta-1}}d\xi + \frac{\phi(\|L-x_k\|)(x-L)^{-\beta}}{\Gamma(1-\beta)} + \\ \frac{\phi'_\xi(\|\xi-x_k\|)|_{\xi=L}(x-L)^{1-\beta}}{\Gamma(2-\beta)}. \end{cases}$$

$s_m^n = s(x_m, t_n), d_m^n = d(x_m, t_n)$.

For $\int_L^{x_m} \frac{\phi''_{\xi\xi}(\|\xi - x_k\|)}{(x_m-\xi)^{\beta-1}} d\xi$, if we let $\xi = \frac{x_m - L}{2} y + \frac{x_m + L}{2}$, it can be rewritten as

$$\frac{x_m - L}{2^{2-\beta}} \int_{-1}^{1} \frac{\phi''_{yy}(\| \frac{x_m - L}{2} y + \frac{x_m + L}{2} - x_k \|)}{.} (Ly - L + x_m - x_m y)^{\beta - 1} dy \tag{7}$$

(7) is a standard Gaussian integration, it can be computed by

$$\frac{x_m - L}{2^{2-\beta}} \sum_{j=1}^{K} w_j \frac{\phi''_{yy}(\| \frac{x_m - L}{2} \hat{y}_j + \frac{x_m + L}{2} - x_k \|)}{(L\hat{y}_j - L + x_m - x_m \hat{y}_j)^{\beta - 1}}.$$

where K denotes the number of Gaussian points, $\hat{y}_j, (j = 1, \cdots, K)$ are Gaussian points, $w_j, (j = 1, \cdots, K)$ are corresponding integration coefficients.

Substituting (7) into (6) and combining with the initial and boundary conditions, a full discrete scheme is obtained as:

$$A_1 \vec{\lambda}(t_{n+1}) = A_2 \vec{\lambda}(t_n) + \vec{s}(t_n). \tag{8}$$

Remarks: as is show in [29], the invertible of matrix A_1 is observable,

$$A_1 = \begin{pmatrix} \phi(x_1, x_1) & \phi(x_1, x_2) & \cdots & \phi(x_1, x_M) \\ \phi(x_2, x_1) & \phi(x_2, x_2) & \cdots & \phi(x_2, x_M) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(x_M, x_M) & \phi(x_M, x_2) & \cdots & \phi(x_M, x_M) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ f(x_2, x_1, t_n, \beta) & f(x_2, x_2, t_n, \beta) & \cdots & f(x_2, x_M, t_n, \beta) \\ \vdots & \vdots & \vdots & \vdots \\ f(x_{M-1}, x_1, t_n, \beta) & f(x_{M-1}, x_2, t_n, \beta) & \cdots & f(x_{M-1}, x_M, t_n, \beta) \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{aligned} & f(x_m, x_k, t_n, \beta) \\ = {} & \phi(\|x_m - x_k\|) + d_m^n \Delta t \left\{ \begin{array}{l} \frac{1}{\Gamma(2-\beta)} \int_L^{x_m} \frac{\phi''_{\xi\xi}(\|\xi - x_k\|)}{(x_m - \xi)^{\beta-1}} d\xi + \frac{\phi(\|L - x_k\|)(x_m - L)^{-\beta}}{\Gamma(1-\beta)} \\ + \frac{\phi'_\xi(\|\xi - x_k\|)|_{\xi = L}(x_m - L)^{1-\beta}}{\Gamma(2-\beta)} \end{array} \right\}. \end{aligned}$$

$$\vec{s}(t_n) = \begin{pmatrix} \varpi_1(t_n) & s_2^n \Delta t & \cdots & s_{M-1}^n \Delta t & \varpi_2(t_n) \end{pmatrix}^T.$$

2.2. **Algorithm.** When RBF meshless method is engaged in solving spatial fractional differential equations, (8) needs to be solved. We divide the process to solve (8) into seven steps subjectively.

Step1: set parameters Δt, ε, L, R and $\{x_k\}_{k=1}^M \in [L, R]$.

Step2: aggregate A_1.

Step3: implement RBF interpolation, get $\vec{\lambda}(t_0)$ in conjunction with initial and boundary conditions.

Step4: for $k = 1, \cdots, M$, compute (7) through Gaussian formula.

Step5: for $i = 2, \cdots, N$, aggregate A_2 and $\vec{s}$.

Step6: iterate (8), get $\vec{\lambda}(i)(i = 2, \cdots, M)$.

Step7: compute $U = A_1 \lambda$.

This algorithm deal with spatial fractional differential equations with left sided Riemann-Liouville definition, when facing another definition of fractional derivative, the algorithm still work after adjusting A_2.

2.3. **Local truncation errors, stability and convergence.** In RBF interpolation, if node sets X_1,X_2 hold same density and obey quasi uniformed distribution, the errors of RBF interpolation in the two sets have same order [29].

For simplicity, we set that x_k are uniform distribution nodes in $[L, R]$ and donate $h = x_{k+1} - x_k$, $k = 1, \cdots, M-1$. Combining (6) with (8),

$$\frac{u(x_m, t_{n+1}) - u(x_m, t_n)}{\Delta t} = s_m^n + \frac{1}{\Delta t} A_3(m,:) A_1^{-1}(:,n) u(x_m, t_n). \tag{9}$$

Where $A_3(2 : M-1,:) = A_2(2 : M-1,:) - A_1(2 : M-1,:)$, $A_3(1,:) = A_2(1,:)$, $A_3(M,:) = A_2(M,:)$.

According to RBF theory[29], $\sum\limits_{i=1}^{M} \lambda_i(t_n)\phi''(\|x_m - x_i\|) = u(x,t_n)|_{x=x_m} + O(h^{\kappa-2})$, κ is measurement of RBF smoothness. Thus

$$\begin{cases} |\frac{1}{\Delta t} A_3(m,:)A_1^{-1}(:,n)u(x_m,t_n) - d_m^n \frac{\partial^\beta u(x,t_n)}{\partial x^\beta}|_{x=x_m}| \\ = d_m^n |\frac{1}{\Gamma(2-\beta)} \int_L^{x_m} \frac{O(h^{\kappa-2})}{(x-\xi)^{\beta-1}} d\xi| \\ = d_m^n |\frac{1}{\Gamma(2-\beta)} \int_0^{x_m - L} \frac{O(h^{\kappa-2})}{z^{\beta-1}} dz| \\ = d_m^n \frac{(x_m - L)^{2-\beta}}{\Gamma(3-\beta)} O(h^{\kappa-2}) \\ = C_m^n O(h^{\kappa-2}). \end{cases} \tag{10}$$

Where $d_m^n \frac{(x_m-L)^{2-\beta}}{\Gamma(3-\beta)} = C_m^n$.

According to Taylor expansion,

$$\frac{u(x_m, t_{n+1}) - u(x_m, t_n)}{\Delta t} = u_t(x,t)|_{x=x_m, t=t_n} + O(\Delta t).$$

In summary, the truncation error of (8) can be deduced as

$$R_{local} = O(h^{\kappa-2} + \Delta t).$$

So, the proposed scheme is a consistent scheme. From (9),

$$\begin{cases} u(x_m, t_{n+1}) = u(x_m, t_n) + \Delta t \frac{d_m^n}{h^\beta} \sum\limits_{k=0}^{m} g_k u(x_{m-k+1}, t_n) + \Delta t s_m^n \\ +(A_3(m,:)A_1(:,n)u(x_m,t_n) - \Delta t \frac{d_m^n}{h^\beta} \sum\limits_{k=0}^{m} g_k u(x_{m-k+1}, t_n)) \\ = u(x_m, t_n) + \Delta t \frac{d_m^n}{h^\beta} \sum\limits_{k=0}^{m} g_k u(x_{m-k+1}, t_n) + \Delta t s_m^n \\ +\Delta t (\frac{1}{\Delta t} A_3(m,:)A_1(:,n)u(x_m,t_n) - d_m^n \frac{\partial^\beta u(x,t_n)}{\partial x}|_{x=x_m}) \\ +\Delta t (d_m^n \frac{\partial^\beta u(x,t_n)}{\partial x}|_{x=x_m} - \frac{d_m^n}{h^\beta} \sum\limits_{k=0}^{m} g_k u(x_{m-k+1}, t_n)). \end{cases} \tag{11}$$

Meerschaert and Tadjeran[30] deduced the truncation error when they using Shifted Grünwald formula to discrete left sided Riemann-Liouville fractional derivative.

$$\frac{\partial^\beta u(x,t_n)}{\partial x^\beta}|_{x=x_m} = \frac{1}{h^\beta} \sum_{k=0}^{m} g_k u(x_{m-k+1}, t_n) + O(h). \tag{12}$$

Where, $g_0 = 1, g_k = (-1)^k \frac{\Gamma(\beta+1)}{\Gamma(k+1)\Gamma(\beta-k+1)}, k = 1, 2, \cdots$.

Substituting (12) and (10) into (11), we get

$$\begin{cases} u(x_m, t_{n+1}) = u(x_m, t_n) + \Delta t \frac{d_m^n}{h^\beta} \sum\limits_{k=0}^{m} g_k u(x_{m-k+1}, t_n) + \Delta t s_m^n \\ +\Delta t C_m^n O(h^{\kappa-2}) + \Delta t d_m^n O(h). \end{cases} \tag{13}$$

For (9), it is essential that the radius of $I + A_3 A_1^{-1}$ should fit $\lambda(I + A_3 A_1^{-1}) \leq 1$. Meerschaert and Tadjeran[31] analysed as follows,

$$u_{sg}(x_m, t_{n+1}) = u_{sg}(x_m, t_n) + \Delta t \frac{d_m^n}{h^\beta} \sum_{k=0}^{m} g_k u_{sg}(x_{m-k+1}, t_n) + \Delta t s_m^n. \tag{14}$$

and gave its stability conditions as $\frac{\Delta t}{h^\beta} \leq \frac{1}{\beta d_{max}}$, where $d_{max} = max_{x,t}|d(x,t)|$.

Obviously, the stability of (13) is also $\frac{\Delta t}{h^\beta} \leq \frac{1}{\beta d_{max}}$. So when $\lambda(I + A_3 A_1^{-1}) \leq 1$ and $\frac{\Delta t}{h^\beta} \leq \frac{1}{\beta d_{max}}$, equation (8) is stable.

And now, let $R_m^n = u(x_m, t_n) - u_{sg}(x_m, t_n), n = 1, 2, \cdots$, from (13)-(14), we get

$$R_m^{n+1} = R_m^n + \Delta t \frac{d_m^n}{h^\beta} \sum_{k=0}^{m} g_k R_{m-k+1}^n + \Delta t C_m^n O(h^{\kappa-2}) + \Delta t d_m^n O(h). \tag{15}$$

(15) can be rewritten as matrix form:

$$\vec{R}^{n+1} = B\vec{R}^n + \Delta t \vec{Re}^{\,n}.$$

Where $\vec{R}^{n+1} = [R_1^{n+1} \ R_2^{n+1} \ \cdots \ R_M^{n+1}]^T$, $\vec{Re}^{\,n} = [Re_1^n \ Re_2^n \ \cdots \ Re_{M-1}^n \ Re_M^n]^T$, $Re_m^n = C_m^n O(h^{\kappa-2}) + d_m^n O(h)$, according to boundary conditions $Re_1^n = 0, Re_M^n = 0$, and $\frac{\Delta t}{h^\beta} \leq \frac{1}{\beta d_{max}}$, the stable conditions of (13) means $\|B\|_\infty \leq 1$.

Thus $\|\vec{R}^{n+1}\|_\infty \leq \|B\|_\infty \|\vec{R}^n\|_\infty + \Delta t \|\vec{Re}^{\,n}\|_\infty$, through initial conditions we knows that $\vec{R}^1 = \vec{0}$.

So,

$$\begin{aligned} &\|\vec{R}^{n+1}\|_\infty \leq \|B\|_\infty \|\vec{R}^n\|_\infty + \Delta t(C_{max} O(h^{\kappa-2}) + d_{max} O(h)) \\ \leq\ &\|B\|_\infty^2 \|\vec{R}^{n-1}\|_\infty + \|B\|_\infty \Delta t(C_{max} O(h^{\kappa-2}) + d_{max} O(h)) + \\ &\Delta t(C_{max} O(h^{\kappa-2}) + d_{max} O(h)) \\ \leq\ &\|B\|_\infty^2 \|\vec{R}^{n-1}\|_\infty + 2\Delta t(C_{max} O(h^{\kappa-2}) + d_{max} O(h)) \\ \leq\ &\|B\|_\infty^n \|\vec{R}^1\|_\infty + n\Delta t(C_{max} O(h^{\kappa-2}) + d_{max} O(h)). \end{aligned}$$

In consideration of $n\Delta t \leq T$, $\|\vec{R}^{n+1}\|_\infty \leq T(C_{max} O(h^{\kappa-2}) + d_{max} O(h))$. Therefore, the error between shifted Grünwald formula and RBF meshless method is bounded by $T(C_{max} O(h^{\kappa-2}) + d_{max} O(h))$.

3. Numerical experiments

In(1), when $L = 0$, $R = 1$, $\beta = 1.8$, $d(x,t) = \frac{\Gamma(5-\beta)x^\beta}{24}$, $s(x,t) = -2e^{-t}x^4$, initial conditions and boundary conditions are $u(x,0) = x^4$, $0 < x < 1$, $u(0,t) = 0$ and $u(1,t) = e^{-t}$, its analytic solution is $u(x,t) = e^{-t}x^4$.

MQ function $\phi(r) = \sqrt{r^2 + \varepsilon^2}$ is engaged as RBF, 40 points Gaussian-Legendre formula is used to compute (7), $\{x_i\}_{i=1}^N$ are uniformly distributed in [0,L]. There is no general rule to choose shape parameter ε in MQ, so the choice of shape parameter in each case is by manual trial. After implementing the proposed method on the equation, some figures and tables of numerical results are displayed. Figure 1 shows

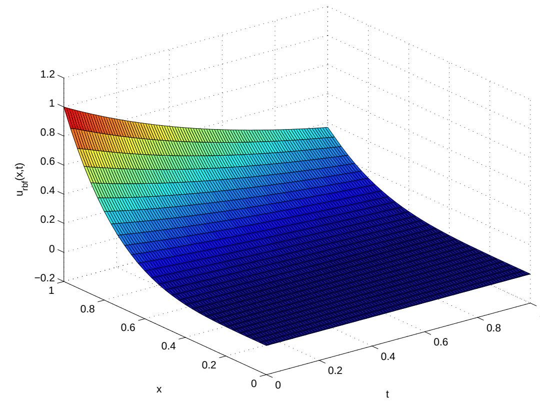


FIGURE 1. Curve of numerical solution when $T = 1, \Delta t = 0.01, h = 1/30, \varepsilon = 0.015$

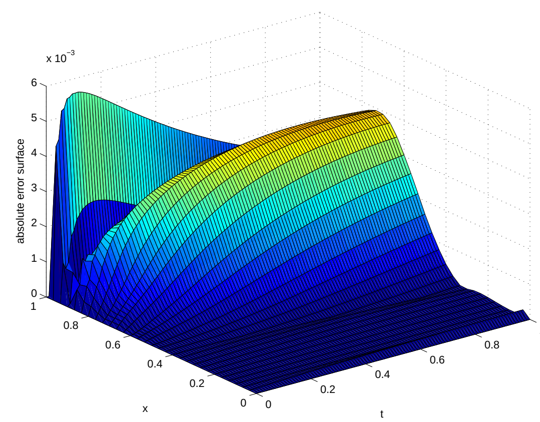


FIGURE 2. Curve of absolute error when $T = 1, \Delta t = 0.01, h = 1/30, \varepsilon = 0.009$

the curve of numerical solution when $T = 1,\ \Delta t = 0.01,\ h = 1/30,\ \varepsilon = 0.015$. Figure 2 shows the curve of absolute errors when $T = 1,\ \Delta t = 0.01,\ h = 1/30,\ \varepsilon = 0.009$. Figure 3 shows the curve of absolute errors when $T = 2,\ \Delta t = 0.01,\ h = 1/30,\ \varepsilon = 0.009$. Figure 4 shows the comparision between numerical solution and analytic solution when $T = 1,\ \Delta t = 0.01,\ \varepsilon = 0.015$ at $t = 0.25,\ 0.50,\ 0.75,\ 1.00$. Figure 5 shows the comparision between numerical solution and analytic solution when $T = 1,\ \Delta t = 0.01,\ \varepsilon = 0.015$ at $x = 0.25,\ 0.50,\ 0.75$. Table 1 matches our theoretical analysis completely, all the Δt and h satisfies $\frac{\Delta t}{h^\beta} \le \frac{1}{\beta d_{max}}$ and the maximum error behavior is observed amostly as expected holding at least the same order as shifted Grünwald formula.

4. CONCLUSION

A truly meshless method is proposed to solve spatial fractional differential equations. This method is an unifying method for spatial fractional differential equations, no matter which kind of definition of fractional derivative it contains. For

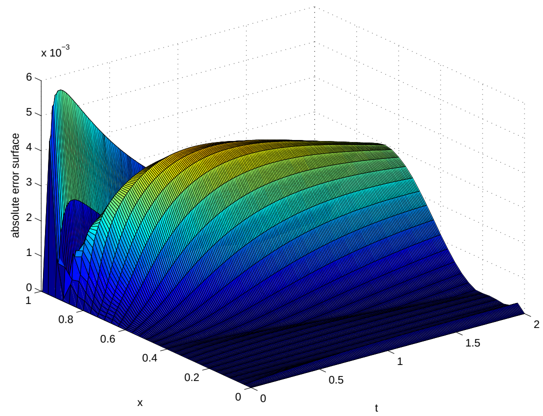
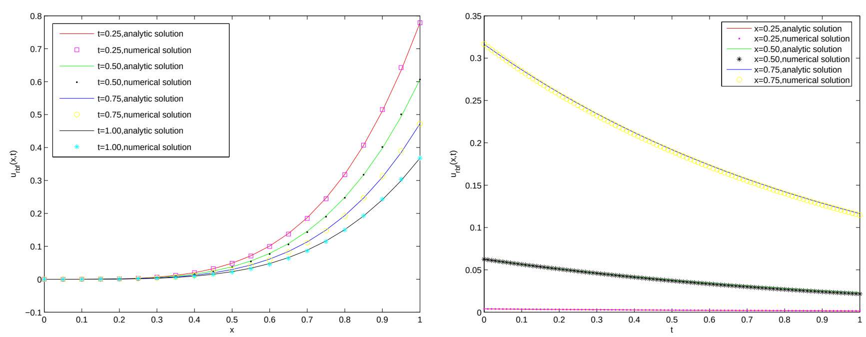


FIGURE 3. Curve of absolute error when $T = 2, \Delta t = 0.01, h = 1/30, \varepsilon = 0.009$



(a) $t = 0.25, 0.50, 0.75, 1.00$ (b) $x = 0.25, 0.50, 0.75$

FIGURE 4. Comparision between numerical solutions and analytic solutions when $T = 1, \Delta t = 0.01, h = 1/20, \varepsilon = 0.015$ at (a) $t = 0.25, 0.50, 0.75, 1.00$, (b) $x = 0.25, 0.50, 0.75$

TABLE 1. When $T = 1$,the relation among space step,time step, shape parameter,radius of iterative matrix, maximum absolute error and convergence rate.

space step	time step	$\frac{\Delta t}{h^\beta}$	$\frac{1}{\beta d_{max}}$	ε	$\lambda(I + A_3 A_1^{-1})$	maximum absolute error	convergence rate
h=1/20	0.01	2.197	5.500	0.015	1	8.872e-3	
h=1/25	$0.01 \times \frac{20}{25}$	2.627	5.500	0.012	1	6.640e-3	1.30
h=1/30	$0.01 \times \frac{20}{30}$	3.039	5.500	0.009	1	5.579e-3	0.95
h=1/35	$0.01 \times \frac{20}{35}$	3.438	5.500	0.007	1	4.009e-3	2.14
h=1/40	$0.01 \times \frac{20}{40}$	3.825	5.500	0.005	1	2.985e-3	2.21

the characters of RBF interpolation, this method can be easily extended to high dimensional cases. This method can only be used in linear equations now, in further work, we want to investigate the behavior of this method in nonlinear cases.

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