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STABILITY ANALYSIS AND CHAOS CONTROL OF THE DISCRETIZED FRACTIONAL-ORDER MACKEY-GLASS EQUATION

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Abstract. This paper is devoted to analyze the dynamics of the fractional-order Mackey-Glass equation without delay before and after applying a discretization process to it. Local bifurcations of fixed points in the discretized equation are discussed. Lyapunov exponent is plotted as an indicator to chaos. Finally, chaos in the discretized fractional-order Mackey-Glass equation is being controlled using a modified delayed feedback control (DFC) method. Since fractional-order differential equations (FODE) possess memory, the delay in the original Mackey-Glass model is substituted with the fractional-order differentiation which led to a richer dynamic behavior.

1. Introduction

The concept of derivative is traditionally associated to an integer; given a function, we can derive it one, two, three times and so on. It could have an interest to investigate the possibility to derive a real number of times a function. The main idea is to examine the properties of the ordinary derivative and see where and how it is possible to generalize the concepts.

Fractional Calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to $\frac{1}{2}$ order. Despite generalized would be a better option, the name fractional is used for denoting this kind of derivative.
Recalling the basic definitions (Caputo) and properties of fractional-order differentiation and integration.

Definition 1. *The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$ is defined by*

$$I_a^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds,$$

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and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D_a^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}.$$

In addition, the following results are the main in fractional calculus. Let $\beta, \gamma \in \mathbb{R}^+$, $\alpha \in (0,1)$,

- $I_a^\beta : L^1 \to L^1$, and if $f(x) \in L^1$, then$I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
- $\lim_{\beta \to n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on $[a,b]$, $n = 1, 2, 3, ...$, where $I_a^1 f(x) = \int_a^x f(s) ds$.
- $\lim_{\beta \to 0} I_a^\beta f(x) = f(x)$ weakly.
- If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1} D_a^\alpha f(x) = \frac{df(x)}{dx}$.

For the main properties of the fractional derivatives and integrals, one can see [15], [16], [17], and [19].
To solve fractional-order differential equations there are two famous methods: frequency domain methods [23] and time domain methods [7]. In recent years it has been shown that the second method is more effective because the first method is not always reliable in detecting chaos [24] and [25].
Often it is not desirable to solve a differential equation analytically, and one turns to numerical or computational methods.

In [20], a numerical method for nonlinear fractional-order differential equations with constant or time-varying delay was devised. It should be noticed that the fractional differential equations tend to lower the dimensionality of the differential equations in question, however, introducing delay in differential equations makes it infinite dimensional. So, even a single ordinary differential equation with delay could display chaos.

2. Fractional-order Mackey-Glass equation

The Mackey-Glass equation is the nonlinear time delay differential equation, which was proposed as a model of hematopoiesis, given by

$$\frac{dx}{dt} = \frac{\rho x_\tau}{1 + x_\tau^c} - \gamma x, \qquad \gamma, c, \rho > 0, \tag{2.1}$$

where γ, c, ρ, τ are real parameters, and x_τ represents the value of the variable x at time $(t - \tau)$. Depending on the values of the parameters, this equation displays a range of periodic and chaotic dynamics.
In this work, we will show that considering a fractional-order differentiation in equation (2.1) will exhibits more complex and richer dynamics.

Consider the fractional-order Mackey-Glass equation given in the form.

$$D^\alpha x(t) = \frac{\rho x}{1 + x^c} - \gamma x, \qquad t \in (0, T], \tag{2.2}$$

with the initial condition

$$x(0) = x_0 \tag{2.3}$$

where $\alpha \in (0, 1], \gamma = 1$ and $\rho, c > 0$.

2.1. **Stability of equilibrium points.** Equilibrium points of the fractional-order Mackey-Glass equation (2.2) can be easily obtained by solving the algebraic equation

$$D^{\alpha}x(t)=0, \quad \Rightarrow \quad x=\frac{\rho x}{1+x^{c}}.$$

Indeed, there are only two equilibrium points of equation (2.2) namely, $x_1^*=0$, and $x_2^*=\sqrt[c]{\rho-1}$. To study the stability of these equilibrium points we calculate the first derivative of the right hand side of equation (2.2) at each equilibrium point [2] and[3].

$$f^{'}(x)=\frac{\rho+\rho x^{c}(1-c)}{(1+x^{c})^{2}}-1.$$

That is, $f^{'}(0)=\rho-1$,and $f^{'}(\sqrt[c]{\rho-1})=\frac{c(1-\rho)}{\rho}$.

Now the solution of the initial value problem

$$D^{\alpha}\varepsilon(t)=f^{'}(x_1^*)\varepsilon(t)=(\rho-1)\varepsilon(t) \quad , \ t>0 \text{ and } \varepsilon(0)=x_0,$$

is given by [16].

$$\varepsilon(t)=\sum_{n=0}^{\infty}\frac{(\rho-1)^{n}t^{\alpha n}}{\Gamma(\alpha n+1)}\ (x_0-0). \tag{2.4}$$

So the equilibrium point $x_1^*=0$ if($\rho > 1$) is unstable. Similarly, for the equilibrium point $x_2^*=0$ we have the initial value problem

$$\begin{aligned} D^{\alpha}\varepsilon(t)&=f^{'}(x_2^*)\varepsilon(t)=\frac{c(1-\rho)}{\rho}\varepsilon(t), \qquad t>0,\\ \varepsilon(0)&=x_0-x_2^*. \end{aligned} \tag{2.5}$$

and its solution is given by

$$\varepsilon(t)=\sum_{n=0}^{\infty}\frac{(\frac{c(1-\rho)}{\rho})^{n}t^{\alpha n}}{\Gamma(\alpha n+1)}\ (x_0-\sqrt[c]{\rho-1}). \tag{2.6}$$

So, the equilibrium point $x_2^*=\sqrt[c]{\rho-1}$ is asymptotically stable.

2.2. **Existence and uniqueness.** Let $I=[0,T]$, $T<\infty$ and $C(I)$ be the class of all continuous functions defined on I. For the existence of a unique solution $x\in C[0,T]$ of the problem (2.2)-(2.3), we have the following theorem.

Theorem 1. *If ($\frac{\rho\ T^{\alpha}}{2\Gamma(1+\alpha)}<1$), then the initial value problem(2.2)-(2.3) has a unique solution $x\in C[0,T]$.*

Proof.
The proof follows directly from Theorem 1 of [14].

2.3. **Numerical results.** An Adams-type predictor-corrector method has been introduced in [5], [7] and [6] and investigated further in [1], [12], and [19]. In this work we apply an Adams-type predictor-corrector method for the numerical solution of a fractional integral equation. The numerical solution of (2.2)-(2.3) is obtained by applying the PECE (Predict,Evaluate, Correct, Evaluate) method and is displayed in Figure (1) for different values of α and we take $\rho=0.5$, $c=6$ and $x_0=0.85$.

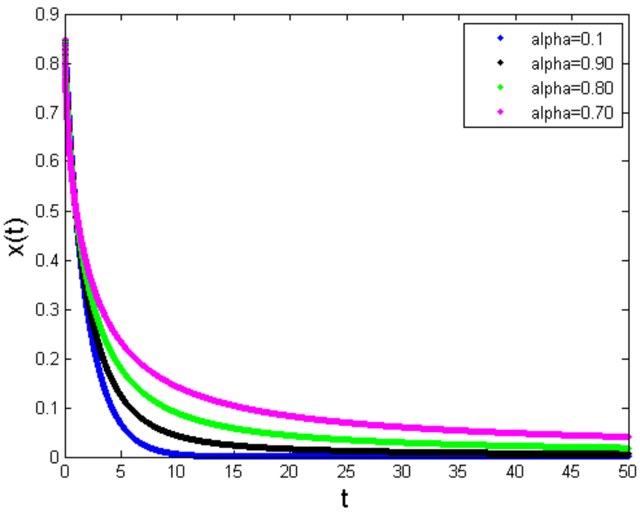


FIGURE 1. Numerical solution of (2.2)-(2.3) with different α using PECE method.

2.4. **Bifurcation and chaos.** In this part of the paper, bifurcation and chaos in the Fractional-order Mackey-Glass system (2.2)-(2.3) are numerically investigated. First we fix $c = 6$ and $\alpha = 0.90$ and we let ρ varies form -2 to 30. The initial state of the Fractional-order Mackey-Glass system is taken as $x(0) = 0.9$. The resulting bifurcation diagram is shown in Figure (2). Second, we fix $c = 6$, and $\rho = 20$ and we let the fractional-order α varies form 0 to 1. The resulting bifurcation diagram is shown in Figure (3).

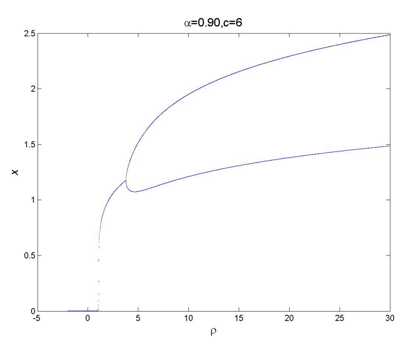


FIGURE 2. Bifurcation diagram of (2.2) as a function of ρ with $\alpha = 0.90$.

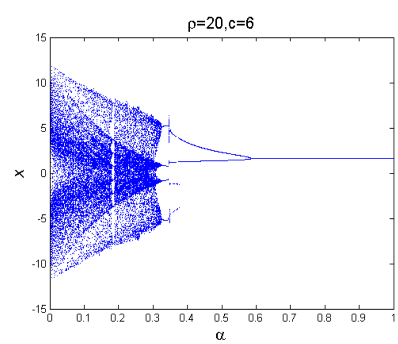


FIGURE 3. Bifurcation diagram of (2.2) as a function of α with $\rho = 20$.

3. Discretization process

In this section we will apply a discretization process mentioned in [8], [9] ,and [10] for discretizing fractional-order Mackey-Glass equation(2.2)-(2.3). It is worth to mention here that many discretization methods have been applied to fractional-order systems such as Euler's method and predictor-corrector method. Euler's method discretization is an approximation for the derivative while the predictor-corrector method is an approximation for the integral. However, our proposed dicretization method here is an approximation for the right hand side as we will see. Consider the fractional-order Mackey-Glass equation (2.2) with piecewise constant arguments given by

$$D^{\alpha}x(t)=\frac{\rho x([\frac{t}{r}]r)}{1+x^{c}([\frac{t}{r}]r)}-x([\frac{t}{r}]r), \tag{3.1}$$

with initial condition (2.3).

The steps of the discretization process are as follows
let $t \in [0,r)$, then $\frac{t}{r} \in [0,1)$. That is,

$$D^{\alpha}x(t)=\frac{\rho x_0}{1+x_0^c}-x_0, \tag{3.2}$$

and the solution of (3.2) is given by

$$\begin{aligned} x(t) &= x_0+I^{\alpha}(\frac{\rho x_0}{1+x_0^c}-x_0), \\ &= x_0+(\frac{\rho x_0}{1+x_0^c}-x_0)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds, \\ &= x_0+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\ (\frac{\rho x_0}{1+x_0^c}-x_0). \end{aligned}$$

let $t \in [r,2r)$, then $\frac{t}{r} \in [1,2)$. That is,

$$D^{\alpha}x(t)=\frac{\rho x_1}{1+x_1^c}-x_1, \qquad t\in [r,2r), \tag{3.3}$$

and the solution of (3.3) is given by

$$\begin{aligned} x(t) &= x_1(r)+I_r^{\alpha}(\frac{\rho x_1}{1+x_1^c}-x_1), \\ &= x_1(r)+(\frac{\rho x_1}{1+x_1^c}-x_1)\int_r^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds. \\ &= x_1(r)+\tfrac{(t-r)^{\alpha}}{\Gamma(1+\alpha)}\ (\tfrac{\rho x_1(r)}{1+x_1^c(r)}-x_1(r)). \end{aligned}$$

Repeating the process we can easily get

$$x(t)=x_n(nr)+\tfrac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}(\tfrac{\rho x_n(nr)}{1+x_n^c(nr)}-x_n(nr)),\ \ t\in\ [nr,(n+1)r).$$

Let $t \to (n+1)r$, we obtain the discretization

$$x_{n+1}((n+1)r)=x_n(nr)+\tfrac{(r)^{\alpha}}{\Gamma(1+\alpha)}\ (\tfrac{\rho x_n(nr)}{1+x_n^c(nr)}-x_n(nr)).$$

That is,

$$x_{n+1}=x_n+\frac{r^{\alpha}}{\Gamma(1+\alpha)}\ (\frac{\rho x_n}{1+x_n^c}-x_n). \tag{3.4}$$

It should be noticed that formula (3.4) is the fractional Euler's discretization formula for our model (2.2)-(2.3) (see [11]).

3.1. **Fixed points and stability.** Now we study the fixed points of the system (3.4) which has two fixed points, namely

$$x_{fix1} = 0, \text{ and } \quad x_{fix2} = \sqrt[c]{\rho - 1},$$

which are obtained by solving the equation

$$x = x + \frac{r^\alpha}{\Gamma(1+\alpha)} \left(\frac{\rho x}{1+x^c} - x\right).$$

To study the stability of these fixed points we relay on the following theorem

Theorem 2. [18] *Let f be a smooth map on* $\mathbb{R}$*, and assume that* x^* *is a fixed point of f.*

1. *If* $| f^{'}(x^*) | < 1, \;$ *then* x^* *is stable.*
2. *If* $| f^{'}(x^*) | > 1, \;$ *then* x^* *is unstable.*

That is, x_{fix1} is stable if

$$1 - \frac{2\Gamma(1+\alpha)}{r^\alpha} < \rho < 1,$$

while x_{fix2} is stable if

$$1 < \rho < \frac{r^\alpha c}{r^\alpha c - 2\Gamma(1+\alpha)}.$$

Since the Lyapunov exponent (LE) is a good indicator for existence of chaos and play a key role in the study of nonlinear dynamical systems and they are a measure of the sensitivity of the solutions of a given dynamical system to small changes in the initial conditions. One feature of chaos is the sensitive dependence on initial conditions; for a chaotic dynamical system at least one LE must be positive. Since for non-chaotic systems all LEs are non-positive, the presence of a positive LE has often been used to help determine if a system is chaotic or not.
Indeed, Lyapunov exponent for (3.4) is given by [26]

$$Lya.exp = \lim_{n\to\infty} \Sigma log_2 \left(1 + \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho + \rho x^c(1-c)}{(1+x^c)^2} - 1\right)\right).$$

When $\alpha = 1$, the Lyapunov exponent for the discrete system

$$x_{n+1} = x_n + r\left(\frac{\rho x_n}{1+x_n^c} - x_n\right),$$

is obtained. Figure (4) shows the lyapunov exponent for the system(3.4) for different values of the fractional-order parameter α and $r = 0.50$.

4. Bifurcation and Chaos

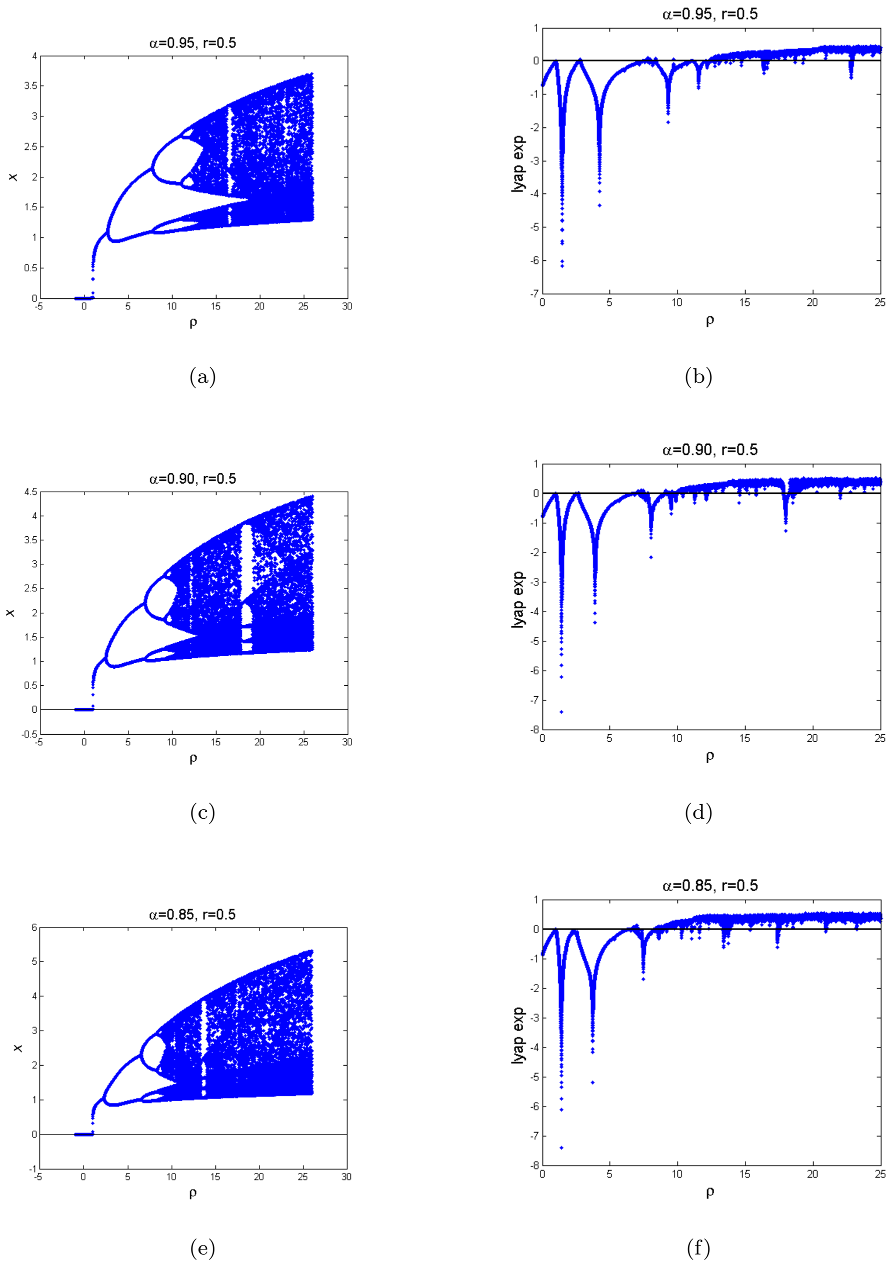
In this section we study the local bifurcation of the fixed points of system (3.4). From equation (3.4), $f(x) = x + \frac{r^\alpha}{\Gamma(1+\alpha)} \left(\frac{\rho x}{1+x^c} - x\right)$. So, the eigenvalues associated to f are $\lambda_1 = 1 + \frac{r^\alpha}{\Gamma(1+\alpha)}(\rho - 1)$,
and $\lambda_2 = 1 + \frac{r^\alpha c}{\Gamma(1+\alpha)\rho}(1 - \rho)$.

When $\rho = 1$, $\lambda_1 = \lambda 2 = 1$, $x_{fix1} = x_{fix2} = 0$. That is, $(x_{fix}, \rho_c) = (0, 1)$ is a non-hyperbolic fixed point. In fact, it is a transcritical bifurcation point of f. On the other hand, at $\rho = \frac{r^\alpha c}{r^\alpha c - 2\Gamma(1+\alpha)}$, $\lambda_2 = -1$ and $x_{fix2} = \sqrt[c]{\frac{2\Gamma(1+\alpha)}{r^\alpha c - 2\gamma(1+\alpha)}}$.

That is, $(x_{fix}, \rho_c) = (\sqrt[c]{\frac{2\Gamma(1+\alpha)}{r^\alpha c - 2\Gamma(1+\alpha)}}, \frac{r^\alpha c}{r^\alpha c - 2\Gamma(1+\alpha)})$ is a non-hyperbolic fixed point. In fact, it is a period doubling (supercritical flip) bifurcation point of f.

In all numerical simulations we take $c = 6$ and $r = 0.50$. Figure (4) shows the bifurcation diagrams of system (3.4) as a function of the control parameter ρ with different values of α and $r = 0.50$. If we take, for example, $\alpha = 0.95$, $c = 6$, and $r = 0.50$, we can see clearly in Figure (4)(a) the the bifurcation from a stable fixed point to a stable orbit of period two at $\rho = 2.71$, and then the bifurcation from period two to period four at $\rho \simeq 7.8$. The further period doubling occur at decreasing increments in ρ, and the orbit becomes chaotic for $\rho \simeq 9.4$. Note the intriguing window just beyond $\rho = 13$. It is pretty clear from Figure (4) that when $\alpha \to 0$, the stability region for fixed points $x_{fix_{1,2}}$ is being shrinked. To be more clear, let $\alpha = 1$, so x_{fix1} is stable if $-2 < \rho < 1$. While if $\alpha = 0.90$, x_{fix1} is stable if $-1.7234 < \rho < 1$ and so on. The same conclusion can be said to x_{fix2}. That is, when $\alpha = 1$, x_{fix2} is stable if $1 < \rho < 3$. While if $\alpha = 0.90$, x_{fix2} is stable if $1 < \rho < 2.4890$ and so on.

Meanwhile, Figure (5) shows the bifurcation diagrams of system (3.4) as a function of α with different values of ρ.



(a) (b)

(c) (d)

(e) (f)

FIGURE 4. Bifurcation diagrams and Lyapunov exponent for system (3.4) with different values of the fractional-order parameter α and $r = 0.5$.

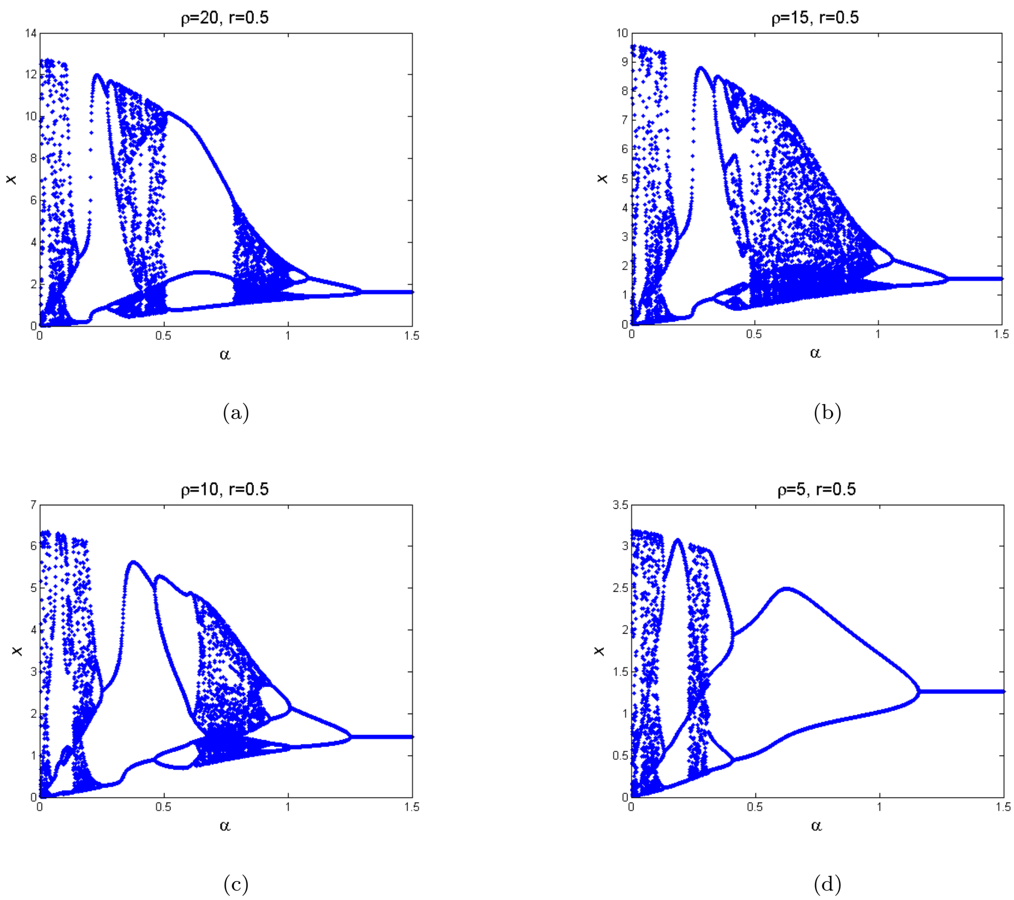


Figure 5. Bifurcation diagrams for system (3.4) as a function of α with different values of the parameter ρ and $r = 0.5$.

If we compare Figure (2) with Figure (4)(c), we can see clearly that the discretized fractional-order Mackey-Glass (3.4) has the same tendency as the original fractional-order system (2.2)-(2.3). Moreover, if we compare Figure (3) with Figure (5) (a), we will also see the same tendency of the dynamic behavior of the two systems (2.2) and (3.4) but not with the same values of α.

5. Chaos control

Chaotic behavior occurs in many fields such as physics, chemistry, biology, econometrics and engineering, etc. However, these irregular and complex phenomena are often undesirable, subtle and elusive. In many practical situation, in order to improve system performance or avoid fatigue failures of mechanical systems, we must control a chaotic system to a regular orbits, such as a periodic orbit or a steady

state. Therefore studying how to control chaotic systems has received increased attention. In chaos theory, control of chaos is based on the fact that any chaotic attractor contains an infinite number of unstable periodic orbits. Chaotic dynamics then consists of a motion where the system state moves in the neighborhood of one of these orbits for a while, then falls close to a different unstable periodic orbit where it remains for a limited time, and so forth. This results in a complicated and unpredictable wandering over longer periods of time. Control of chaos is the stabilization, by means of small system perturbations, of one of these unstable periodic orbits. The result is to render an otherwise chaotic motion more stable and predictable, which is often an advantage. The perturbation must be tiny, to avoid significant modification of the system's natural dynamics. Several techniques have been devised for chaos control, but most are developments of two basic approaches: the OGY (Ott, Grebogi and Yorke) method, and Pyragas continuous control. Both methods require a previous determination of the unstable periodic orbits of the chaotic system before the controlling algorithm can be designed.
In this section we are going to control the period-1 orbit (i.e. fixed point) of the map (3.4) via a modified delayed feedback control algorithm that allows one to stabilize unstable target states of chaotic systems for any initial conditions placed on a strange attractor. The algorithm is based on ergodicity of chaotic systems and has been mentioned in [22]. Ergodicity is the universal property of chaotic systems. This feature means that the chaotic trajectory visits the close neighborhood of any orbit with finite probability.

The main idea of the delayed feedback control method (DFC) is that we let the system to evolve unperturbed until it approaches a close neighborhood of the target steady state. At this moment we activate the DFC perturbation that stabilizes the target state. The algorithm does not require a knowledge of location of the target.

Our main is to stabilize the non-zero fixed point $x_{fix2} = \sqrt[c]{\rho - 1}$ of the map (3.4) by using the DFC algorithm

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\,(\frac{\rho x_n}{1+x_n^c} - x_n) + k(x_n - x_{n-1}),$$

where K is the feedback gain. Introducing the auxiliary variable $y_n = x_{n-1}$, we get the 2-D map

$$\begin{aligned} x_{n+1} &= x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\,(\frac{\rho x_n}{1+x_n^c} - x_n) + k(x_n - y_n), \\ y_{n+1} &= x_n. \end{aligned} \tag{5.1}$$

The map (5.1) has two fixed points, namely: $(x^*, y^*)_1 = (0, 0)$, and $(x^*, y^*)_2 = (\sqrt[c]{\rho - 1}, \sqrt[c]{\rho - 1})$. To study the stability of the non-zero fixed point, we calculate the Jacobian matrix at it.

$$J(\sqrt[c]{\rho - 1}, \sqrt[c]{\rho - 1}) = \begin{bmatrix} 1 + \frac{r^\alpha}{\Gamma(1+\alpha)}\frac{c(1-\rho)}{\rho} + k & -k \\ 1 & 0 \end{bmatrix},$$

with eigenvalues given by

$$\lambda_{1,2} = \frac{(1+m+k) \pm \sqrt{(1+m+k)^2 - 4k}}{2}, \tag{5.2}$$

where $m = \frac{r^{\alpha}c(1-\rho)}{\Gamma(1+\alpha)\rho}$.

The optimal value of the feedback gain, which leads to the fastest convergence of nearby initial conditions towards the desired fixed point, is given by

$$K_{\pm} = \frac{r^{\alpha}c(\rho-1)}{\Gamma(1+\alpha)\rho} - 1 \pm \sqrt{4\frac{r^{\alpha}c(\rho-1)}{\Gamma(1+\alpha)\rho}}.$$

The root K_{-} corresponds to the case for which the magnitudes of $|\lambda_{1,2}|$ are minimal, and thus $k_{op} = k_{-}$. Taking $r = 0.5, \alpha = 0.90, c = 6$, and $\rho = 11$ (which is the minimum value of ρ generates chaos with the mentioned parameter values in Figure (4)(c)), then $K_{op} = -1.4474$. Thus we have controlled the non-zero fixed point $(x^{*}, y^{*})_2 = (1.4678, 1.4678)$ in Figure (4)(c). Figure (6) shows the controlled orbit of the fixed point $(x^{*}, y^{*})_2 = (1.4678, 1.4678)$ with initial condition $(x_0, y_0) = (1, 1)$.

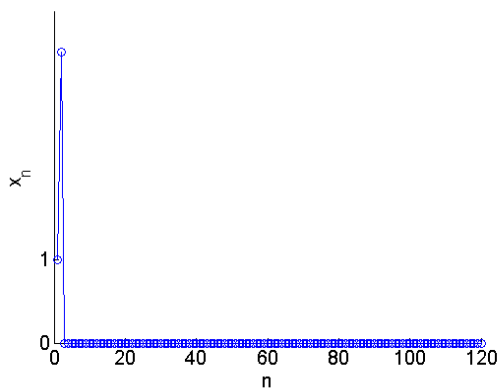


FIGURE 6. Controlled period-one orbit of system (3.4) with $r = 0.5, \alpha = 0.90, c = 6$, and $\rho = 11$ with modified DFC method.

6. Conclusion

In this work we studied the dynamics of the fractional-order Mackey-Glass equation before and after applying a simple discretization scheme to it. Local bifurcation of fixed points of the discretized equation was studied and illustrated via bifurcation diagrams. Chaos has been detected via lyapunov exponent for different values of the fractional-order parameter α. Finally, chaos in the discretized system was controlled using the modified delayed feedback control method.

References

[1] E. Ahmed, A.M.A. El-Sayed, E.M. El-Mesiry, H.A.A. El-Saka, Numerical solution for the fractional replicator equation, Internat. Modern Phys. C. 16 (7) (2005) 19.

[2] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, On some RouthHurwitz conditions for fractional order differential equations and their applications in Lorenz, Rossler, Chua and Chen systems, Phys. Lett. A 358 (1) (2006).

[3] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predatorprey and rabies models, J. Math. Anal. Appl. 325 (2007) 542553.

[4] Y.Q. Chen, B.M. Vinagre, I. Podlubny, A new discretization method for fractional-order differentiators via continued fraction expansion, in: Proceedings of DETC, vol. 3, pp. 761769.

[5] K. Diethelm, A. Freed, On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity, in: F. Keil, W. Mackens, H. Vo, J. Werther (Eds.), Scientific Computing in Chemical Engineering IIComputational Fluid Dynamics, Reaction Engineering, and Molecular Properties, Springer, Heidelberg, 1999, pp. 217224.

[6] K. Diethelm, A. Freed, The FracPECE subroutine for the numerical solution of differential equations of fractional order, in: S. Heinzel, T. Plesser (Eds.), Forschung und wissenschaftliches Rechnen 1998, Gesellschaft fur Wisseschaftliche Datenverarbeitung, Gottingen, 1999, pp. 5771.

[7] k.Diethelm, N.J. Ford, and Freed, A.D., A predictor-corrector approach for the numerical solution of fractional differential equations, Journal of Nonlinear Dynamics, 29, pp. 3-22.

[8] A. M. A.El-Sayed, El-Raheem, Z. F., Salman, S. M., Discretization of forced Duffing system with fractional-order damping, Journal of Advances in difference equations 2014,2014:66.

[9] A. M. A. El-Sayed, Salman, S. M., On a discretization process of fractional-order Riccati differential equation, Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, pp. 251-259.

[10] A. M. A.El-Sayed, Agarwal, R. P., Salman, S. M., Fractional-order Chua's system: discretization, bifurcation and chaos, Journal of Advances in difference equations, 2013, 2013:320.

[11] Z. M. Odibat, and S. Momani, An algorithm for the numerical solution of differential equations of fractional order, J. Appl. Math. & Informatics Vol. 26(2008), No. 1-2, pp. 15-27.

[12] A.M.A. El-Sayed, Fractional differentialdifference equations, J. Fract. Calc. 10 (1996) 101106.

[13] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, On the fractional-order logistic equation, J. Appl. Math. Letters Vol. 20 817-823, 2007.

[14] A. M. A. EL-Sayed, E. Ahmed, M. A. E. Herzallah, On the fractional-order differential Games with non-uniform interaction rate and Asymmetric games. Journal of Fractional Calculus and Applications,Vol. 1. July 2011, No.1, pp. 1-9.

[15] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal.: Theory, Methods Appl. 33 (1998), pp. 181-186.

[16] A.M.A. El-Sayed, Gaafar, F.M., Fractional order differential equations with memory and fractional-order relaxation oscillation model, (PUMA) Pure Math. Appl. 12(2001).

[17] A.M.A. El-Sayed, Gaafar, F.M., Hashem, H.H., On the mayimal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations, Math. Sci. Res. J. 8 (2004), pp. 336-348.

[18] S. N. Elaidy, An introduction to difference equations, Third Edition, Undergradute Texts in Mathematics, Springer, New York, 2005.

[19] R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi(Eds.),Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wien, 1997, pp. 223276.

[20] K. S. Miller, and B. Ross, An Introduction to The Fractional Calculus and Fractional Differential Equations , John-Wiley and Sons Inc, 1993.

[21] I. Podlubny, Fractional Differential Equations (Academic Press, New York, 1999).

[22] V.Pyragas, and Pyragas, K., Modification of delayed feedback control using ergodicity of chaotic systems, Lithuanian Journal of Physics, Vol. 50, No. 3, pp. 305?316 (2010).

[23] H. Sun, V. Abdelwahed, and B. Onaral, Linear approximation for transfer function with a pole of fractional-order, IEEE Trans. Autom. Contr. 29, pp. 441-444.

[24] M.S. Tavazoei, and Haeri, H., Unreliability of frequency domain approximation in recognizing chaos in fractional-order systems, IET sign. process 1, pp. 171-181.

[25] M.S. Tavazoei, and Haeri, H., limitation of frequency domain approximation for detecting chaos in fractional-order systems, Journal of Nonlinear analysis Th. Meth. Apply. 69, pp. 1299-1320.

[26] F. E.Udwadia, von Bremen, H., A Note on the Computation of the Largest p-Lyapunov Characteristic Exponents for Nonlinear Dynamical Systems, J App Math and Comp 2000, 114:205-214.

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