Journal of Fractional Calculus and Applications
Vol. 8(1) Jan. 2017, pp. 173-185.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

SYMMETRY IDENTITIES FOR GENERALIZED HERMITE-BASED APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

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Abstract. Recently, Pathan and Khan introduced a new class of generalized Hermite-Apostol-Euler polynomials and Hermite-Apostol-Genocchi polynomials. Motivated by the works of Pathan and Khan, in the present paper, we define a symmetric identities for generalized Hermite-Apostol-Euler polynomials and Hermite-Apostol-Genocchi polynomials. These results extend some known summation and identities for generalized Apostol type polynomials.

1. Introduction

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [3] and [4] defined as

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y = -1$ and x is replaced by $2x$.

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see for details[1], [2], [5], [15-20] and the references cited therein):

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \ (\mid t \mid < 2\pi; 1^{\alpha} = 1) \tag{1.3}$$

2010 *Mathematics Subject Classification.* 05A10, 11B65, 11B68, 33C45, 33C99.
Key words and phrases. Hermite polynomials, Hermite-Apostol-Bernoulli polynomials, Hermite- Apostol-Euler polynomials, Hermite-Apostol-Genocchi polynomials, Symmetric identities.
Submitted June 6, 2016.

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!},\ (\mid t\mid<\pi; 1^{\alpha}=1) \tag{1.4}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!},\ (\mid t\mid<\pi; 1^{\alpha}=1) \tag{1.5}$$

So that

$$B_n(x)=B_n^{(1)}(x); E_n(x)=E_n^{(1)}(x); G_n(x)=G_n^{(1)}(x),\ n\in\mathbb{N}_0=N\cup\{0\}. \tag{1.6}$$

In particular, Luo and Srivastava [6, 7] and Luo [8-14] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{C}$, the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{C}$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{C}$ defined as follows:

Definition 1.1. The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, (\mid t\mid<2\pi, if\lambda=1; \mid t\mid<\mid\log\lambda\mid, if\lambda\neq 1; 1^{\alpha}=1) \tag{1.7}$$

with

$$B_n^{(\alpha)}(x)=B_n^{(\alpha)}(x;1)$$

and

$$B_n^{(\alpha)}(\lambda)=B_n^{(\alpha)}(0;\lambda) \tag{1.8}$$

where $B_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Bernoulli numbers of order α.

Definition 1.2. The generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, (\mid t\mid<\mid log(-\lambda)\mid<\pi, 1^{\alpha}=1) \tag{1.9}$$

with

$$E_n^{(\alpha)}(x)=E_n^{(\alpha)}(x;1)$$

and

$$E_n^{(\alpha)}(\lambda)=E_n^{(\alpha)}(0;\lambda) \tag{1.10}$$

where $E_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Euler numbers of order α.

Definition 1.3. The generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α are defined by means of the generating function

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, (\mid t\mid<\mid log(-\lambda)\mid<\pi, 1^{\alpha}=1) \tag{1.11}$$

with

$$G_n^{(\alpha)}(x)=G_n^{(\alpha)}(x;1), G_n^{(\alpha)}(\lambda)=G_n^{(\alpha)}(0;\lambda) \tag{1.12}$$

where $G_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Genocchi numbers of order α.

Very recently, Pathan and Khan [20] studied a new family of generalized Hermite-Apostol-Bernoulli, Hermite-Apostol-Euler and Hermite-Apostol-Genocchi polynomials of order α in the following form:

Definition 1.4. For arbitrary real or complex parameter α and for $a, c \in \Re^+$, the generalized Hermite-Apostol-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(x; a, c, \lambda)$, $m \in \mathbb{N}, \lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t\log(a) <\mid \log(-\lambda) \mid$, by means of the following generating function:

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\frac{t^n}{n!} \tag{1.13}$$

where

$$A(\lambda, a; t) = \left(\lambda a^t - \sum_{h=0}^{m-1} \frac{(t\log a)^h}{h!}\right)^{-1} \tag{1.14}$$

It is easy to see that if we set $y = 0$ in (1.13), we arrive at a recent result of Tremblay et al. [21;p.3, Eq.(1.8)] involving the generalized Apostol-Bernoulli polynomials

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\frac{t^n}{n!} \tag{1.15}$$

For $c = e$ in (1.13) gives

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x, y; a, e, \lambda)\frac{t^n}{n!} \tag{1.16}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (1.13), we arrive at the following result

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{[0,\alpha]}(x; , e, e, \lambda)\frac{t^n}{n!}, (\mid t \mid < 2\pi, 1^{\alpha} = 1) \tag{1.17}$$

This is the generating function for the generalized Apostol-Bernoulli polynomials of order α. Thus we have

$$B_n^{[0,\alpha]}(x; , e, e, \lambda) = B_n^{[\alpha]}(x; \lambda) \tag{1.18}$$

Definition 1.5. For arbitrary real or complex parameter α and $a, c \in \Re^+$, the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in \mathbb{N}, \lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t\log a \mid<\mid \log(-\lambda) \mid$ by means of generating function

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\frac{t^n}{n!} \tag{1.19}$$

where

$$B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t\log a)^h}{h!}\right)^{-1} \tag{1.20}$$

It is easy to see that if we set $y = 0$ in (1.19), we arrive at a recent result of Tremblay et al. [21;p.3, Eq.(2.1)] involving the generalized Apostol-Euler polynomials

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x; a, c, \lambda)\frac{t^n}{n!} \tag{1.21}$$

For $c = e$ in (1.19) gives

$$2^{m\alpha}[B(\lambda,a;t)]^{\alpha}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(x,y;a,e,\lambda)\frac{t^n}{n!} \tag{1.22}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (1.19), we arrive at the following result

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{[0,\alpha]}(x;,e,e,\lambda)\frac{t^n}{n!}, (\mid t \mid< \pi, 1^{\alpha} = 1) \tag{1.23}$$

This is the generating function for the generalized Apostol-Euler polynomials of order α. Thus we have

$$E_n^{[0,\alpha]}(x;,e,e,\lambda) = E_n^{[\alpha]}(x;\lambda) \tag{1.24}$$

Definition 1.6. For arbitrary real or complex parameter α and $a, c \in \Re^+$, the generalized Hermite-Apostol-Genocchi polynomials ${}_HG_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$, $m \in \mathbb{N}, \lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t\log a \mid<\mid \log(-\lambda) \mid$ by means of generating function

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{n!} \tag{1.25}$$

where $B(\lambda, a;t)$ is given by equation (1.20). It is easy to see that if we set $y = 0$ in (1.25), we arrive at a recent result of Tremblay et al. [21;p.5, Eq.(2.4)] involving the generalized Apostol-Genocchi polynomials

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x;a,c,\lambda)\frac{t^n}{n!} \tag{1.26}$$

For $c = e$ in (1.25) gives

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(x,y;a,e,\lambda)\frac{t^n}{n!} \tag{1.27}$$

Obviously if we set $y = 0$, $m = 1$, $a = c = e$ in (1.25), we arrive at the following result

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{[0,\alpha]}(x;,e,e,\lambda)\frac{t^n}{n!}, (\mid t \mid< \pi, 1^{\alpha} = 1) \tag{1.28}$$

This is the generating function for the generalized Apostol-Genocchi polynomials of order α. Thus we have

$$G_n^{[0,\alpha]}(x;,e,e,\lambda) = G_n^{[\alpha]}(x;\lambda) \tag{1.29}$$

In this paper, we established a symmetry identities for generalized Hermite-Apostol-Euler and Hermite-Apostol-Genocchi polynomials. These results extend known summation and identities of Apostol-Hermite-Euler and Apostol-Hermite-Genocchi polynomials studied by Khan and Pathan and Khan.

2. Symmetry identities for generalized Hermite-Apostol-Euler polynomials

In this section, we give general symmetry identities for the generalized Hermite-Apostol-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ by applying the generating functions (1.19) and (1.21). Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 2.1. Let for all integers $a>0, b>0, c>0$, $a\neq b$, $x,y\in\mathbb{R}$ and $n\geq 0$, the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\,{}_HE_{n-k}^{[\alpha,m-1]}(bx,b^2y;c;\lambda)\,{}_HE_{k}^{[\alpha,m-1]}(ax,a^2y;c,\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}\,{}_HE_{n-k}^{[\alpha,m-1]}(ax,a^2y;c,\lambda)\,{}_HE_{k}^{[\alpha,m-1]}(bx,b^2y;c,\lambda). \tag{2.1}$$

Proof. Start with

$$G(t):=\left(\frac{2^{2m}}{(\lambda c^{at}+\sum\limits_{h=0}^{m-1}\frac{(t\log a)^h}{h!})(\lambda c^{bt}+\sum\limits_{h=0}^{m-1}\frac{(t\log b)^h}{h!})}\right)^{\alpha}c^{abxt+a^2b^2yt^2} \tag{2.2}$$

Then the expression for $G(t)$ is symmetric in a and b and we can expand $G(t)$ into series in two ways to obtain

$$G(t)=\frac{1}{(ab)^{\alpha m}}\sum_{n=0}^{\infty}{}_HE_n^{[\alpha,m-1]}(bx,b^2y;c,\lambda)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}{}_HE_k^{[\alpha,m-1]}(ax,a^2y;c,\lambda)\frac{(bt)^k}{k!}$$
$$=\frac{1}{(ab)^{\alpha m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}{}_HE_{n-k}^{[\alpha,m-1]}(bx,b^2y;c,\lambda)\frac{(a)^{n-k}}{(n-k)!}\,{}_HE_k^{[\alpha,m-1]}(ax,a^2y;c,\lambda)\frac{b^k}{k!}t^n \tag{2.3}$$

On the similar lines we can show that

$$G(t):=\frac{1}{(ab)^{\alpha m}}\sum_{n=0}^{\infty}{}_HE_n^{[\alpha,m-1]}(ax,a^2y;c,\lambda)\frac{b^{n-k}}{(n-k)!}\,{}_HE_k^{[\alpha,m-1]}(bx,b^2y;c,\lambda)\frac{a^k}{k!}t^n \tag{2.4}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive the desired result.

Remark 2.1. For $\lambda=1$, $c=e$, Theorem (2.1) reduces to a known result of Pathan and Khan [18.p.104.,Theorem 4.1]. Further by taking $m=1$ in Theorem 2.1, we immediately deduce the following result

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\,{}_HE_{n-k}^{(\alpha)}(bx,b^2y;c,\lambda)\,{}_HE_{k}^{(\alpha)}(ax,a^2y;c,\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}\,{}_HE_{n-k}^{(\alpha)}(ax,a^2y;c,\lambda)\,{}_HE_{k}^{(\alpha)}(bx,b^2y,c,\lambda). \tag{2.5}$$

Remark 2.2. By setting $b = 1$ in Theorem (2.1), we get the following result

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}{}_HE_{n-k}^{[\alpha,m-1]}(x,y;c,\lambda){}_HE_{k}^{[\alpha,m-1]}(ax,a^2y;c,\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}{}_HE_{n-k}^{[\alpha,m-1]}(ax,a^2y;c,\lambda){}_HE_{k}^{[\alpha,m-1]}(x,y;c,\lambda). \quad (2.6)$$

Theorem 2.2. Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}{}_HE_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j,b^2z;c;\lambda\right)E_{k}^{(\alpha)}(ay;c,\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}{}_HE_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2z;c,\lambda\right)E_{k}^{(\alpha)}(by;c,\lambda). \quad (2.7)$$

Proof. Let

$$G(t):=\frac{(2a)^{\alpha}(2b)^{\alpha}(\lambda c^{abt}+1)^2c^{ab(x+y)t+a^2b^2zt^2}}{(\lambda c^{at}+1)^{\alpha+1}(\lambda c^{bt}+1)^{\alpha+1}}$$

$$G(t):=\left(\frac{2a}{\lambda c^{at}+1}\right)^{\alpha}c^{abxt+a^2b^2zt^2}\left(\frac{\lambda c^{abt}+1}{\lambda c^{bt}+1}\right)\left(\frac{2b}{\lambda c^{bt}+1}\right)^{\alpha}c^{abyt}\left(\frac{\lambda c^{abt}+1}{\lambda c^{at}+1}\right)$$
$$=\left(\frac{2a}{\lambda c^{at}+1}\right)^{\alpha}c^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}(-\lambda)^{i}c^{bti}\left(\frac{2b}{\lambda c^{bt}+1}\right)^{\alpha}c^{abyt}\sum_{j=0}^{b-1}(-\lambda)^{j}c^{atj} \quad (2.8)$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}{}_HE_{n}^{(\alpha)}\left(bx+\frac{b}{a}i+j,b^2z;c,\lambda\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}E_{k}^{(\alpha)}(ay;c,\lambda)\frac{(bt)^k}{(k)!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}{}_HE_{n-k}^{(\alpha)}\left(ax+\frac{b}{a}i+j,b^2z;c,\lambda\right)E_{k}^{(\alpha)}(ay;c,\lambda)\right)\frac{t^n}{n!} \quad (2.9)$$

On the other hand

$$G(t):=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}{}_HE_{n-k}^{(\alpha)}\left(bx+\frac{a}{b}i+j,a^2z,c,\lambda\right)E_{k}^{(\alpha)}(by;c,\lambda)\right)\frac{t^n}{n!} \quad (2.10)$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

Remark 2.3. For $\lambda = 1$, $c = e$, Theorem (2.2) reduces to known result of Pathan and Khan [18,p.105.,Theorem 4.2].

Theorem 2.3. For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} {}_{H}E_{n-k}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right) E_{k}^{(\alpha)}(ay + \frac{a}{b} j; c, \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} {}_{H}E_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right) E_{k}^{(\alpha)}(by + \frac{b}{a} j; c, \lambda). \tag{2.11}$$

Proof. The proof is analogous to Theorem (2.2) but we need to write equation (2.8) in the form

$$G(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} {}_{H}E_{n}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} E_{k}^{(\alpha)}(ay + \frac{a}{b} j; c, \lambda) \frac{(bt)^k}{k!} \tag{2.12}$$

On the other hand equation (2.8) can be shown equal to

$$G(t) := \sum_{n=0}^{\infty} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} {}_{H}E_{n}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right) \frac{(bt)^n}{n!} \sum_{k=0}^{\infty} E_{k}^{(\alpha)}(by + \frac{b}{a} j; c, \lambda) \frac{(at)^k}{k!} \tag{2.13}$$

Next making change of index and by equating the coefficients of $\frac{t^n}{n!}$ to zero in (2.12) and (2.13), we get the result

Remark 2.4. For $\lambda = 1$, $c = e$, Theorem (2.3) reduces to known result of Pathan and Khan [18.,p.106.,Theorem 4.3].

Remark 2.5. On setting $y = 0$ in Theorem (2.3), we immediately get the following result

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} {}_{H}E_{n-k}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right) E_{k}^{(\alpha)}(\frac{a}{b} j; c, \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} {}_{H}E_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right) E_{k}^{(\alpha)}(\frac{b}{a} j; c, \lambda). \tag{2.14}$$

Theorem 2.4. Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} E_{n-k}^{(\alpha)}(ay; c, \lambda) \sum_{i=0}^{a-1} (-\lambda)^{i} {}_{H}E_{k}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right)$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} E_{n-k}^{(\alpha)}(by; c, \lambda) \sum_{i=0}^{b-1} (-\lambda)^{i} {}_{H}E_{k}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right). \tag{2.15}$$

Proof. Let

$$G(t) := \frac{(2a)^{\alpha} (2b)^{\alpha} (1 + \lambda(-1)^{a+1} c^{abt}) c^{ab(x+y)t + a^2 b^2 z t^2}}{(\lambda c^{at} + 1)^{\alpha} (\lambda c^{bt} + 1)^{\alpha+1}} \tag{2.16}$$

$$G(t) := \left(\frac{2a}{\lambda c^{at}+1}\right)^{\alpha} c^{abxt+a^2b^2zt^2}\left(\frac{1-\lambda(-c^{bt})^a}{\lambda c^{bt}+1}\right)\left(\frac{2b}{\lambda c^{bt}+1}\right)^{\alpha} c^{abyt}$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{a-1}(-\lambda)^i {}_HE_k^{(\alpha)}\left(bx+\frac{b}{a}i, b^2z; c, \lambda\right)\frac{(at)^k}{k!}\sum_{n=0}^{\infty}E_n^{(\alpha)}(ay; c, \lambda)\frac{(bt)^n}{(n)!}$$

Replacing n by $n-k$ in above equation, we have

$$G(t) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^k\sum_{i=0}^{a-1}(-\lambda)^i {}_HE_k^{(\alpha)}\left(bx+\frac{b}{a}i, b^2z; c, \lambda\right)E_{n-k}^{(\alpha)}(ay; c, \lambda)\right)\frac{t^n}{n!} \tag{2.17}$$

On the other hand

$$G(t) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^k\sum_{i=0}^{b-1}(-\lambda)^i {}_HE_k^{(\alpha)}\left(ax+\frac{a}{b}i, a^2z; c, \lambda\right)E_{n-k}^{(\alpha)}(by; c, \lambda)\right)\frac{t^n}{n!} \tag{2.18}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

Theorem 2.5. Let $a, b, c > 0, m \geq 1$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^kE_{n-k}^{(\alpha,m)}(ay; c, \lambda)\sum_{i=0}^{a-1}(-\lambda)^i {}_HE_k^{(\alpha,m)}\left(bx+\frac{b}{a}i, b^2z; c, \lambda\right)$$

$$= \sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^kE_{n-k}^{(\alpha,m)}(by; c, \lambda)\sum_{i=0}^{b-1}(-\lambda)^i {}_HE_k^{(\alpha,m)}\left(ax+\frac{a}{b}i, a^2z; c, \lambda\right). \tag{2.19}$$

3. Symmetry identities for generalized Hermite-Apostol-Genocchi polynomials

In this section, we give general symmetry identities for the generalized Hermite-Apostol-Genocchi polynomials ${}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)$ by applying the generating functions (1.25) and (1.26). Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 3.1. For all integers $a > 0, b > 0, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^k {}_HG_{n-k}^{[\alpha,m-1]}(bx, b^2y; c; \lambda) {}_HG_k^{[\alpha,m-1]}(ax, a^2y; c, \lambda)$$

$$= \sum_{k=0}^{n}\binom{n}{k} a^kb^{n-k} {}_HG_{n-k}^{[\alpha,m-1]}(ax, a^2y; c, \lambda) {}_HG_k^{[\alpha,m-1]}(bx, b^2y; c, \lambda). \tag{3.1}$$

Proof. Start with

$$H(t) := \left(\frac{2^{2m}t^{2m}}{(\lambda c^{at}+\sum_{h=0}^{m-1}\frac{(t\log a)^h}{h!})(\lambda c^{bt}+\sum_{h=0}^{m-1}\frac{(t\log b)^h}{h!})}\right)^{\alpha} c^{abxt+a^2b^2yt^2} \tag{3.2}$$

Then the expression for $H(t)$ is symmetric in a and b and we can expand $H(t)$ into series in two ways to obtain

$$H(t) := \frac{1}{(ab)^{\alpha m}} \sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(bx, b^2y; c, \lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} {}_HG_k^{[\alpha,m-1]}(ax, a^2y; c, \lambda) \frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_HG_{n-k}^{[\alpha,m-1]}(bx, b^2y; c, \lambda) \frac{a^{n-k}}{(n-k)!} {}_HG_k^{[\alpha,m-1]}(ax, a^2y; c, \lambda) \frac{b^k}{k!} t^n \tag{3.3}$$

On the similar lines we can show that

$$H(t) := \frac{1}{(ab)^{\alpha m}} \sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(ax, a^2y; c, \lambda) \frac{b^{n-k}}{(n-k)!} {}_HG_k^{[\alpha,m-1]}(bx, b^2y; c, \lambda) \frac{a^k}{k!} t^n \tag{3.4}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

Remark 3.1. For $m = 1$ in Theorem 3.1, we immediately deduce the following result

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_HG_{n-k}^{(\alpha)}(bx, b^2y; c, \lambda) {}_HG_k^{(\alpha)}(ax, a^2y; c, \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_HG_{n-k}^{(\alpha)}(ax, a^2y; c, \lambda) {}_HG_k^{(\alpha)}(bx, b^2y; c, \lambda). \tag{3.5}$$

Remark 3.2. On setting $b = 1$ in Theorem (3.1), we immediately get the following result

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} {}_HG_{n-k}^{[\alpha,m-1]}(x, y; c, \lambda) {}_HG_k^{[\alpha,m-1]}(ax, a^2y; c, \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^k {}_HG_{n-k}^{[\alpha,m-1]}(ax, a^2y; c, \lambda) {}_HG_k^{[\alpha,m-1]}(x, y; c, \lambda). \tag{3.6}$$

Theorem 3.2. Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} {}_HG_{n-k}^{(\alpha)} \left(bx + \frac{b}{a} i + j, b^2 z; c, \lambda \right) G_k^{(\alpha)}(ay; c, \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} {}_HG_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i + j, a^2 z; c, \lambda \right) G_k^{(\alpha)}(by; c, \lambda). \tag{3.7}$$

Proof. Let

$$H(t) := \frac{(2at)^\alpha (2bt)^\alpha (\lambda c^{abt} + 1)^2 c^{ab(x+y)t + a^2b^2zt^2}}{(\lambda c^{at} + 1)^{\alpha+1} (\lambda c^{bt} + 1)^{\alpha+1}}$$

$$H(t) := \left(\frac{2at}{\lambda c^{at} + 1} \right)^\alpha c^{abxt + a^2b^2zt^2} \left(\frac{\lambda c^{abt} + 1}{\lambda c^{bt} + 1} \right) \left(\frac{2bt}{\lambda c^{bt} + 1} \right)^\alpha c^{abyt} \left(\frac{\lambda c^{abt} + 1}{\lambda c^{at} + 1} \right)$$

$$= \left(\frac{2at}{\lambda c^{at}+1}\right)^{\alpha} c^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}(-\lambda)^i c^{bti}\left(\frac{2bt}{\lambda c^{bt}+1}\right)^{\alpha} c^{abyt}\sum_{j=0}^{b-1}(-\lambda)^j c^{atj} \quad (3.8)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}\,{}_HG_{n-k}^{(\alpha)}\left(ax+\frac{b}{a}i+j,b^2z;c,\lambda\right)G_k^{(\alpha)}(ay;c,\lambda)\right)\frac{t^n}{n!} \quad (3.9)$$

On the other hand

$$H(t) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^k\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}\,{}_HG_{n-k}^{(\alpha)}\left(bx+\frac{a}{b}i+j,a^2z,c,\lambda\right)G_k^{(\alpha)}(by;c,\lambda)\right)\frac{t^n}{n!} \quad (3.10)$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

Theorem 3.3. For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}(a)^{n-k}(b)^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}\,{}_HG_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i,b^2z;c,\lambda\right)G_k^{(\alpha)}(ay+\frac{a}{b}j;c,\lambda)$$

$$= \sum_{k=0}^{n}\binom{n}{k}(b)^{n-k}(a)^k\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}\,{}_HG_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i,a^2z;c,\lambda\right)G_k^{(\alpha)}(by+\frac{b}{a}j;c,\lambda). \quad (3.11)$$

Proof. The proof is analogous to Theorem (3.2) but we need to write equation (3.8) in the form

$$H(t) := \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}\,{}_HG_n^{(\alpha)}\left(bx+\frac{b}{a}i,b^2z;c,\lambda\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}G_k^{(\alpha)}(ay+\frac{a}{b}j;c,\lambda)\frac{(bt)^k}{k!} \quad (3.12)$$

On the other hand equation (3.8) can be shown equal to

$$H(t) := \sum_{n=0}^{\infty}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}\,{}_HG_n^{(\alpha)}\left(ax+\frac{a}{b}i,a^2z;c,\lambda\right)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty}G_k^{(\alpha)}(by+\frac{b}{a}j;c,\lambda)\frac{(at)^k}{k!} \quad (3.13)$$

Next making change of index and by equating the coefficients of $\frac{t^n}{n!}$ to zero in (3.12) and (3.13), we get the result.

Remark 3.3. By setting $y = 0$ in Theorem 3.3, we immediately get the following result

$$\sum_{k=0}^{n}\binom{n}{k}(a)^{n-k}(b)^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}\,{}_HG_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i,b^2z;c,\lambda\right)G_k^{(\alpha)}(\frac{a}{b}j;c,\lambda)$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (b)^{n-k} (a)^k \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} {}_H G_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right) G_k^{(\alpha)} (\frac{b}{a} j; c, \lambda). \tag{3.14}$$

Theorem 3.4. Let $a, b, c > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) b^{n-k} a^k G_{n-k}^{(\alpha)}(ay; c, \lambda) \sum_{i=0}^{a-1} (-\lambda)^i {}_H G_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right)$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k G_{n-k}^{(\alpha)}(by; c, \lambda) \sum_{i=0}^{b-1} (-\lambda)^i {}_H G_k^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right). \tag{3.15}$$

Proof. Let

$$H(t) := \frac{(2at)^\alpha (2bt)^\alpha (1 + \lambda(-1)^{a+1} c^{abt}) c^{ab(x+y)t + a^2 b^2 z t^2}}{(\lambda c^{at} + 1)^\alpha (\lambda c^{bt} + 1)^{\alpha+1}} \tag{3.16}$$

$$H(t) := \left(\frac{2at}{\lambda c^{at} + 1} \right)^\alpha c^{abxt + a^2 b^2 z t^2} \left(\frac{1 - \lambda(-c^{bt})^a}{\lambda c^{bt} + 1} \right) \left(\frac{2bt}{\lambda c^{bt} + 1} \right)^\alpha c^{abyt}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-\lambda)^i {}_H G_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right) \frac{a^k}{k!} \sum_{n=0}^{\infty} G_n^{\alpha}(ay; c, \lambda) b^n \frac{t^{n+k}}{(n)!}$$

Replacing n by $n-k$ in above equation, we have

$$H(t) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) b^{n-k} a^k \sum_{i=0}^{a-1} (-\lambda)^i {}_H G_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right) G_{n-k}^{(\alpha)}(ay; c, \lambda) \right) \frac{t^n}{n!} \tag{3.17}$$

On the other hand

$$H(t) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k \sum_{i=0}^{b-1} (-\lambda)^i {}_H G_k^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z, c, \lambda \right) G_{n-k}^{(\alpha)}(by; c, \lambda) \right) \frac{t^n}{n!} \tag{3.18}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

In view of Theorems (3.1) to (3.5), we easily obtain the following general symmetry identity

Theorem 3.5. Let $a, b, c > 0, m \geq 1$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) b^{n-k} a^k G_{n-k}^{(\alpha,m)}(ay; c, \lambda) \sum_{i=0}^{a-1} (-\lambda)^i {}_H G_k^{(\alpha,m)} \left(bx + \frac{b}{a} i, b^2 z; c, \lambda \right)$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k G_{n-k}^{(\alpha,m)}(by; c, \lambda) \sum_{i=0}^{b-1} (-\lambda)^i {}_H G_k^{(\alpha,m)} \left(ax + \frac{a}{b} i, a^2 z; c, \lambda \right). \tag{3.19}$$

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