Journal of Fractional Calculus and Applications,
Vol. 8(1) Jan. 2017, pp. 91-98.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON MEROMORPHIC FUNCTIONS WITH A FIXED POINT INVOLVING SRIVASTAVA-ATTIYA OPERATOR

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ABSTRACT. Making use of the familiar differential subordination structure in this paper,we investigate a new class of meromorphic functions with a fixed point w involving Srivastava-Attiya operator. Some results connected to sharp coefficient bounds, distortion theorem and other important properties are obtained.

1. Introduction

Let w be a fixed point in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Denote by H the class of functions which are regular and

$$A(w) = \{f \in H : f(w) = f'(w) - 1 = 0\}.$$

Also denote by

$$N_w = \{f \in A(w) : f \ \text{ is univalent in } \Delta \ \}$$

the subclass $A(w)$ consist of the functions of the form

$$f(z) = (z - w) + \sum_{n=2}^{\infty} a_n (z - w)^n, \tag{1}$$

that are analytic in the open unit disk. Note that $N_0 = N$ be a subclass of $A(w)$ consisting of univalent functions in Δ. By $N_w^* \ (\beta)$and $C_w \ (\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions.

$$Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > \beta, \ Re\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right) > \beta$$

and $z \in \Delta$ for some β $(0 \leq \beta < 1)$, introduced and studied by Kanas and Ronning [11].The class $N_w^*(0)$ is defined by geometric property that the image of any circular arc centered at w is starlike with respect to$f(w)$ and the corresponding class $C_w(0)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by

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2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Subordination, meromorphic function, coefficient estimate, distortion bound and radii of starlikeness.

Submitted Jun 20, 2016.

Goodman in [9] and [8] for uniformly starlike and convex functions, except that in this case the point w is fixed. In particular, $C_0(0) = C$ and $N_0^*(0) = N^*$ respectively, are the well-known standard class of convex and starlike functions (see [22]).

Let $\sum$ denoted the subclass of meromorphic functions$f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \ , \tag{2}$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $\sum_w$ be the subclass of $A(w)$ consist of the functions of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \qquad (a_n \geq 0; z \neq w). \tag{3}$$

A functions $f(z)$ of the form (3) is in the class of meromorphic starlike of order $\beta(0 \leq \beta < 1)$ denoted by $\sum_w(\beta)$, if

$$-\,Re\left(\frac{(z-w)\,f'(z)}{f(z)}\right) > \beta \qquad (z - w \in \Delta := \Delta^* \cup \{0\}). \tag{4}$$

and is in the class of meromorphic convex of order β $(0 \leq \beta < 1)$ denoted by $\sum_w^C(\beta)$, if

$$-\,Re\left(1 + \frac{(z-w)\,f''(z)}{f'(z)}\right) > \beta \qquad (z - w \in \Delta := \Delta^* \cup \{0\}) \tag{5}$$

We recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by [24]

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$(a \in D \setminus \left\{\mathbb{Z}_0^-\right\}; s \in \mathbb{C}, R(s) > 1 \text{ and } |z| = 1)$$

where, as usual $\mathbb{Z}_0^- := \mathbb{Z}/\{\mathbb{N}\}\,(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\};\ N := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [5], Lin and Srivastava [13], Lin et al. [14], and see the references stated therein.

For the class of analytic functions denote by A consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \Delta).$$

Srivastava and Attiya [23] introduced and investigated the linear operator:

$$J_{s,b} : A \to A$$

defined in terms of the Hadamard product (or convolution) by

$$J_{s,b} f(z) = G_{s,b} * f(z) \tag{6}$$

where, for convenience,

$$G_{s,b}(z) := (1 + b)^s [\Phi(z, s, a) - b^{-s}] \tag{7}$$

$(z \in \Delta; b \in \mathbb{C}\setminus\{\mathbb{Z}_0^-\}; s \in \mathbb{C}; f \in A)$. For $f \in A$ it is easy to observe from (6) and (7) that

$$J_{s,b}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)a_n z^n, \quad (z \in \Delta). \tag{8}$$

It is well known that the Srivastava-Attiya operator $J_{s,b}$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [1], Libera [12], Bernardi [4], and Jung et al. [10].

Motivated essentially by the above mentioned Srivastava-Attiya operator, Murugusundaramoorthy and Janani [20] introduced a new linear operator

$$J_b^s : \sum_w \to \sum_w$$

in terms of Hadamard product given by

$$J_b^s f(z) = \vartheta_{b,p}^s * f(z) \tag{9}$$

$$(z - w \in \Delta := \Delta^* \cup \{0\}; b \in \mathbb{C}\setminus\{\mathbb{Z}_0^-\}; s \in \mathbb{C}; f \in_w \ )$$

where, for convenience

$$\vartheta_{b,p}^s(z) := (1+b)^s[\Phi(z,s,b) - b^{-s}] \tag{10}$$

and

$$\Phi(z,s,b) = \frac{1}{b^s} + \frac{(z-w)^{-1}}{(1+b)^s} + \frac{(z-w)}{(2+b)^s} + ... \ .$$

For $f \in_w$, it is easy to observe from the above equations (9) and (10) that

$$J_b^s f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} C_b^s(n) a_n (z-w)^n, \qquad (z - w \in \Delta := \Delta^* \cup \{0\}) \tag{11}$$

where

$$C_b^s(n) = \left|\left(\frac{(1+b)}{n+1+b}\right)^s\right| \tag{12}$$

and (throughout this paper unless otherwise mentioned) the parameters s, b are constrained as $b \in \mathbb{C}\setminus\{\mathbb{Z}_0^-\}; s \in \mathbb{C}$. Motivated by earlier works on meromorphic functions by function theorists(see [2, 3, 6, 15, 16, 17, 18, 19, 21, 25]).

Now we defined the class $\Psi^w(A,B)$ consisting the functions $f(z) \in_w$ such that

$$-\frac{(z-w)\left[J_b^s f(z)\right]''}{\left[J_b^s f(z)\right]'} \prec H(z) \tag{13}$$

where $H(z) = 2\frac{1+A(z-w)}{1+B(z-w)}$, $A = B + (C-B)(1-\lambda), -1 \le B < C \le 1, 0 \le \lambda < 1$ and "$\prec$" denotes the subordination symbol [7, 26] .

2. Main Results

In this section we find sharp coefficient estimates and integral representation for the class $\Psi^w(A,B)$.

Let $f(z) \in_w$, then $f(z) \in \Psi^w(A,B)$ if and only if

$$\sum_{n=1}^{\infty} C_b^s(n)n\left[(1+B)(n+1)+2(C-B)(1-\lambda)\right]a_n < 2(C-B)(1-\lambda). \tag{14}$$

The result is sharp for the function $h(z)$ given by

$$h(z)=\frac{1}{z-w}+\frac{2(C-B)(1-\lambda)}{C_b^s(n)n\left[(1+B)(n+1)+2(C-B)(1-\lambda)\right]}(z-w)^n \ , \ \ n=1,2,... \tag{15}$$

Proof. Let $f(z) \in \Psi^w(A,B)$, then the inequality (11) or the inequality

$$\left|\frac{(z-w)\left[J_b^s f(z)\right]^{''}+2\left[J_b^s f(z)\right]'}{B(z-w)\left[J_b^s f(z)\right]^{''}+2\left[B+(C-B)(1-\lambda)\right]\left[J_b^s f(z)\right]'}\right|<1 \tag{16}$$

holds true, therefore by using (3)

$$\left|\frac{\sum_{n=1}^{\infty} C_b^s(n)n(n+1)a_n(z-w)^n}{2(C-B)(1-\lambda)-\xi}\right|<1$$

where
$\xi=\sum_{n=1}^{\infty} C_b^s(n)n\left[B(n-1)+2\left[B+(C-B)(1-\lambda)\right]\right]a_n(z-w)^n$.
Since $Re(z) \le |z|$ for all z, therefore

$$Re\left\{\frac{\sum_{n=1}^{\infty} C_b^s(n)n(n+1)a_n(z-w)^n}{2(C-B)(1-\lambda)-\xi}\right\}<1.$$

where
$\xi=\sum_{n=1}^{\infty} C_b^s(n)n\left[B(n-1)+2\left[B+(C-B)(1-\lambda)\right]\right]a_n(z-w)^n$.
By letting $(z-w)\to 1$ through real values, we have

$$\sum_{n=1}^{\infty} C_b^s(n)n\left[(1+B)(n+1)+2(C-B)(1-\lambda)\right]a_n < 2(C-B)(1-\lambda).$$

Conversely, let (14) holds true, if we let $(z-w) \in \partial\Delta^*$where $\partial\Delta^*$denotes the boundary of Δ^*,then we have

$$\left|\frac{(z-w)\left[J_b^s f(z)\right]^{''}+2\left[J_b^s f(z)\right]'}{B(z-w)\left[J_b^s f(z)\right]^{''}+2\left[B+(C-B)(1-\lambda)\right]\left[J_b^s f(z)\right]'}\right|$$

$$\le \quad \frac{\sum_{n=1}^{\infty} C_b^s(n)n(n+1)a_n(z-w)^n}{2(C-B)(1-\lambda)-\sum_{n=1}^{\infty} C_b^s(n)n\left[B(n-1)+2\left[B+(C-B)(1-\lambda)\right]\right]a_n}<1.$$

Thus by the Maximum modulus theorem, we conclude $f(z) \in \Psi^w(A,B)$. □

If $f(z) \in \Psi^w(A,B)$, then

$$J_b^s f(z)=\int_0^z\left[\exp\int_0^z \frac{2\left[AM(t)-1\right]}{(t-w)\left[1-M(t)B\right]}dt\right]ds$$

where $A=B+(C-B)(1-\lambda)$ and $|M(z)|<1$.

Proof. Since $f(z) \in \Psi^w(A,B)$, so (11) or equivalently (16) holds true. Hence

$$\frac{(z-w)\left[J_b^s f(z)\right]'' + 2\left[J_b^s f(z)\right]'}{B(z-w)\left[J_b^s f(z)\right]'' + 2A\left[J_b^s f(z)\right]'} = M(z)$$

where $|M(z)| < 1, z \in \Delta^*$ and $A = B + (C-B)(1-\lambda)$.

This yields

$$\frac{\left[J_b^s f(z)\right]''}{\left[J_b^s f(z)\right]'} = \frac{2\left[AM(z)-1\right]}{(z-w)\left[1-M(z)B\right]},$$

after integration we obtain the required result. □

Theorem 2show that if $f(z) \in \Psi^w(A,B)$, then

$$|a_n| < \frac{2(C-B)(1-\lambda)}{C_b^s(n)n\left[(1+B)(n+1)+2(C-B)(1-\lambda)\right]}, \quad n = 1,2,3,... \tag{17}$$

3. Distortion Bounds and Extreme points

In this section we investigate about distortion and extreme point of the class$\Psi^w(A,B)$.

Let $f(z) \in \Psi^w(A,B)$, then

$$\frac{1}{r} - \frac{(C-B)(1-\lambda)}{(1+B)+(C-B)(1-\lambda)}r < \left|J_b^s f(z)\right| < \frac{1}{r} + \frac{(C-B)(1-\lambda)}{(1+B)+(C-B)(1-\lambda)}r$$

where $0 < |z-w| = r < 1$.

Proof. By Theorem 2 and (17) we have

$$\begin{aligned} \left|J_b^s f(z)\right| &= \frac{1}{z-w} + \sum_{n=1}^{\infty} C_b^s(n) a_n (z-w)^n \\ &\le \frac{1}{r} + \sum_{n=1}^{\infty} C_b^s(n)\left|a_n\right| r^n \qquad (18) \\ &< \frac{1}{r} + \frac{(C-B)(1-\lambda)}{(1+B)+(C-B)(1-\lambda)}r \qquad (19) \end{aligned}$$

similarly we obtain

$$\left|J_b^s f(z)\right| \ge \frac{1}{r} - \frac{(C-B)(1-\lambda)}{(1+B)+(C-B)(1-\lambda)}r.$$

□

The function $f(z)$ of the from (3) belongs to $\Psi^w(A,B)$ if and only if it can be expressed by

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z), \quad \lambda_n \ge 0, \qquad n = 1,2,... \tag{20}$$

where $f_0(z) = \frac{1}{z-w}$,

$$f_n(z) = \frac{1}{z-w} + \frac{2(C-B)(1-\lambda)}{C_b^s(n)n\left[(1+B)(n+1)+2(C-B)(1-\lambda)\right]}(z-w)^n, \qquad n = 1,2,...$$

and

$$\sum_{n=0}^{\infty} \lambda_n = 1.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \lambda_0 f_0(z) +$$

$$\sum_{n=1}^{\infty} \lambda_n \left[\frac{1}{z-w} + \frac{2(C-B)(1-\lambda)}{C_b^s(n) n \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]} (z-w)^n \right]$$

$$= \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{2(C-B)(1-\lambda)}{C_b^s(n) n \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]} \lambda_n (z-w)^n .$$

Now by using Theorem 2 we conclude that $f(z) \in \Psi^w(A,B)$. Conversely, if $f(z)$ given by (3) belongs to $\Psi^w(A,B)$. By letting $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$ where

$$\lambda_n = \frac{C_b^s(n) n \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]}{2(C-B)(1-\lambda)} a_n , \quad n = 1, 2, ...$$

we conclude the required result. □

4. Radii of Starlikeness and convexity

In the last section we introduce the radii of starlikeness for functions in the class $\Psi^w(A,B)$.

If $f(z) \in \Psi^w(A,B)$, then f is starlike of order δ $(0 \le \delta < 1)$ in disk $|z-w| < r_1$, and it is convex of order δ in disk $|z-w| < r_2$ where

$$r_1 = \inf_{n \ge 1} \left\{ \frac{(1-\delta) C_b^s(n) n \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]}{2(n+2-\delta)(C-B)(1-\lambda)} \right\}^{\frac{1}{n+1}} \tag{21}$$

and

$$r_2 = \inf_{n \ge 1} \left\{ \frac{(1-\delta) C_b^s(n) \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]}{2(n+2-\delta)(C-B)(1-\lambda)} \right\}^{\frac{1}{n+1}} .$$

Proof. For starlikeness it is enough to show that

$$\left| \frac{(z-w) f'(z)}{f(z)} + 1 \right| \le 1 - \delta,$$

but

$$\begin{aligned} \left| \frac{(z-w) f'(z)}{f(z)} + 1 \right| &= \left| \frac{\sum_{n=1}^{\infty} (n+1) a_n (z-w)^n}{\frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n} \right| \\ &\le \frac{\sum_{n=1}^{\infty} (n+1) a_n |z-w|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z-w|^{n+1}} . \end{aligned} \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{(n+2-\delta)}{1-\delta} a_n |(z-w)|^{n+1} \le 1$$

by using (14) we obtain

$$\sum_{n=1}^{\infty} \frac{(n+2-\delta)}{1-\delta} a_n |(z-w)|^{n+1} \le \sum_{n=1}^{\infty} \frac{C_b^s(n) n \left[(1+B)(n+1) + 2(C-B)(1-\lambda)\right]}{2(C-B)(1-\lambda)} a_n \le 1.$$

So it enough to suppose

$$|(z-w)|^{n+1} = \frac{(1-\delta)\,C_b^s(n)n\,[(1+B)(n+1)+2(C-B)\,(1-\lambda)]}{2\,(n+2-\delta)\,(C-B)\,(1-\lambda)}. \tag{23}$$

For convexity by using the fact that "$f(z)$ is convex if and only if $zf'(z)$ is starlike" and by an easy calculation we conclude the required result. □

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