Journal of Fractional Calculus and Applications
Vol. 8(1) Jan. 2017, pp. 108-117.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR A COMPREHENSIVE SUBCLASS OF m-FOLD SYMMETRIC ANALYTIC BI-UNIVALENT FUNCTIONS

SERAP BULUT

Abstract. In this work, considering a general subclass of m-fold symmetric analytic bi-univalent functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in this class. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coeffcient bounds.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f)\,;\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [32], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [6] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [34] proved that $|a_2| \leq 4/3$. Brannan and

---

1991 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic functions; Univalent functions; Bi-univalent functions; m-fold symmetric bi-univalent functions; Faber polynomials.
Submitted July 28, 2016.

Taha [7] and Taha [42] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [38] (see also [7]). In fact, the aforecited work of Srivastava *et al.* [38] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [22], Xu *et al.* [44, 45], Hayami and Owa [28], and others (see, for example, [4, 8, 9, 10, 11, 12, 13, 23, 33, 35, 37]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behavior of the coefficients of the function f and f^{-1} unpredictable.

On the other hand, the Faber polynomials introduced by Faber [21] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [18], [24] and [27] applying the Faber polynomial expansions to meromorphic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$ motivated us to apply this technique to classes of analytic bi-univalent functions, see [2, 3, 14, 15, 16, 25, 29, 31, 40].

Let $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$. A domain E is said to be m-fold symmetric if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric ($m \in \mathbb{N}$) if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

In particular every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{3}$$

Srivastava *et al.* [39] defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. For normalized form of f given by (3), they obtained the series expansion for f^{-1} as following:

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_{m+1} w^{m+1} + \left[(m+1) a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \\ &= w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1}. \end{aligned} \tag{4}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$ given by (3). For $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. For some examples of m-fold symmetric bi-univalent functions, see [39].

The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favorite subjects of geometric function theory in these days, see [17, 26, 39, 41]. Here, in this paper, we use the Faber polynomial expansions for a general subclass of m-fold symmetric analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_{mk+1}|$.

2. The Class $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$

Firstly, we consider a comprehensive class of m-fold symmetric analytic bi-univalent functions defined by Bulut [17].

Definition 1. (*see* [17]) *For $\lambda \geq 1$ and $\mu \geq 0$, a function $f \in \Sigma_m$ given by* (3) *is said to be in the class $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$ if the following conditions are satisfied:*

$$\Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)>\alpha \tag{5}$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)>\alpha \tag{6}$$

where $0 \leq \alpha < 1$; $m \in \mathbb{N}$; $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (4).

Remark 1. In the following special cases of Definition 1, we show how the class of analytic bi-univalent functions $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$ for suitable choices of λ, μ and m lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.

(**i**) For $\mu = 1$, we obtain the m-fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^{1}(\alpha,\lambda)=\mathcal{A}_{\Sigma,m}^{\lambda}(\alpha)$$

introduced by Sümer Eker [41]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{1}(\alpha,\lambda)=\mathcal{B}_{\Sigma}(\alpha,\lambda)$$

introduced by Frasin and Aouf [22].

(**ii**) For $\mu = 1$ and $\lambda = 1$, we have the m-fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^{1}(\alpha,1)=\mathcal{H}_{\Sigma,m}(\alpha)$$

introduced by Srivastava *et al.* [39]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{1}(\alpha,1)=\mathcal{H}_{\Sigma}(\alpha)$$

introduced by Srivastava *et al.* [38].

(**iii**) For $\mu = 0$ and $\lambda = 1$, we get the class

$$\mathcal{N}_{\Sigma,m}^{0}(\alpha,1)$$

of m-fold symmetric bi-starlike functions of order α (see [26]). In addition, for $m = 1$ we have the bi-starlike function class

$$\mathcal{N}_{\Sigma,1}^{0}(\alpha,1)=\mathcal{S}_{\Sigma}^{*}(\alpha)$$

introduced by Brannan and Taha [7].

(**iv**) For $\lambda = 1$, we have a new class

$$\mathcal{N}_{\Sigma,m}^{\mu}(\alpha,1)=\mathcal{P}_{\Sigma,m}(\alpha,\mu)$$

which consists of m-fold symmetric bi-Bazilevič functions.

(**v**) For $m = 1$, we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{\mu}(\alpha,\lambda)=\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$$

introduced by Çağlar *et al.* [19].

3. Coefficient estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [1]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n, \tag{7}$$

where K_{n-1}^{-n} is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, [5]. In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2} = -2a_2, \qquad K_2^{-3} = 3\left(2a_2^2 - a_3\right), \qquad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, for any $n \geq 2$ and for any $p \in \mathbb{R}$, an expansion of K_n^p is as, [1],

$$K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!\,3!} D_n^3 + \cdots + \frac{p!}{(p-n)!\,n!} D_n^n,$$

where $D_n^l = D_n^l(a_2, a_3, \ldots, a_n)$, and by [43],

$$D_n^l(a_2, a_3, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{l!}{i_1! \ldots i_{n-1}!} a_2^{i_1} \ldots a_n^{i_{n-1}},$$

and the sum is taken over all non-negative integers $i_1, \ldots, i_{n-1}$ satisfying

$$\begin{aligned} i_1 + i_2 + \cdots + i_{n-1} &= l \\ i_1 + 2i_2 + \cdots + (n-1) i_{n-1} &= n-1. \end{aligned}$$

It is clear that $D_n^n(a_2, \ldots, a_n) = a_2^n$.

Similarly, using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (3), that is,

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} = z + \sum_{k=1}^{\infty} K_k^{\frac{1}{m}}(a_2, a_3, \ldots, a_{k+1}) z^{mk+1},$$

the coefficients of its inverse map $g = f^{-1}$ may be expressed as:

$$g(w) = f^{-1}(w) = w + \sum_{k=1}^{\infty} \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}) w^{mk+1}. \tag{8}$$

Consequently, for functions $f \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$ of the form (3), we can write:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + \sum_{k=1}^{\infty} F_k(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}) z^{mk}, \tag{9}$$

where

$$\begin{aligned} F_k(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}) &= [\mu + mk\lambda] \times [(\mu-1)!] \\ &\quad \times \left[\sum_{i_1+2i_2+\cdots+ki_k=k} \frac{a_{m+1}^{i_1} a_{2m+1}^{i_2} \cdots a_{mk+1}^{i_k}}{i_1! i_2! \ldots i_k! \left[\mu - (i_1 + i_2 + \cdots + i_k)\right]!}\right] \end{aligned} \tag{10}$$

is a Faber polynomial of degree k. In particular, the first three terms of $F_k\left(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}\right)$ are

$$\begin{aligned}
F_1 &= (\mu + m\lambda)\, a_{m+1} \\
F_2 &= (\mu + 2m\lambda)\left[\frac{\mu-1}{2}a_{m+1}^2 + a_{2m+1}\right] \\
F_3 &= (\mu + 3m\lambda)\left[\frac{(\mu-1)(\mu-2)}{3!}a_{m+1}^3 + (\mu-1)\, a_{m+1}a_{2m+1} + a_{3m+1}\right].
\end{aligned}$$

Our first theorem introduces an upper bound for the coefficients $|a_{mk+1}|$ of m-fold symmetric analytic bi-univalent functions in the class $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$.

Theorem 1. *For $\lambda \geq 1$, $\mu \geq 0$, $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$ be given by (3). If $a_{mj+1} = 0$ $(1 \leq j \leq k-1)$, then*

$$|a_{mk+1}| \leq \frac{2(1-\alpha)}{\mu + mk\lambda} \qquad (k \geq 2).$$

Proof. For the function $f \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$ of the form (3), we have the expansion (9) and for the inverse map $g = f^{-1}$, considering (4) and (8), we obtain

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + \sum_{k=1}^{\infty} F_k\left(A_{m+1}, A_{2m+1}, \ldots, A_{mk+1}\right) w^{mk}, \tag{11}$$

with

$$A_{mk+1} = \frac{1}{mk+1}K_k^{-(mk+1)}\left(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}\right) \qquad (k \geq 1). \tag{12}$$

On the other hand, since $f \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$ and $g = f^{-1} \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha,\lambda)$, by definition, there exist two positive real part functions

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^{mk} \in \mathcal{A} \qquad \text{and} \qquad q(w) = 1 + \sum_{k=1}^{\infty} d_k w^{mk} \in \mathcal{A},$$

where

$$\Re(p(z)) > 0 \quad \text{and} \quad \Re(q(w)) > 0$$

in $\mathbb{U}$ so that

$$\begin{aligned}
& (1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \\
=\; & \alpha + (1-\alpha)\, p(z) \\
=\; & 1 + (1-\alpha)\sum_{k=1}^{\infty} K_k^1\left(c_1, c_2, \ldots, c_k\right) z^{mk}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
& (1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \\
=\; & \alpha + (1-\alpha)\, q(w) \\
=\; & 1 + (1-\alpha)\sum_{k=1}^{\infty} K_k^1\left(d_1, d_2, \ldots, d_k\right) w^{mk},
\end{aligned} \tag{14}$$

respectively. Note that, by the Caratheodory lemma (e.g., [20]),

$$|c_k| \leq 2 \qquad \text{and} \qquad |d_k| \leq 2 \quad (k \in \mathbb{N}).$$

Comparing the corresponding coefficients of (9) and (13), for any $k \geq 1$, yields

$$F_k(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}) = (1-\alpha) K_k^1(c_1, c_2, \ldots, c_k), \tag{15}$$

and similarly, from (11) and (14) we find

$$F_k(A_{m+1}, A_{2m+1}, \ldots, A_{mk+1}) = (1-\alpha) K_k^1(d_1, d_2, \ldots, d_k). \tag{16}$$

Note that for $a_{mj+1} = 0 \ (1 \leq j \leq k-1)$, we have

$$A_{mk+1} = -a_{mk+1}$$

and so

$$\begin{aligned} (\mu + mk\lambda) a_{mk+1} &= (1-\alpha) c_k, \\ -(\mu + mk\lambda) a_{mk+1} &= (1-\alpha) d_k. \end{aligned}$$

Taking the absolute values of the above equalities, we obtain

$$|a_{mk+1}| = \frac{(1-\alpha)|c_k|}{\mu + mk\lambda} = \frac{(1-\alpha)|d_k|}{\mu + mk\lambda} \leq \frac{2(1-\alpha)}{\mu + mk\lambda},$$

which completes the proof of the Theorem 1. □

By setting $\mu = 0$ and $\lambda = 1$ in Theorem 1, we obtain the following consequence.

Corollary 2. [26] *For $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma,m}^{0}(\alpha, 1)$ be given by (3). If $a_{mj+1} = 0 \ (1 \leq j \leq k-1)$, then*

$$|a_{mk+1}| \leq \frac{2(1-\alpha)}{mk} \qquad (k \geq 2).$$

Remark 2. By setting $m = 1$ in Theorem 1, we get [14, Theorem 1].

Theorem 3. *For $\lambda \geq 1$, $\mu \geq 0$, $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$ be given by (3). Then one has the following*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{(\mu+2m\lambda)(\mu+m)}} & , \quad 0 \leq \alpha < \frac{m\left(\mu+2m\lambda-m\lambda^2\right)}{(\mu+2m\lambda)(\mu+m)} \\ \frac{2(1-\alpha)}{\mu+m\lambda} & , \quad \frac{m\left(\mu+2m\lambda-m\lambda^2\right)}{(\mu+2m\lambda)(\mu+m)} \leq \alpha < 1 \end{cases}, \tag{17}$$

$$|a_{2m+1}| \leq \begin{cases} \min\left\{\frac{2(m+1)(1-\alpha)}{(\mu+2m\lambda)(\mu+m)}, \frac{2(m+1)(1-\alpha)^2}{(\mu+m\lambda)^2} + \frac{2(1-\alpha)}{\mu+2m\lambda}\right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\alpha)}{\mu+2m\lambda} & , \quad \mu \geq 1 \end{cases},$$

$$\left| a_{2m+1} - \frac{\mu + 2m + 1}{2} a_{m+1}^2 \right| \leq \frac{2(1-\alpha)}{\mu + 2m\lambda}. \tag{18}$$

Proof. If we set $k = 1$ and $k = 2$ in (15) and (16), respectively, we get

$$(\mu + m\lambda) a_{m+1} = (1-\alpha) c_1, \tag{19}$$

$$(\mu + 2m\lambda)\left[\frac{\mu - 1}{2} a_{m+1}^2 + a_{2m+1}\right] = (1-\alpha) c_2, \tag{20}$$

$$-(\mu + m\lambda) a_{m+1} = (1-\alpha) d_1, \tag{21}$$

$$(\mu+2m\lambda)\left[\frac{\mu+2m+1}{2}a_{m+1}^{2}-a_{2m+1}\right]=(1-\alpha)\,d_{2}. \tag{22}$$

From (19) and (21), we find (by the Caratheodory lemma)

$$|a_{m+1}|=\frac{(1-\alpha)\,|c_{1}|}{\mu+m\lambda}=\frac{(1-\alpha)\,|d_{1}|}{\mu+m\lambda}\leq\frac{2\,(1-\alpha)}{\mu+m\lambda}. \tag{23}$$

Also from (20) and (22), we obtain

$$(\mu+2m\lambda)\,(\mu+m)\,a_{m+1}^{2}=(1-\alpha)\,(c_{2}+d_{2})\,. \tag{24}$$

Using the Caratheodory lemma, we get

$$|a_{m+1}|\leq\sqrt{\frac{4\,(1-\alpha)}{(\mu+2m\lambda)\,(\mu+m)}},$$

and combining this with the inequality (23), we obtain the desired estimate on the coefficient $|a_{m+1}|$ as asserted in (17).

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (22) from (20). We thus get

$$(\mu+2m\lambda)\left[-(m+1)\,a_{m+1}^{2}+2a_{2m+1}\right]=(1-\alpha)\,(c_{2}-d_{2})$$

or

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^{2}+\frac{(1-\alpha)\,(c_{2}-d_{2})}{2\,(\mu+2m\lambda)}. \tag{25}$$

Upon substituting the value of a_{m+1}^{2} from (19) into (25), it follows that

$$a_{2m+1}=\frac{m+1}{2}\frac{(1-\alpha)^{2}\,c_{1}^{2}}{(\mu+m\lambda)^{2}}+\frac{(1-\alpha)\,(c_{2}-d_{2})}{2\,(\mu+2m\lambda)}.$$

We thus find (by the Caratheodory lemma) that

$$|a_{2m+1}|\leq\frac{2\,(m+1)\,(1-\alpha)^{2}}{(\mu+m\lambda)^{2}}+\frac{2\,(1-\alpha)}{\mu+2m\lambda}. \tag{26}$$

On the other hand, upon substituting the value of a_{m+1}^{2} from (24) into (25), it follows that

$$a_{2m+1}=\frac{1-\alpha}{2\,(\mu+2m\lambda)\,(\mu+m)}\left[(\mu+2m+1)\,c_{2}+(1-\mu)\,d_{2}\right].$$

Consequently, (by the Caratheodory lemma) we have

$$|a_{2m+1}|\leq\frac{1-\alpha}{(\mu+2m\lambda)\,(\mu+m)}\left[(\mu+2m+1)+|1-\mu|\right]. \tag{27}$$

Combining (26) and (27), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in (18).

Finally, from (22), we deduce (by the Caratheodory lemma) that

$$\left|a_{2m+1}-\frac{\mu+2m+1}{2}a_{m+1}^{2}\right|=\frac{(1-\alpha)\,|d_{2}|}{\mu+2m\lambda}\leq\frac{2\,(1-\alpha)}{\mu+2m\lambda}.$$

This evidently completes the proof of Theorem 3. □

By setting $\mu=1$ in Theorem 3, we obtain the following consequence.

Corollary 4. *For $\lambda \geq 1$, $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{A}_{\Sigma,m}^{\lambda}(\alpha)$ be given by* (3)*. Then one has the following*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{(1+2m\lambda)(1+m)}} & , \quad 0 \leq \alpha < \frac{m\left(1+2m\lambda-m\lambda^2\right)}{(1+2m\lambda)(1+m)} \\ \frac{2(1-\alpha)}{1+m\lambda} & , \quad \frac{m\left(1+2m\lambda-m\lambda^2\right)}{(1+2m\lambda)(1+m)} \leq \alpha < 1 \end{cases},$$

$$|a_{2m+1}| \leq \frac{2(1-\alpha)}{1+2m\lambda},$$

$$\left|a_{2m+1} - (m+1)\, a_{m+1}^2\right| \leq \frac{2(1-\alpha)}{1+2m\lambda}.$$

Remark 3. Corollary 4 is an improvement of the estimates obtained by Sümer Eker [41, Theorem 2].

By setting $\mu = 1$ and $\lambda = 1$ in Theorem 3, we obtain the following consequence.

Corollary 5. *For $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{H}_{\Sigma,m}(\alpha)$ be given by* (3)*. Then one has the following*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{(1+2m)(1+m)}} & , \quad 0 \leq \alpha < \frac{m}{1+2m} \\ \frac{2(1-\alpha)}{1+m} & , \quad \frac{m}{1+2m} \leq \alpha < 1 \end{cases},$$

$$|a_{2m+1}| \leq \frac{2(1-\alpha)}{1+2m},$$

$$\left|a_{2m+1} - (m+1)\, a_{m+1}^2\right| \leq \frac{2(1-\alpha)}{1+2m}.$$

Remark 4. Corollary 5 is an improvement of the estimates obtained by Srivastava *et al.* [39, Theorem 3].

By setting $\mu = 0$ and $\lambda = 1$ in Theorem 3, we obtain the following consequence.

Corollary 6. (*see also* [26]) *For $m \in \mathbb{N}$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma,m}^{0}(\alpha, 1)$ be given by* (3)*. Then one has the following*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{m^2}} & , \quad 0 \leq \alpha < \frac{1}{2} \\ \frac{2(1-\alpha)}{m} & , \quad \frac{1}{2} \leq \alpha < 1 \end{cases},$$

$$|a_{2m+1}| \leq \begin{cases} \frac{(m+1)(1-\alpha)}{m^2} & , \quad 0 \leq \alpha < \frac{2m+1}{2(m+1)} \\ \frac{2(m+1)(1-\alpha)^2}{m^2} + \frac{1-\alpha}{m} & , \quad \frac{2m+1}{2(m+1)} \leq \alpha < 1 \end{cases},$$

$$\left|a_{2m+1} - \frac{2m+1}{2} a_{m+1}^2\right| \leq \frac{1-\alpha}{m}.$$

References

[1] H. Airault and A. Bouali, Differential calculus on the Faber polynomials, Bull. Sci. Math. **130** (3) (2006), 179–222.

[2] G. Akın and S. Sümer Eker, Coefficient estimates for a certain class of analytic and bi-univalent functions defined by fractional derivative, C. R. Math. Acad. Sci. Paris **352** (12) (2014), 1005–1010.

[3] Ş. Altınkaya and S. Yalçın, Faber polynomial coefficient bounds for a subclass of bi-univalent functions, C. R. Math. Acad. Sci. Paris, Ser. I **353** (12) (2015), 1075–1080.

[4] M.K. Aouf, R.M. El-Ashwah and A.M. Abd-Eltawab, New subclasses of biunivalent functions involving Dziok-Srivastava operator, ISRN Math. Anal. **2013**, Art. ID 387178, 5 pp.

[5] H. Airault and J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bull. Sci. Math. **126** (5) (2002), 343–367.

[6] D.A. Brannan, J.G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.

[7] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, in Mathematical Analysis and Its Applications (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors) (Kuwait; February 18–21, 1985), KFAS Proceedings Series, Vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also Studia Univ. Babeş-Bolyai Math. 31 (2) (1986), 70–77.

[8] S. Bulut, Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator, The Scientific World Journal **2013**, Art. ID 171039, 6 pp.

[9] S. Bulut, Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator, J. Funct. Spaces Appl. **2013**, Art. ID 181932, 7 pp.

[10] S. Bulut, Coefficient estimates for a new subclass of analytic and bi-univalent functions, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), *in press.*

[11] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math. **43** (2) (2013), 59–65.

[12] S. Bulut, Coefficient estimates for a new subclass of analytic and bi-univalent functions defined by Hadamard product, J. Complex Anal. **2014**, Art. ID 302019, 7 pp.

[13] S. Bulut, Coefficient estimates for a new subclass of analytic and bi-univalent functions defined by convolution, Miskolc Math. Notes **17** (1) (2016), 101–110.

[14] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I **352** (6) (2014), 479–484.

[15] S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions, Filomat **30** (6) (2016), 1567–1575.

[16] S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions defined by Sălăgean differential operator, Mat. Vesnik **67** (3) (2015), 185–193.

[17] S. Bulut, Coefficient estimates for general subclasses of m-fold symmetric analytic bi-univalent functions, Turkish J. Math., *in press.*

[18] S. Bulut, N. Magesh and V.K. Balaji, Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I **353** (2) (2015), 113–116.

[19] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat **27** (7) (2013), 1165–1171.

[20] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.

[21] G. Faber, Über polynomische Entwickelungen, Math. Ann. **57** (3) (1903) 389–408.

[22] B.A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011) 1569–1573.

[23] S.P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, J. Egyptian Math. Soc. **20** (2012), 179–182.

[24] S.G. Hamidi, S.A. Halim and J.M. Jahangiri, Coefficient estimates for a class of meromorphic bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I **351** (9-10) (2013), 349–352.

[25] S.G. Hamidi and J.M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, C. R. Acad. Sci. Paris, Ser. I, **352** (1) (2014), 17–20.
[26] S.G. Hamidi and J.M. Jahangiri, Unpredictability of the coefficients of m-fold symmetric bi-starlike functions, Internat. J. Math. **25** (7) (2014), 1450064, 1–8.
[27] S.G. Hamidi, T. Janani, G. Murugusundaramoorthy and J.M. Jahangiri, Coefficient estimates for certain classes of meromorphic bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I **352** (4) (2014), 277–282.
[28] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, Pan Amer. Math. J. **22** (4) (2012), 15–26.
[29] J.M. Jahangiri and S.G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, Int. J. Math. Math. Sci. **2013**, Art. ID 190560, 4 pp.
[30] J.M. Jahangiri and S.G. Hamidi, Faber polynomial coefficient estimates for analytic bi-Bazilevič functions, Mat. Vesnik **67** (2) (2015), 123–129.
[31] J.M. Jahangiri, S.G. Hamidi and S.A. Halim, Coefficients of bi-univalent functions with positive real part derivatives, Bull. Malays. Math. Sci. Soc. (2) **37** (3) (2014), 633–640.
[32] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18** (1967), 63–68.
[33] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, Abstr. Appl. Anal. **2013**, Art. ID 573017, 3 pp.
[34] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.
[35] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, J. Egyptian Math. Soc. **21** (3) (2013), 190–193.
[36] S. Prema and B.S. Keerthi, Coefficient bounds for certain subclasses of analytic functions, J. Math. Anal. **4** (1) (2013), 22–27.
[37] H.M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat **27** (5) (2013), 831–842.
[38] H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010) 1188–1192.
[39] H.M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Math. J. **7** (2) (2014), 1–10.
[40] H.M. Srivastava, S. Sümer Eker and R.M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat **29** (8) (2015), 1839–1845.
[41] S. Sümer Eker, Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions, Turkish J. Math. **40** (3) (2016), 641–646.
[42] T.S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, 1981.
[43] P.G. Todorov, On the Faber polynomials of the univalent functions of class Σ, J. Math. Anal. Appl. **162** (1) (1991), 268–276.
[44] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. **25** (2012) 990–994.
[45] Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput. **218** (2012) 11461–11465.

Serap BULUT, Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, TURKEY
E-mail address: serap.bulut@kocaeli.edu.tr