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A COMPREHENSIVE CLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS BY MEANS OF CHEBYSHEV POLYNOMIALS

SERAP BULUT, NANJUNDAN MAGESH AND CHINNASWAMY ABIRAMI

Abstract. In this present paper, we introduce a subclass $\mathcal{B}_{\Sigma}^{\mu}(\lambda, t)$ of analytic and bi-univalent functions using the Chebyshev polynomials expansions and obtain the initial coefficient bounds and Fekete-Szegö problem. Further we discuss its consequences.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further, by $\mathcal{S}$ we will showdenote the family of all functions in $\mathcal{A}$ which are univalent in Δ.

For two functions f and g, analytic in Δ, we say that the function $f(z)$ is subordinate to $g(z)$ in Δ, and write

$$f(z) \prec g(z) \qquad (z \in \Delta)$$

if there exists a Schwarz function $w(z)$, analytic in Δ, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \Delta)$$

such that

$$f(z) = g(w(z)) \qquad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1). Several recent investigations (see, for example, [1, 3, 5, 6, 9, 10, 12, 13, 14, 15, 17]) provide the detailed study of bi-univalent functions.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in Δ. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha; \quad z \in \Delta; \quad 0 \le \alpha < 1 \right\} \quad (1.3)$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha; \quad z \in \Delta; \quad 0 \le \alpha < 1 \right\}. \quad (1.4)$$

For $0 \le \alpha < 1$, a function $f \in \Sigma$ is in the class $S^*_\Sigma(\alpha)$ of bi-starlike function of order α, or $\mathcal{K}_\Sigma(\alpha)$ of bi-convex function of order α if both f and f^{-1} are respectively starlike or convex functions of order α.

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev. For a brief history of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [7, 8, 11, 2]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad (-1 < t < 1)$$

where n denotes the polynomial degree and $t = \cos\theta$.

Definition 1. For $\lambda \ge 1$, $\mu \ge 0$ and $t \in (1/2, 1]$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$ if the following subordinations hold for all $z, w \in \Delta$:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) := \frac{1}{1 - 2tz + z^2} \quad (1.5)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec H(w,t) := \frac{1}{1 - 2tw + w^2}, \quad (1.6)$$

where the function $g = f^{-1}$ is defined by (1.2).

We note that if $t = \cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in \Delta).$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots \quad (z \in \Delta).$$

From [16], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad (z \in \Delta, \quad t \in (-1,1))$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N})$$

are the Chebyshev polynomials of the second kind and we have

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2-1, \quad U_3(t) = 8t^3-4t, \quad U_4(t) = 16t^4-12t^2+1, \ldots . \tag{1.7}$$

The generating function of the first kind of Chebyshev polnomial $T_n(t)$, $t \in [-1,1]$, is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in \Delta).$$

The first kind of Chebyshev polnomial $T_n(t)$ and second kind of Chebyshev polnomial $U_n(t)$ are connected by:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Remark 1. (i) For $\mu = 1$, we get the class $\mathcal{B}_\Sigma^1(\lambda,t) = \mathcal{B}_\Sigma(\lambda,t)$ consists of functions $f \in \Sigma$ satisfying the condition

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (1.2). This class introduced and studied by Bulut et al. [4].

(ii) For $\lambda = 1$, we have a class $\mathcal{B}_\Sigma^\mu(1,t) = \mathcal{B}_\Sigma^\mu(t)$ consists of bi-Bazilevič functions:

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (1.2).

(iii) For $\lambda = 1$ and $\mu = 1$, we have the class $\mathcal{B}_\Sigma^1(1,t) = \mathcal{B}_\Sigma(t)$ consists of functions f satisfying the condition

$$f'(z) \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$g'(w) \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (1.2).

(iv) For $\lambda = 1$ and $\mu = 0$, we have the class $\mathcal{B}_\Sigma^0(1,t) = \mathcal{S}_\Sigma^*(t)$ consists of functions f satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$\frac{wg'(w)}{g(w)} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (1.2).

In this present paper, we define a subclass $\mathcal{B}_{\Sigma}^{\mu}(\lambda, t)$ of analytic and bi-univalent functions using the Chebyshev polynomials expansions and obtain the initial coefficient bounds and Fekete-Szegö problem. Further we discuss its consequences.

2. Main results

Theorem 1. *For $\lambda \geq 1$, $\mu \geq 0$ and $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\mu}(\lambda, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|(\mu+\lambda)^2 - 2\left[2(\mu+\lambda)^2 - (\mu+2\lambda)(\mu+1)\right]t^2\right|}}, \tag{2.1}$$

$$|a_3| \leq \frac{4t^2}{(\mu+\lambda)^2} + \frac{2t}{\mu+2\lambda}, \tag{2.2}$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2t}{\mu+2\lambda} & , \quad |\eta-1| \leq \frac{\left|(\mu+\lambda)^2 - 2\left[2(\mu+\lambda)^2 - (\mu+2\lambda)(\mu+1)\right]t^2\right|}{4(\mu+2\lambda)t^2} \\ \frac{8|\eta-1|t^3}{\left|(\mu+\lambda)^2 - 2[2(\mu+\lambda)^2 - (\mu+2\lambda)(\mu+1)]t^2\right|} & , \quad |\eta-1| \geq \frac{\left|(\mu+\lambda)^2 - 2\left[2(\mu+\lambda)^2 - (\mu+2\lambda)(\mu+1)\right]t^2\right|}{4(\mu+2\lambda)t^2} \end{cases}. \tag{2.3}$$

Proof. Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\mu}(\lambda, t)$. From (1.5) and (1.6), we have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + U_1(t)p(z) + U_2(t)p^2(z) + \cdots \tag{2.4}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + U_1(t)q(w) + U_2(t)q^2(w) + \cdots \tag{2.5}$$

for some analytic functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \Delta), \tag{2.6}$$

and

$$q(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \qquad (w \in \Delta), \tag{2.7}$$

such that $p(0) = q(0) = 0$, $|p(z)| < 1$ $(z \in \Delta)$ and $|q(w)| < 1$ $(w \in \Delta)$. It is well-known that if $|p(z)| < 1$ and $|q(w)| < 1$, then

$$|c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.8}$$

From (2.4), (2.5), (2.6) and (2.7), we have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + U_1(t)c_1 z + \left[U_1(t)c_2 + U_2(t)c_1^2\right]z^2 + \cdots \tag{2.9}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=1+U_1(t)d_1w+\left[U_1(t)d_2+U_2(t)d_1^2\right]w^2+\cdots. \tag{2.10}$$

Equating the coefficients in (2.9) and (2.10), we get

$$(\mu+\lambda)a_2=U_1(t)c_1 \tag{2.11}$$

$$(\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2+a_3\right]=U_1(t)c_2+U_2(t)c_1^2 \tag{2.12}$$

$$-(\mu+\lambda)a_2=U_1(t)d_1 \tag{2.13}$$

and

$$(\mu+2\lambda)\left[\frac{\mu+3}{2}a_2^2-a_3\right]=U_1(t)d_2+U_2(t)d_1^2. \tag{2.14}$$

From (2.11) and (2.13) , we obtain

$$c_1=-d_1 \tag{2.15}$$

and

$$2\left(\mu+\lambda\right)^2a_2^2=U_1^2(t)\left(c_1^2+d_1^2\right). \tag{2.16}$$

Also, by using (2.12) and (2.14) , we obtain

$$\left(\mu+2\lambda\right)\left(\mu+1\right)a_2^2=U_1(t)\left(c_2+d_2\right)+U_2(t)\left(c_1^2+d_1^2\right). \tag{2.17}$$

By using (2.16) in (2.17) , we get

$$\left[\left(\mu+2\lambda\right)\left(\mu+1\right)-\frac{2U_2(t)}{U_1^2(t)}\left(\mu+\lambda\right)^2\right]a_2^2=U_1(t)\left(c_2+d_2\right). \tag{2.18}$$

From (1.7), (2.8) and (2.18) , we have the desired inequality (2.1) . Next, by subtracting (2.14) from (2.12) , we have

$$2\left(\mu+2\lambda\right)a_3-2\left(\mu+2\lambda\right)a_2^2=U_1(t)\left(c_2-d_2\right)+U_2(t)\left(c_1^2-d_1^2\right). \tag{2.19}$$

Further, in view of (2.15), we obtain

$$a_3=a_2^2+\frac{U_1(t)}{2\left(\mu+2\lambda\right)}\left(c_2-d_2\right). \tag{2.20}$$

Hence using (2.16) and applying (1.7) , we get desired inequality (2.2) .

Now, by using (2.18) and (2.20) for some $\eta\in\mathbb{R}$, we get

$$\begin{aligned} a_3-\eta a_2^2 &= (1-\eta)\left[\frac{U_1^3(t)(c_2+d_2)}{\left(\mu+2\lambda\right)\left(\mu+1\right)U_1^2(t)-2\left(\mu+\lambda\right)^2U_2(t)}\right]+\frac{U_1(t)\left(c_2-d_2\right)}{2\left(\mu+2\lambda\right)} \\ &= U_1(t)\left[\left(h(\eta)+\frac{1}{2\left(\mu+2\lambda\right)}\right)c_2+\left(h(\eta)-\frac{1}{2\left(\mu+2\lambda\right)}\right)d_2\right], \end{aligned}$$

where

$$h(\eta)=\frac{U_1^2(t)(1-\eta)}{\left(\mu+2\lambda\right)\left(\mu+1\right)U_1^2(t)-2\left(\mu+\lambda\right)^2U_2(t)}.$$

So, we conclude that

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{2t}{\mu+2\lambda}, & 0\leq|h(\eta)|\leq\frac{1}{2(\mu+2\lambda)}\\ 4\left|h(\eta)\right|t, & |h(\eta)|\geq\frac{1}{2(\mu+2\lambda)}\end{cases}.$$

This completes the proof of Theorem 1. □

Taking $\mu = 1$ in Theorem 1, we get the following consequence.

Corollary 1. [4] *For $\lambda \geq 1$ and $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\lambda, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}},$$

$$|a_3| \leq \frac{4t^2}{(1+\lambda)^2} + \frac{2t}{1+2\lambda},$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2t}{1+2\lambda} & , \quad |\eta - 1| \leq \frac{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}{4(1+2\lambda)t^2} \\ \frac{8|\eta-1|t^3}{|(1+\lambda)^2 - 4\lambda^2 t^2|} & , \quad |\eta - 1| \geq \frac{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}{4(1+2\lambda)t^2} \end{cases}.$$

Taking $\lambda = 1$ in Theorem 1, we get the following consequence.

Corollary 2. *For $\mu \geq 0$ and $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\mu}(t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|(\mu+1)^2 - 2\mu(\mu+1)t^2\right|}},$$

$$|a_3| \leq \frac{4t^2}{(\mu+1)^2} + \frac{2t}{\mu+2},$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2t}{\mu+2} & , \quad |\eta - 1| \leq \frac{\left|(\mu+1)^2 - 2\mu(\mu+1)t^2\right|}{4(\mu+2)t^2} \\ \frac{8|\eta-1|t^3}{\left|(\mu+1)^2 - 2\mu(\mu+1)t^2\right|} & , \quad |\eta - 1| \geq \frac{\left|(\mu+1)^2 - 2\mu(\mu+1)t^2\right|}{4(\mu+2\lambda)t^2} \end{cases}.$$

Taking $\lambda = 1$ and $\mu = 1$ in Theorem 1, we get the following consequence.

Corollary 3. *For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1-t^2}},$$

$$|a_3| \leq t^2 + \frac{2t}{3},$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2t}{3} & , \quad |\eta - 1| \leq \frac{1-t^2}{3t^2} \\ \frac{2|\eta-1|t^3}{1-t^2} & , \quad |\eta - 1| \geq \frac{1-t^2}{3t^2} \end{cases}.$$

Taking $\lambda = 1$ and $\mu = 0$ in Theorem 1, we get the following consequence.

Corollary 4. *For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{S}_{\Sigma}^{*}(t)$. Then*

$$|a_2| \leq 2t\sqrt{2t},$$
$$|a_3| \leq 4t^2 + t,$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} t & , \quad |\eta - 1| \leq \frac{1}{8t^2} \\ 8\,|\eta - 1|\, t^3 & , \quad |\eta - 1| \geq \frac{1}{8t^2} \end{cases}.$$

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Serap BULUT
Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, TURKEY
E-mail address: `serap.bulut@kocaeli.edu.tr`

Nanjundan MAGESH
P. G. and Research Department of Mathematics, Govt Arts College for Men, Krishnagiri-635001, INDIA
E-mail address: `nmagi_2000@yahoo.co.in`

Chinnaswamy ABIRAMI
Faculty of Engineering and Technology, SRM University, Kattankulathur-603203, Tamilnadu, INDIA
E-mail address: shreelekha07@yahoo.com