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ON FEJÉR TYPE INEQUALITIES VIA LOCAL FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish some weighted version of the generalized Hermite- Hadamard type, so-called Hermite-Hadamard-Fejér type, inequalities for local fractional integrals. Then, we obtain several inequalities related both left and rigt side of this inequality using the local fractional integrals and generalized convex functions.

1. Introduction

Theorem 1 (Hermite-Hadamard inequality). *Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds* [9]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [2] and [3], Dragomir et al. proved the following results connected with the Hermite-Hadamard inequality:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq M\frac{(b-a)^2}{24} \tag{1.2}$$

and

$$m\frac{(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq M\frac{(b-a)^2}{24}. \tag{1.3}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [5], [10]-[20]). In [4], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

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Theorem 3. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \frac{1}{b-a}\int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \quad (1.4)$$

holds, where $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [6], Minculete and Mitroi presented the following important inequalities;

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\le f''\le M$. Then, the following inequalities hold:*

$$\begin{aligned} m\frac{\lambda(1-\lambda)}{2}(b-a)^2 &\le \lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b) \quad (1.5)\\ &\le M\frac{\lambda(1-\lambda)}{2}(b-a)^2 \end{aligned}$$

and

$$\begin{aligned} & m\frac{(1-2\lambda)^2}{8}(b-a)^2 \quad (1.6)\\ \le\ & \frac{f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)}{2}-f\left(\frac{a+b}{2}\right)\\ \le\ & M\frac{(1-2\lambda)^2}{8}(b-a)^2 \end{aligned}$$

for $\lambda\in[0,1]$.

And using the Theorem 4, some inequalities of Hermite-Hadamard-Fejer type for differentiable mappings were proved as follows:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\le f''\le M$. Assume $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$. Then, the following inequalities hold:*

$$\begin{aligned} \frac{m}{2}\int_a^b (t-a)(b-t)g(t)dt &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)dt-\int_a^b f(t)g(t)dt \quad (1.7)\\ &\le \frac{M}{2}\int_a^b (t-a)(b-t)g(t)dt \end{aligned}$$

and

$$\begin{aligned} \frac{m}{8}\int_a^b (2t-a-b)^2 g(t)dt &\le \int_a^b f(t)g(t)dt-f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt \quad (1.8)\\ &\le \frac{M}{8}\int_a^b (2t-a-b)^2 g(t)dt. \end{aligned}$$

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [21, 22] and so on.

Recently, the theory of Yang's fractional sets [21] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;
(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [21] *A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2. [21] *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 3. [21] *Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 4 (Generalized convex function)**.** [21] *Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality*

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;

(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 6. *Let $f\in D_{\alpha}(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1,x_2\in I$,

$$f(x_2)-f(x_1)\geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}(x_2-x_1)^{\alpha}.$$

Corollary 1. *Let $f\in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x)\geq 0\left(or\ f^{(2\alpha)}(x)\leq 0\right)$$

for all $x\in(a,b)$.

Lemma 1. [21]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x)=g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, then we have

$${}_aI_b^{(\alpha)}f(x)=g(b)-g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x),g(x)\in D_{\alpha}[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, then we have

$${}_aI_b^{(\alpha)}f(x)g^{(\alpha)}(x)=\left.f(x)g(x)\right|_a^b-{}_aI_b^{(\alpha)}f^{(\alpha)}(x)g(x).$$

Lemma 2. [21] *We have*

i) $\dfrac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)}\int\limits_a^b x^{k\alpha}(dx)^{\alpha}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right)$, $k\in R$.

Lemma 3. [21] *Suppose that $f(x)\in C_{\alpha}[a,b]$, then*

$$\frac{d^{\alpha}\left({}_aI_x^{(\alpha)}f(t)\right)}{dx^{\alpha}}=f(x)\quad a<x<b.$$

Lemma 4 (Generalized Hölder's inequality)**.** [21] *Let $f,g\in C_{\alpha}[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|f(x)g(x)|(dx)^{\alpha}\leq\left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|f(x)|^p(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|g(x)|^q(dx)^{\alpha}\right)^{\frac{1}{q}}.$$

In [7], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 7 (Gereneralized Hermite-Hadamard's inequality)**.** *Let* $f(x) \in I_x^{\alpha}[a,b]$ *be generalized convex function on* $[a,b]$ *with* $a<b$*. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{(\alpha)}f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}.$$

The interested reader is refer to [1],[7],[8],[11]-[15],[21]-[25] for paper related to local freactional.

In this paper, we firstly establish generalized Hermite–Hadamard-Fejer inequality via local fractional integrals. Then, we obtain several inequalities related both left and right side of this inequality using the local fractional integrals and generalized convex functions.

3. Main Results

Now, we give the following our results:

Theorem 8 (Hermite–Hadamard-Fejer inequality)**.** *Let* $f(x) \in I_x^{(\alpha)}[a,b]$ *be a generalized convex function on* $[a,b]$ *with* $a<b$*. If* $g:[a,b]\to R^{\alpha}$ *is nonnegative, local fractional integrable and symmetric* $\frac{a+b}{2}$*, then the following inequalities for local fractional integrals hold*

$$f\left(\frac{a+b}{2}\right)\ {}_aI_b^{(\alpha)}g(x) \leq\ {}_aI_b^{(\alpha)}f(x)\,g(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}\ {}_aI_b^{(\alpha)}g(x)\,. \tag{3.1}$$

Proof. Since f be a generalized convex on $[a,b]$, we have for all $t\in[0,1]$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{at+(1-t)b+tb+(1-t)a}{2}\right) \\ &\leq \frac{1}{2^{\alpha}}\left[f(at+(1-t)b)+f(tb+(1-t)a)\right]. \end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{1}{\Gamma(1+\alpha)}g(tb+(1-t)a)$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(tb+(1-t)a)\,(dt)^{\alpha} \\ \leq\ & \frac{1}{2^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(tb+(1-t)a)\left[f(at+(1-t)b)+f(tb+(1-t)a)\right](dt)^{\alpha}. \end{aligned} \tag{3.3}$$

Setting $x=tb+(1-t)a$, and $(dx)^{\alpha}=(b-a)^{\alpha}(dt)^{\alpha}$ gives

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b g(x)(dx)^{\alpha} \\
\leq\ & \frac{1}{2^{\alpha}}\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\left\{\int_a^b f(a+b-x)\,g(x)(dx)^{\alpha}+\int_a^b f(x)\,g(x)(dx)^{\alpha}\right\} \\
=\ & \frac{1}{2^{\alpha}}\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\left\{\int_a^b f(x)\,g(a+b-x)(dx)^{\alpha}+\int_a^b f(x)\,g(x)(dx)^{\alpha}\right\} \\
=\ & \frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b f(x)\,g(x)(dx)^{\alpha}
\end{aligned}
$$

which completes proof of the first inequality in (3.1). For the proof of the second inequality in (3.1), we first note that if f is a generalized convex function, then, for all $t\in[0,1]$, it yields

$$f(at+(1-t)b)+f(tb+(1-t)a)\leq f(a)+f(b). \tag{3.4}$$

Multiplying (3.4) by $\frac{1}{\Gamma(1+\alpha)}g(tb+(1-t)a)$, then integrating the resulting inequality with respect to t over $[0,1]$, we get:

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)}\int_0^1 g(tb+(1-t)a)f(at+(1-t)b)(dt)^{\alpha} \\
& +\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(tb+(1-t)a)f(tb+(1-t)a)(dt)^{\alpha} \\
\leq\ & [f(a)+f(b)]\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(tb+(1-t)a)(dt)^{\alpha}.
\end{aligned}
$$

This implies that

$$\frac{2^{\alpha}}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b f(x)\,g(x)(dx)^{\alpha}\leq[f(a)+f(b)]\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b g(x)(dx)^{\alpha}$$

which proves the second inequality in (3.1). □

Theorem 9. *Let $f(x) \in D_{2\alpha}[a,b]$ such that there exist constants $m,M \in R^{\alpha}$ so that $m \leq f^{(2\alpha)}(x) \leq M$ for $x \in [a,b]$. Then*

$$\begin{aligned} & \frac{mt^{\alpha}(1-t)^{\alpha}}{2^{\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha} \\ \leq \; & t^{\alpha}f(a)+(1-t)^{\alpha}f(b)-f(at+(1-t)b) \\ \leq \; & \frac{Mt^{\alpha}(1-t)^{\alpha}}{2^{\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha} \end{aligned} \tag{3.5}$$

for all $t \in [0,1]$.

Proof. We consider the function $g:[0,1] \to R^{\alpha}$ defined by

$$g(t) = t^{\alpha}f(a)+(1-t)^{\alpha}f(b)-f(at+(1-t)b)-\frac{mt^{\alpha}(1-t)^{\alpha}}{2^{\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha}.$$

Since

$$g^{(2\alpha)}(t) = (a-b)^{2\alpha}\left[m - f^{(2\alpha)}(at+(1-t)b)\right] \leq 0,$$

then g is a generalized concave function. Therefore, since $g(0)=g(1)=0$,

$$0 = t^{\alpha}g(1)+(1-t)^{\alpha}g(0) \leq g(1.t+(1-t).0) = g(t)$$

for all $t \in [0,1]$. Thus,we obtain the first part of inequality (3.5).

To see that the later inequality holds, we take the generalized convex function $h:[0,1] \to R^{\alpha}$ defined by

$$h(t) = t^{\alpha}f(a)+(1-t)^{\alpha}f(b)-f(at+(1-t)b)-\frac{Mt^{\alpha}(1-t)^{\alpha}}{2^{\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha}.$$

Since $h(0)=h(1)=0$, which implies that

$$0 = t^{\alpha}h(1)+(1-t)^{\alpha}h(0) \geq h(1.t+(1-t).0) = h(t)$$

for all $t \in [0,1]$. This completes the proof of the theorem. □

Corollary 2. *Suppose that the assumptions of Theorem 9 are satisfied, then we have the inequality*

$$\begin{aligned} & \frac{m(1-2t)^{\alpha}}{2^{3\alpha}\Gamma^2(1+\alpha)}(b-a)^{2\alpha} \\ \leq \; & \frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2^{\alpha}} - f\left(\frac{a+b}{2}\right) \\ \leq \; & \frac{M(1-2t)^{\alpha}}{2^{3\alpha}\Gamma^2(1+\alpha)}(b-a)^{2\alpha} \end{aligned} \tag{3.6}$$

for all $t \in [0,1]$.

Proof. According to Theorem 9 for $t=\frac{1}{2}$, we obtain the following result,

$$\frac{m}{2^{3\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha} \leq \frac{f(a)+f(b)}{2^{\alpha}} - f\left(\frac{a+b}{2}\right) \leq \frac{M}{2^{3\alpha}\Gamma^2(1+\alpha)}(a-b)^{2\alpha}. \tag{3.7}$$

Therefore we consider the above inequality (3.7) replacing $a \to ta + (1-t)b$ and $b \to (1-t)a + tb$ (the hypothesis $m \leq f^{(\alpha)} \leq M$ is still working on the interval with these endpoints because it is contained by $[a,b]$) and we get the claimed result. □

Theorem 10. *Let* $f(x) \in D_{2\alpha}[a,b]$ *such that there exist constants* $m, M \in R^{\alpha}$ *so that* $m \leq f^{(2\alpha)}(x) \leq M$ *for* $x \in [a,b]$*. Assume* $g : [a,b] \to R^{\alpha}$ *is nonnegative, local fractional integrable and symmetric* $\frac{a+b}{2}$*, then the following inequalities for local fractional integrals hold*

$$\frac{m}{2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(x-b)^{\alpha}(a-x)^{\alpha}g(x)(dx)^{\alpha} \tag{3.8}$$

$$\leq \frac{f(a)+f(b)}{2^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x) - {}_{a}I_{b}^{\alpha}f(x)g(x)$$

$$\leq \frac{M}{2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(x-b)^{\alpha}(a-x)^{\alpha}g(x)(dx)^{\alpha}$$

and

$$\frac{m}{2^{3\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(a+b-2x)^{\alpha}g(x)(dx)^{\alpha} \tag{3.9}$$

$$\leq {}_{a}I_{b}^{\alpha}f(x)g(x) - f\left(\frac{a+b}{2}\right)\ {}_{a}I_{b}^{\alpha}g(x)$$

$$\leq \frac{M}{2^{3\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(a+b-2x)^{\alpha}g(x)(dx)^{\alpha}.$$

Proof. Multiplying both sides of (3.5) by $\frac{1}{\Gamma(1+\alpha)}g(ta+(1-t)b)$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\frac{m}{2^{\alpha}\Gamma^{3}(1+\alpha)}(a-b)^{2\alpha}\int_{0}^{1}t^{\alpha}(1-t)^{\alpha}g(ta+(1-t)b)(dt)^{\alpha}$$

$$\leq \frac{f(a)}{\Gamma(1+\alpha)}\int_{0}^{1}t^{\alpha}g(ta+(1-t)b)(dt)^{\alpha}$$

$$+\frac{f(b)}{\Gamma(1+\alpha)}\int_{0}^{1}(1-t)^{\alpha}g(ta+(1-t)b)(dt)^{\alpha}$$

$$-\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f(at+(1-t)b)g(ta+(1-t)b)(dt)^{\alpha}$$

$$\leq \frac{M}{2^{\alpha}\Gamma^{3}(1+\alpha)}(a-b)^{2\alpha}\int_{0}^{1} t^{\alpha}(1-t)^{\alpha}g(ta+(1-t)b)\,(dt)^{\alpha}.$$

This implies that, by using the change of the variable $ta+(1-t)b=u$,

$$\frac{m}{(a-b)^{\alpha}\,2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(u-b)^{\alpha}(a-u)^{\alpha}g(u)\,(du)^{\alpha} \tag{3.10}$$

$$\leq \frac{f(a)}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(u-b)^{\alpha}g(u)\,(du)^{\alpha}$$

$$+\frac{f(b)}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(a-u)^{\alpha}g(u)\,(du)^{\alpha}$$

$$-\frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(u)g(u)\,(du)^{\alpha}$$

$$\leq \frac{M}{(a-b)^{\alpha}\,2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(u-b)^{\alpha}(a-u)^{\alpha}g(u)\,(du)^{\alpha}.$$

On the other hand, because of the symmetry of g, for $u=a+b-x$, we also have

$$\frac{m}{(a-b)^{\alpha}\,2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(x-b)^{\alpha}(a-x)^{\alpha}g(x)\,(dx)^{\alpha} \tag{3.11}$$

$$\leq \frac{f(a)}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(a-x)^{\alpha}g(x)\,(dx)^{\alpha}$$

$$+\frac{f(b)}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(x-b)^{\alpha}g(x)\,(dx)^{\alpha}$$

$$-\frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)g(x)\,(dx)^{\alpha}$$

$$\leq \frac{M}{(a-b)^{\alpha}\,2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}(x-b)^{\alpha}(a-x)^{\alpha}g(x)\,(dx)^{\alpha}.$$

Adding (3.10) and (3.11) we obtain (3.8).

The second part is established in a similar manner, we follow same steps as above, using (3.6) instead of (3.5). The computation is straightforward, taking into account the symmetry of g (applied now as $g(ta+(1-t)b)=g((1-t)a+tb)$). We omit the details. This completes the proof. □

We will establish a new result connected with the left-hand sides of (3.1) used the following Theorem.

Theorem 11. *Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \to R^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^0$ with $a < b$ and $g : [a,b] \to R^\alpha$ is nonnegative and local fractional integrable, then the following equality for local fractional integrals holds:*

$$ {}_aI_b^{(\alpha)} f(x)\, g(x) - f\left(\frac{a+b}{2}\right) {}_aI_b^{(\alpha)} g(x) = \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)} \int_0^1 k(t) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \tag{3.12}$$

where

$$k(t) = \begin{cases} \frac{1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha, & t \in \left[0, \frac{1}{2}\right) \\ \\ \frac{1}{\Gamma(1+\alpha)} \int_1^t g(sa+(1-s)b)\,(ds)^\alpha, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} & K \qquad (3.13) \\ = & \frac{1}{\Gamma(1+\alpha)} \int_0^1 k(t) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ = & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha \right) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ & + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 \left(\frac{1}{\Gamma(1+\alpha)} \int_1^t g(sa+(1-s)b)\,(ds)^\alpha \right) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ = & K_1 + K_2. \end{aligned}$$

Using the local fractional integration by parts, we have

$$\begin{aligned} K_1 = & \left(\frac{1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha \right) \frac{f(ta+(1-t)b)}{(a-b)^\alpha} \Bigg|_0^{\frac{1}{2}} \qquad (3.14) \\ & - \frac{1}{(a-b)^\alpha \Gamma(1+\alpha)} \int_0^{\frac{1}{2}} g(ta+(1-t)b) f(ta+(1-t)b)\,(dt)^\alpha \end{aligned}$$

$$= \frac{1}{(a-b)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}} g(sa+(1-s)b)\,(ds)^{\alpha}\right) f\left(\frac{a+b}{2}\right) - \frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}} g(ta+(1-t)b)f(ta+(1-t)b)\,(dt)^{\alpha}$$

and similarly,

$$K_2 = \frac{1}{(a-b)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1} g(sa+(1-s)b)\,(ds)^{\alpha}\right) f\left(\frac{a+b}{2}\right) - \frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1} g(ta+(1-t)b)f(ta+(1-t)b)\,(dt)^{\alpha}. \tag{3.15}$$

Adding (3.14) and (3.15) in (3.13) and using the changing variable $x = ta + (1-t)b$ for $t \in [0, 1]$, we get

$$\begin{aligned} K &= K_1 + K_2 \qquad (3.16)\\ &= \frac{1}{(a-b)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1} g(sa+(1-s)b)\,(ds)^{\alpha}\right) f\left(\frac{a+b}{2}\right) \\ &\quad - \frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_{0}^{1} g(ta+(1-t)b)f(ta+(1-t)b)\,(dt)^{\alpha} \\ &= \frac{1}{(a-b)^{2\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1} g(x)\,(dx)^{\alpha}\right) f\left(\frac{a+b}{2}\right) \\ &\quad - \frac{1}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{0}^{1} g(x)f(x)\,(dx)^{\alpha} \\ &= \frac{1}{(a-b)^{2\alpha}} f\left(\frac{a+b}{2}\right) {}_aI_b^{(\alpha)} g(x) - \frac{1}{(a-b)^{2\alpha}}\, {}_aI_b^{(\alpha)} f(x)\, g(x). \end{aligned}$$

Multiplying the both sides of (3.16) by $(a-b)^{2\alpha}$, we obtain the desired result, which completes the proof. □

Corollary 3. *Under assumption of Theorem 11, if we take $g(x) = 1^{\alpha}$, then we have*

$$\begin{aligned}
&{}_{a}I_{b}^{(\alpha)}f(x)-\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\\
=\;&\frac{(b-a)^{2\alpha}}{[\Gamma(1+\alpha)]^{2}}\left[\int_{0}^{\frac{1}{2}}t^{\alpha}f^{(\alpha)}(ta+(1-t)b)\,(dt)^{\alpha}+\int_{\frac{1}{2}}^{1}(t-1)^{\alpha}f^{(\alpha)}(ta+(1-t)b)\,(dt)^{\alpha}\right].
\end{aligned}$$

Theorem 12. *Let $I \subseteq R$ be an interval, $f : I^{0} \subseteq R \to R^{\alpha}$ (I^{0} is the interior of I) such that $f \in D_{\alpha}(I^{0})$ and $f^{(\alpha)} \in C_{\alpha}[a,b]$ for $a, b \in I^{0}$ with $a < b$ and $g : [a,b] \to R^{\alpha}$ is nonnegative, local fractional integrable and symmetric $\frac{a+b}{2}$. If $\left|f^{(\alpha)}\right|$ is the generalized convex on $[a,b]$, then the following inequality for local fractional integrals holds:*

$$\begin{aligned}
&\left|{}_{a}I_{b}^{\alpha}f(x)\,g(x)-f\left(\frac{a+b}{2}\right)\,{}_{a}I_{b}^{\alpha}g(x)\right| \qquad (3.17)\\
\leq\;&\frac{\left[\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right]}{(b-a)^{\alpha}\Gamma(1+2\alpha)}\left(\int_{\frac{a+b}{2}}^{b}g(x)\left[(x-a)^{2\alpha}-(b-x)^{2\alpha}\right](dx)^{\alpha}\right).
\end{aligned}$$

Proof. Taking modulus in Theorem 11 and using the generalized convexity of $\left|f^{(\alpha)}\right|$, we have

$$\begin{aligned}
&\left|{}_{a}I_{b}^{(\alpha)}f(x)\,g(x)-f\left(\frac{a+b}{2}\right)\,{}_{a}I_{b}^{(\alpha)}g(x)\right| \qquad (3.18)\\
=\;&(b-a)^{2\alpha}\left\{\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}g(sa+(1-s)b)\,(ds)^{\alpha}\right)\right.\right.\\
&\left.\times\left[t^{\alpha}\left|f^{(\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(b)\right|\right]\right](dt)^{\alpha}\\
&+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_{t}^{1}g(sa+(1-s)b)\,(ds)^{\alpha}\right)\right.\\
&\left.\left.\times\left[t^{\alpha}\left|f^{(\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(b)\right|\right]\right](dt)^{\alpha}\right\}\\
=\;&(b-a)^{2\alpha}\left[Q_{1}+Q_{2}\right].
\end{aligned}$$

By change of the order of integration, we get

$$\begin{aligned} Q_1 &= \frac{\left|f^{(\alpha)}(a)\right|}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha \right) t^\alpha\,(dt)^\alpha \\ &\quad + \frac{\left|f^{(\alpha)}(b)\right|}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha \right) (1-t)^\alpha\,(dt)^\alpha \\ &= \frac{\left|f^{(\alpha)}(a)\right|}{[\Gamma(1+\alpha)]^2} \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} g(sa+(1-s)b) t^\alpha\,(dt)^\alpha\,(ds)^\alpha \\ &\quad + \frac{\left|f^{(\alpha)}(b)\right|}{[\Gamma(1+\alpha)]^2} \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} g(sa+(1-s)b)(1-t)^\alpha\,(dt)^\alpha\,(ds)^\alpha. \end{aligned}$$

Using the Lemma 2, we have

$$\begin{aligned} Q_1 &= \frac{1}{\Gamma(1+2\alpha)} \left[\left|f^{(\alpha)}(a)\right| \int_0^{\frac{1}{2}} g(sa+(1-s)b) \left[\frac{1}{4^\alpha} - s^{2\alpha}\right] (ds)^\alpha \right. \\ &\quad \left. + \left|f^{(\alpha)}(b)\right| \int_0^{\frac{1}{2}} g(sa+(1-s)b) \left[(1-s)^{2\alpha} - \frac{1}{4^\alpha}\right] (ds)^\alpha \right], \end{aligned}$$

and using changing variable $x = sa + (1-s)$ for $s \in \left[0, \frac{1}{2}\right]$, we obtain

$$\begin{aligned} Q_1 &= \frac{1}{4^\alpha (b-a)^{3\alpha} \Gamma(1+2\alpha)} \\ &\quad \times \left[\left|f^{(\alpha)}(a)\right| \int_{\frac{a+b}{2}}^b g(x) \left[(b-a)^{2\alpha} - 4^\alpha (b-x)^{2\alpha}\right] (dx)^\alpha \right. \\ &\quad \left. + \left|f^{(\alpha)}(b)\right| \int_{\frac{a+b}{2}}^b g(x) \left[4^\alpha (x-a)^{2\alpha} - (b-a)^{2\alpha}\right] (dx)^\alpha \right]. \end{aligned} \tag{3.19}$$

If we follow the same steps as above, then we obtain

$$\begin{aligned} Q_2 &= \frac{1}{4^\alpha (b-a)^{3\alpha} \Gamma(1+2\alpha)} \left[\left|f^{(\alpha)}(a)\right| \int_a^{\frac{a+b}{2}} g(x) \left[4^\alpha (b-x)^{2\alpha} - (b-a)^{2\alpha}\right] (dx)^\alpha \right. \\ &\quad \left. + \frac{\left|f^{(\alpha)}(b)\right|}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} g(x) \left[(b-a)^{2\alpha} - 4^\alpha (x-a)^{2\alpha}\right] (dx)^\alpha \right]. \end{aligned}$$

Since $g(x)$ is symmetric to $x=\frac{a+b}{2}$, we have $g(x)=g(a+b-x)$. Using changing variable, it follows that

$$\begin{aligned} Q_2 &= \frac{1}{4^{\alpha}(b-a)^{3\alpha}\Gamma(1+2\alpha)} \\ &\times\left[\left|f^{(\alpha)}(a)\right|\int\limits_{\frac{a+b}{2}}^{b} g(x)\left[4^{\alpha}(x-a)^{2\alpha}-(b-a)^{2\alpha}\right](dx)^{\alpha}\right. \\ &\left.+\frac{\left|f^{(\alpha)}(b)\right|}{\Gamma(1+\alpha)}\int\limits_{\frac{a+b}{2}}^{b} g(a+b-x)\left[(b-a)^{2\alpha}-4^{\alpha}(b-x)^{2\alpha}\right](dx)^{\alpha}\right]. \end{aligned} \tag{3.20}$$

Substituting the equalities (3.19) and (3.20) in (3.18), then we obtain required result. □

Corollary 4. *Under assumption of Theorem 12, if we take $g(x)=1^{\alpha}$, then we have*

$$\begin{aligned} &\left| {}_aI_b^{(\alpha)}f(x)-\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{3^{\alpha}(b-a)^{2\alpha}}{2^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left[\frac{\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|}{2^{\alpha}}\right]. \end{aligned}$$

Now, we will give a new result connected with the right-hand sides of (3.1) used the following Theorem.

Theorem 13. *Let $I\subseteq R$ be an interval, $f:I^0\subseteq R\to R^{\alpha}$ (I^0 is the interior of I) such that $f\in D_{\alpha}(I^0)$ and $f^{(\alpha)}\in C_{\alpha}[a,b]$ for $a,b\in I^0$ with $a<b$ and $g:[a,b]\to R^{\alpha}$ is nonnegative and local fractional integrable, then the following equality for local fractional integrals holds:*

$$\frac{f(a)+f(b)}{2^{\alpha}}\,{}_aI_b^{(\alpha)}g(x)-{}_aI_b^{(\alpha)}f(x)g(x)=\frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\int\limits_0^1 p(t)f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \tag{3.21}$$

where

$$p(t)=\frac{1}{\Gamma(1+\alpha)}\left[\int\limits_t^1 g(sa+(1-s)b)(ds)^{\alpha}-\int\limits_0^t g(sa+(1-s)b)(ds)^{\alpha}\right].$$

Proof. It suffices to note that

$$\begin{aligned} & L \qquad (3.22) \\ = & \frac{1}{\Gamma(1+\alpha)} \int_0^1 k(t) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ = & \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{1}{\Gamma(1+\alpha)} \int_t^1 g(sa+(1-s)b)\,(ds)^\alpha \right) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ & + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{-1}{\Gamma(1+\alpha)} \int_0^t g(sa+(1-s)b)\,(ds)^\alpha \right) f^{(\alpha)}(ta+(1-t)b)\,(dt)^\alpha \\ = & L_1 + L_2. \end{aligned}$$

Using the local fractional integration by parts, we have

$$\begin{aligned} L_1 &= \left(\frac{1}{\Gamma(1+\alpha)} \int_t^1 g(sa+(1-s)b)\,(ds)^\alpha \right) \frac{f(ta+(1-t)b)}{(a-b)^\alpha} \Bigg|_0^1 \qquad (3.23) \\ & \quad + \frac{1}{(a-b)^\alpha \Gamma(1+\alpha)} \int_0^1 g(ta+(1-t)b) f(ta+(1-t)b)\,(dt)^\alpha \\ &= \frac{-1}{(a-b)^\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 g(sa+(1-s)b)\,(ds)^\alpha \right) f(b) \\ & \quad + \frac{1}{(a-b)^\alpha \Gamma(1+\alpha)} \int_0^1 g(ta+(1-t)b) f(ta+(1-t)b)\,(dt)^\alpha . \end{aligned}$$

Similarly, we get

$$\begin{aligned} L_2 &= \frac{-1}{(a-b)^\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 g(sa+(1-s)b)\,(ds)^\alpha \right) f(a) \qquad (3.24) \\ & \quad + \frac{1}{(a-b)^\alpha \Gamma(1+\alpha)} \int_0^1 g(ta+(1-t)b) f(ta+(1-t)b)\,(dt)^\alpha . \end{aligned}$$

Substituting the equalities (3.23) and (3.24) in (3.22) and using the changing variable $x = ta + (1-t)b$ for $t \in [0,1]$, we have

$$\begin{aligned} L &= L_1 + L_2 \qquad (3.25) \\ &= \frac{-1}{(a-b)^{\alpha}} \left(\frac{1}{(1+\alpha)} \int_0^1 g(sa+(1-s)b)\,(ds)^{\alpha} \right) [f(a)+f(b)] \\ &\quad + \frac{2^{\alpha}}{(a-b)^{\alpha}\Gamma(1+\alpha)} \int_0^1 g(ta+(1-t)b) f(ta+(1-t)b)\,(dt)^{\alpha} \\ &= \frac{f(a)+f(b)}{(a-b)^{2\alpha}}\ {}_aI_b^{(\alpha)} g(x) - \frac{2^{\alpha}}{(a-b)^{2\alpha}}\ {}_aI_b^{(\alpha)} f(x)\, g(x). \end{aligned}$$

If we multiply both sides of (3.25) by $\frac{(a-b)^{2\alpha}}{2^{\alpha}}$, we obtain the required result. □

Remark 1. *If we take $g(x) = 1^{\alpha}$ in Theorem 13, then we have*

$$\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{(\alpha)} f(x) = \frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(1+\alpha)} \int_0^1 (1-2t)^{\alpha} f^{(\alpha)}(ta+(1-t)b)\,(dt)^{\alpha}$$

which is given by Mo et. al in [8].

Theorem 14. *Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \to R^{\alpha}$ (I^0 is the interior of I) such that $f \in D_{\alpha}(I^0)$ and $f^{(\alpha)} \in C_{\alpha}[a,b]$ for $a, b \in I^0$ with $a < b$ and $g : [a,b] \to R^{\alpha}$ is nonnegative, local fractional integrable and symmetric $\frac{a+b}{2}$. If $\left|f^{(\alpha)}\right|^q$ is the generalized convex on $[a,b]$, then the following inequality for local fractional integrals holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2^{\alpha}}\ {}_aI_b^{(\alpha)} g(x) - {}_aI_b^{(\alpha)} f(x)\, g(x) \right| \qquad (3.26) \\ &\leq \frac{(b-a)^{2\alpha}}{2^{\alpha}} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{1}{q}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 W^p(t)\,(dt)^{\alpha} \right)^{\frac{1}{p}} \\ &\quad \times \left[\left|f^{(\alpha)}(a)\right|^q + \left|f^{(\alpha)}(b)\right|^q \right]^{\frac{1}{q}} \end{aligned}$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$W(t) = \left| \frac{1}{\Gamma(1+\alpha)} \int_{a+(b-a)t}^{b-(b-a)t} g(x)\,(dx)^{\alpha} \right|$$

for $t \in [0,1]$.

Proof. Taking modulus in Theorem 13, we have

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2^{\alpha}} \ {}_{a}I_{b}^{(\alpha)} g(x) - \ {}_{a}I_{b}^{(\alpha)} f(x)\, g(x) \right| \qquad (3.27)\\
\leq\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}} \\
&\times \int_{0}^{1} \left[\left| \int_{t}^{1} g(sa+(1-s)b)\,(ds)^{\alpha} - \int_{0}^{t} g(sa+(1-s)b)\,(ds)^{\alpha} \right| \right. \\
&\left. \times \left| f^{(\alpha)}(ta+(1-t)b) \right| \right] (dt)^{\alpha}.
\end{aligned}$$

Since $g(x)$ is symmetric to $x = \frac{a+b}{2}$, we write

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \left[\int_{t}^{1} g(sa+(1-s)b)\,(ds)^{\alpha} - \int_{0}^{t} g(sa+(1-s)b)\,(ds)^{\alpha} \right] \qquad (3.28)\\
=\ & \frac{1}{\Gamma(1+\alpha)} \int_{a+(b-a)t}^{b-(b-a)t} g(x)\,(dx)^{\alpha},
\end{aligned}$$

for $t \in \left[0, \frac{1}{2}\right]$ and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \left[\int_{t}^{1} g(sa+(1-s)b)\,(ds)^{\alpha} - \int_{0}^{t} g(sa+(1-s)b)\,(ds)^{\alpha} \right] \qquad (3.29)\\
=\ & \frac{-1}{\Gamma(1+\alpha)} \int_{b-(b-a)t}^{a+(b-a)t} g(x)\,(dx)^{\alpha},
\end{aligned}$$

for $t \in \left[\frac{1}{2}, 1\right]$. If we write (3.28) and (3.29) in (3.27), we have

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2^{\alpha}} \ {}_{a}I_{b}^{(\alpha)} g(x) - \ {}_{a}I_{b}^{(\alpha)} f(x)\, g(x) \right| \\
\leq\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)} \int_{0}^{1} W(x) \left| f^{(\alpha)}(ta+(1-t)b) \right| (dt)^{\alpha}
\end{aligned}$$

where $W(t)=\left|\frac{1}{\Gamma(1+\alpha)}\int\limits_{a+(b-a)t}^{b-(b-a)t}g(x)\,(dx)^{\alpha}\right|$. Using generalized Hölder's inequality and generalized convexity of $\left|f^{(\alpha)}\right|^{q}$

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}\ {}_aI_b^{(\alpha)}g(x)-\ {}_aI_b^{(\alpha)}f(x)\,g(x)\right| \\
\leq\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1 W^p(x)\,(dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1 W^p(x)\,(dt)^{\alpha}\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1\left[t^{\alpha}\left|f^{(\alpha)}(a)\right|^{q}+(1-t)^{\alpha}\left|f^{(\alpha)}(b)\right|^{q}\right](dt)^{\alpha}\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1 W^p(t)\,(dt)^{\alpha}\right)^{\frac{1}{p}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}\right)\right]^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 5. *Under assumption of Theorem 14, taking $g(x)=1^{\alpha}$, since*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int\limits_0^1\left|\frac{1}{\Gamma(1+\alpha)}\int\limits_{a+(b-a)t}^{b-(b-a)t}(dx)^{\alpha}\right|^{p}(dt)^{\alpha} \\
=\ & \frac{1}{\Gamma(1+\alpha)}\int\limits_0^1\frac{(b-a)^{p\alpha}}{[\Gamma(1+\alpha)]^{p}}(1-2t)^{p\alpha}(dt)^{\alpha} \\
=\ & \frac{(b-a)^{p\alpha}}{[\Gamma(1+\alpha)]^{p}}\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}
\end{aligned}$$

then we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{(\alpha)}f(x)\right| \\
\leq\ & \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}} \\
&\times\left[\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned}$$

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