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AN EXISTENCE THEOREM FOR A FRACTIONAL DIFFERENTIAL EQUATION USING PROGRESSIVE CONTRACTIONS

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ABSTRACT. In this brief note we present a simple proof of global existence and uniqueness of a solution of a fractional differential equation, $D^q x = f(t, x)$. We require that f be continuous and for each $L > 0$ there is a $K = K(L) > 0$ such that $0 < t \leq L$ and $x, y \in \Re$ imply $|f(t, x) - f(t, y)| \leq K(L)|x - y|$ where $K(L)$ may tend to infinity with L. We then parlay this into a solution on $(0, \infty)$. The proof employs the idea of progressive contractions which avoids the classical methods involving infinitely repeated translations.

1. Introduction and setting

We present a simple and elementary proof of the existence of a global solution of a fractional differential equation of Riemann-Liouville type using a method which we call *progressive contractions*. For the equation $D^q x(t) = f(t, x(t))$ with appropriate initial data we require continuity of f and a Lipschitz condition with "constant" K which can become unbounded as the length of the interval of definition of the solution becomes unbounded. We have used the method of progressive contractions in earlier work not on fractional equations and which has not yet appeared. One project showed uniqueness of solutions of integral equations and the other showed global existence of solutions of a general integral equation defining the sum of two operators. The point is that it is a very flexible idea.

In a sequence of earlier papers ([2], [3], [4]) we have discussed the literature on existence and uniqueness in some depth, commenting on the results found in [6, p. 77-80], [9, pp. 30-34], and [8, p. 165] which will not be repeated here.

Our work begins with the setting and the quotation of two basic results which then launch the study of an extension from the local result in Theorem 1.2 to a global result.

We consider a scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t \downarrow 0} t^{1-q} x(t) = x^0 \tag{1.1}$$

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where $q \in (0,1)$ and $x^0 \neq 0$. It is assumed that

$$f : [0,\infty) \times \Re \to \Re \text{ is continuous} \tag{1.2}$$

and for each $E > 0$ there is a $K = K(E) > 0$ so that

$$0 \leq t \leq E, x, y \in \Re \implies |f(t,x) - f(t,y)| \leq K|x - y|. \tag{1.3}$$

The following theorem can be found in [1, p. 2].

Theorem 1.1 Let $q \in (0,1)$ and $x^0 \neq 0$. Let $f(t,x)$ be a function that is continuous on the set

$$\mathcal{B} : \{(t,x) \in \Re^2 : 0 < t \leq L, x \in I\}$$

where $I \subset \Re$ denotes an unbounded interval. Suppose a function $x : (0,L] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,L]$. Then $x(t)$ satisfies (1.1) on the interval $(0,L]$ if and only if it satisfies the Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds. \tag{1.4}$$

on this same interval.

We then have the following existence theorem found in [2, p. 251].

Theorem 1.2 Let $f : [0,L] \times \Re \to \Re$ be continuous and satisfy the Lipschitz condition (1.3). Then, for each $q \in (0,1)$, there is a $T_0 \in (0,L]$ such that (1.4) has a unique continuous solution ξ on $(0,T_0]$ with

$$\lim_{t\downarrow 0} \int_0^t (t-s)^{q-1} f(s,\xi(s))ds = 0, \quad \lim_{t\downarrow 0} t^{1-q}\xi(t) = x^0. \tag{1.5}$$

Finally, both $\xi(t)$ and $f(t,\xi(t))$ are absolutely integrable.

This theorem asks only continuity and a local Lipschitz condition. This is exactly what we expect for an elementary ordinary differential equation, in marked contrast to what is often required in the literature for (1.4). The purpose of this note is to let $L \to \infty$ and $K = K(L) \to \infty$, obtaining exactly the same result on $[0,\infty)$.

2. Extending the solution to $(0,\infty)$

We begin with the assumptions of Theorem 1.2, asking in addition that f be continuous on $[0,\infty)$ and accepting a unique solution on a fixed interval $(0,T_0]$. Thus, our L here can be arbitrarily large. We use the notation of Theorem 1.2. Let $q \in (0,1)$ be fixed, let $E > T_0$ (of Th. 1.2) be given, let $\alpha \in (0,1)$, let K satisfy (1.3), and find $T > 0$ so that for T_0 of Th. 1.2

$$\frac{K}{\Gamma(q)} \int_{T_0}^T (T-s)^{q-1} ds = \frac{K}{\Gamma(q)} \frac{(T-T_0)^q}{q} < \alpha. \tag{2.1}$$

Divide the interval $[T_0, E]$ into equal parts with end points $T_0, T_1, ..., T_n = E$ with each part having length $S < T - T_0$.

We will need a form related to (2.1). With the change of variable $u = T_i - s$ $(i = 1, \dots, n)$, we have

$$\int_{T_{i-1}}^{T_i} (T_i - s)^{q-1} ds = \int_0^{T_i - T_{i-1}} u^{q-1} du = \frac{S^q}{q} < \frac{(T - T_0)^q}{q} < \frac{\Gamma(q)\alpha}{K} \tag{2.2}$$

since $T_i - T_{i-1} = S$.

Definition 2.1 Let $g : (0, \infty) \to \Re$ be defined by

$$0 < t \le T_0 \implies g(t) = \frac{t^{q-1}}{T_0^{q-1}}$$

and

$$T_0 \le t < \infty \implies g(t) = 1.$$

For any continuous function $\phi : (0, H] \to \Re$ then

$$|\phi|_g := \sup_{0 < t \le H} \frac{|\phi(t)|}{g(t)}$$

provided it exists.

The complete metric space used as in the proof below was introduced in El'sgol'ts [7, p. 16] and repeated in Burton [5, p. 177].

Theorem 1.2 If (1.2) and (1.3) hold for each $E > 0$ then (1.4) has a unique solution on $(0, \infty)$.

Proof

Step 1. Let $(\mathcal{M}_1, |\cdot|_g)$ be the complete metric space of continuous functions $\phi : (0, T_1] \to \Re$ with $\phi(t) = \xi$ of Theorem 1.2 on $(0, T_0]$.

Define $P : \mathcal{M}_1 \to \mathcal{M}_1$ by $\phi \in \mathcal{M}_1$ implies that

$$(P\phi)(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, \phi(s)) ds.$$

As ξ satisfies (1.4) on $(0, T_0]$ for $0 < t \le T_0$ we have

$$(P\phi)(t) = \xi(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, \xi(s)) ds$$

so P does map $\mathcal{M}_1 \to \mathcal{M}_1$.

Now if $\phi_1, \phi_2 \in \mathcal{M}_1$ then

$$|(P\phi_1)(t) - (P\phi_2)(t)| = \left| \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, \phi_1(s)) - f(s, \phi_2(s))] ds \right|$$

(since $\phi_1 = \phi_2 = \xi$ on $(0, T_0]$ and using (1.3) also now let $t > T_0$)

$$\le \frac{K}{\Gamma(q)} \int_{T_0}^t (t - s)^{q-1} |\phi_1(s) - \phi_2(s)| ds$$

$$\le \frac{K}{\Gamma(q)} |\phi_1 - \phi_2|^{[T_0, T_1]} \int_{T_0}^{T_1} (T_1 - s)^{q-1} ds$$

$$\le \alpha |\phi_1 - \phi_2|_g$$

where $|\phi_1 - \phi_2|^{[T_0,T_1]}$ denotes the supremum on that interval. In fact, P is a contraction on $\mathcal{M}_1$ with unique fixed point ξ_1 where ξ_1 agrees with ξ on $(0, T_0]$ because ξ and ξ_1 are both unique on $(0, T_0]$.

Step 2. Let $(\mathcal{M}_2, |\cdot|_g)$ be the complete metric space of continuous functions $\phi : (0, T_2] \to \Re$ with $\phi(t) = \xi_1$ of Step 1 on $(0, T_1]$.

Define $P : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ implies that

$$(P\phi)(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))ds.$$

As ξ_1 satisfies (1.4) on $(0, T_1]$ and in $\mathcal{M}_2$ each $\phi(t) = \xi_1$ on $(0, T_1]$ we have

$$(P\phi)(t) = \xi_1(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \xi_1(s))ds$$

for $0 \le t \le T_1$ so P does map $\mathcal{M}_2 \to \mathcal{M}_2$.

Now if $\phi_1, \phi_2 \in \mathcal{M}_2$ then

$$\begin{aligned}
|(P\phi_1)(t) - (P\phi_2)(t)| &= \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, \phi_1(s)) - f(s, \phi_2(s))]ds \right| \\
&\text{(and since } \phi_1 = \phi_2 = \xi_1 \text{ on } (0, T_1] \text{ and by (1.3))} \\
&\le \frac{K}{\Gamma(q)} \int_{T_1}^t (t-s)^{q-1} |\phi_1(s) - \phi_2(s)|ds \\
&\le \frac{K}{\Gamma(q)} |\phi_1 - \phi_2|^{[T_1,T_2]} \int_{T_1}^{T_2} (T_2 - s)^{q-1} ds \\
&\text{(using (2.1) and (2.2), } T_2 - T_1 = S) \\
&= \frac{K}{\Gamma(q)} |\phi_1 - \phi_2|^{[T_1,T_2]} \int_{T_0}^{T_1} (T_1 - s)^{q-1} ds \\
&\le \alpha |\phi_1 - \phi_2|_g
\end{aligned}$$

where $|\phi_1 - \phi_2|^{[T_1,T_2]}$ denotes the supremum on that interval. In fact, P is a contraction on $\mathcal{M}_2$ with unique fixed point ξ_2 where ξ_2 agrees with ξ_1 on $(0, T_1]$.

We continue stepping S units at a time until we reach E noting on the $i-$th step that after taking into account $\phi_1 = \phi_2$ on $(0, T_{i-1}]$, changing variable, and using (2.2) we obtain

$$\int_{T_{i-1}}^{T_i} (T_i - s)^{q-1} ds = \int_{T_0}^{T_1} (T_1 - u)^{q-1} du \le \frac{\Gamma(q)\alpha}{K}.$$

So we get a contraction on $\mathcal{M}_i$ at each step and a solution ξ_i. Upon completion of the n-steps we have a solution ξ_n on $(0, E]$.

Now, to get a solution on $(0, \infty)$ we get a sequence of solutions $\{\xi_i\}$ on intervals $(0, i]$ and construct the set of functions $\{\xi_i*\}$ which are solutions from the previous sequence, but continued for $t > i$ at the constant value $\xi_i(i)$. This sequence converges uniformly on compact sets to a continuous solution $x(t)$ on $(0, \infty)$ because at each value of t and for $i > t$ it is the case that $x(t)$ coincides with $\xi_i(t)$.

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