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RELATIVE TYPE AND RELATIVE WEAK TYPE BASED GROWTH PROPERTIES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

SANJIB KUMAR DATTA AND TANMAY BISWAS

Abstract. In the paper we wish to introduce the idea of relative type and relative weak type of entire functions of several complex variables with respect to another entire function of several complex variables and prove some related growth properties of it.

1. Introduction, Definitions and Notations

Let f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in view of maximum principal and Hartogs theorem {[8], p. 2, p. 51}, $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2.

The following definition is well known:

Definition 1. {[8], *p. 339,* (*see also* [1])} *The* order ${}_{v_2}\rho_f$ *of an entire function* $f(z_1, z_2)$ *is defined as*

$${}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} = \limsup_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} .$$

We see that the *order* ${}_{v_2}\rho_f$ of an entire function $f(z_1, z_2)$ is defined in terms of the growth of $f(z_1, z_2)$ with respect to the exponential function $\exp(z_1 z_2)$. However, In the same way one can define the *lower order* of $f(z_1, z_2)$ denoted by ${}_{v_2}\lambda_f$ as follows :

$${}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} = \liminf_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} .$$

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An entire function of two complex variables for which *order* and *lower order* are the same is said to be of *regular growth*. Functions which are not of *regular growth* are said to be of *irregular growth*.

The rate of *growth* of an entire function generally depends upon the *order* *(lower order)* of it. The entire function with higher *order* is of faster growth than that of lesser *order*. But if *orders* of two entire functions are the same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their *types* and thus one can define *type* of an entire function $f(z_1, z_2)$ denoted by ${}_{v_2}\sigma_f$ in the following way:

Definition 2. *The* type ${}_{v_2}\sigma_f$ *of an entire function* $f(z_1, z_2)$ *is defined as*

$$ {}_{v_2}\sigma_f = \limsup_{r_1, r_2 \to \infty} \frac{\log M_f(r_1, r_2)}{[r_1 r_2]^{{}_{v_2}\rho_f}}, \ 0 < {}_{v_2}\rho_f < \infty \ . $$

Similarly, the lower type ${}_{v_2}\overline{\sigma}_f$ *of an entire function* $f(z_1, z_2)$ *may be defined as*

$$ {}_{v_2}\overline{\sigma}_f = \liminf_{r_1, r_2 \to \infty} \frac{\log M_f(r_1, r_2)}{[r_1 r_2]^{{}_{v_2}\rho_f}}, \ 0 < {}_{v_2}\rho_f < \infty \ . $$

Analogusly to determine the relative growth of two entire functions of two complex variables having same non zero finite *lower orders* one may introduce the definition of *weak type* ${}_{v_2}\tau_f$ of $f(z_1, z_2)$ of finite positive *lower order* ${}_{v_2}\lambda_f$ in the following way:

Definition 3. *The* weak type ${}_{v_2}\tau_f$ *of an entire function* $f(z_1, z_2)$ *of finite positive* lower order ${}_{v_2}\lambda_f$ *is defined by*

$$ {}_{v_2}\tau_f = \liminf_{r_1, r_2 \to \infty} \frac{\log M_f(r_1, r_2)}{[r_1 r_2]^{{}_{v_2}\lambda_f}}, \ 0 < {}_{v_2}\lambda_f < \infty \ . $$

Similarly, one may define the growth indicator ${}_{v_2}\overline{\tau}_f$ *of an entire function* $f(z_1, z_2)$ *of finite positive* lower order ${}_{v_2}\lambda_f$ *in the following way:*

$$ {}_{v_2}\overline{\tau}_f = \limsup_{r_1, r_2 \to \infty} \frac{\log M_f(r_1, r_2)}{[r_1 r_2]^{{}_{v_2}\lambda_f}}, \ 0 < {}_{v_2}\lambda_f < \infty \ . $$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_f{}^{-1}(s) = \infty$. Bernal {[2], [3]} introduced the definition of relative order of g with respect to f, denoted by $\rho_f(g)$ as follows :

$$ \begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r} \ . \end{aligned} $$

The definition coincides with the classical one [14] if $g(z) = \exp z$.

During the past decades, several authors (see [5],[9],[10],[11],[12],[13]) made closed investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 4. [4] *The relative order between two entire functions of two complex variables denoted by* ${}_{v_2}\rho_g(f)$ *is defined as:*

$$\begin{aligned} {}_{v_2}\rho_g(f) &= \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^{\mu}, r_2^{\mu}); r_1 \geq R(\mu), r_2 \geq R(\mu)\} \\ &= \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \end{aligned}$$

where f *and* g *are entire functions holomorphic in the closed polydisc*

$$U = \{(z_1, z_2) : |z_i| \leq r_i, \ i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1 $\{$*see* [4]$\}$ *if* $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [7] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 5. [7] *Let* $f(z_1, z_2, ..., z_n)$ *and* $g(z_1, z_2, ..., z_n)$ *be any two entire functions of* n *complex variables* $z_1, z_2, ..., z_n$ *with maximum modulus functions* $M_f(r_1, r_2, ..., r_n)$ *and* $M_g(r_1, r_2, ..., r_n)$ *respectively then the relative order of* f *with respect to* g*, denoted by* ${}_{v_n}\rho_g(f)$ *is defined by*

$$\begin{aligned} {}_{v_n}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(r_1^{\mu}, r_2^{\mu}, ..., r_n^{\mu}); \\ \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}. \end{aligned}$$

The above definition can equivalently be written as

$${}_{v_n}\rho_g(f) = \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)}.$$

Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$${}_{v_n}\lambda_g(f) = \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)}.$$

Also an entire function of several complex variables for which order and lower order are the same is said to be of regular growth. The function $\exp(z_1 z_2 ... z_n)$ is an example of regular growth of entire function of several complex variables. Further the functions which are not of regular growth are said to be of irregular growth.

Now in the case of *relative order* of entire functions of several complex variables, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* respectively in order to compare the relative growth of two entire functions of several complex variables having same non zero finite *relative order* or *relative lower order* with respect to another entire function of several complex variables. Their definitions are as follows:

Definition 6. *Let* $f(z_1, z_2, ..., z_n)$ *and* $g(z_1, z_2, ..., z_n)$ *be any two entire functions such that* $0 < {}_{v_n}\rho_g(f) < \infty$*. Then the* relative type ${}_{v_n}\sigma_g(f)$ *of* $f(z_1, z_2, ..., z_n)$ *with respect to* $g(z_1, z_2, ..., z_n)$ *is defined as :*

$$\begin{aligned} {}_{v_n}\sigma_g(f) &= \inf\Big\{k > 0 : M_f(r_1, r_2, ..., r_n) < M_g(kr_1^{{}_{v_n}\rho_g(f)}, kr_2^{{}_{v_n}\rho_g(f)}, ..., kr_n^{{}_{v_n}\rho_g(f)}) \\ &\quad \text{for all sufficiently large values of } r_1, r_2, ..., r_n\}. \end{aligned}$$

Equivalent formula for ${}_{v_n}\sigma_g(f)$ *is*

$${}_{v_n}\sigma_g(f) = \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\rho_g(f)}}.$$

Likewise, one can define the relative lower type *of an entire function* $f(z_1, z_2, ..., z_n)$ *with respect to an entire function* $g(z_1, z_2, ..., z_n)$ *denoted by* ${}_{v_n}\overline{\sigma}_g(f)$ *as follows :*

$$ {}_{v_n}\overline{\sigma}_g(f) = \liminf_{r_1,r_2,...,r_n \to \infty} \frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\rho_g(f)}}, \ 0 < {}_{v_n}\rho_g(f) < \infty \ . $$

Definition 7. *The* relative weak type ${}_{v_n}\tau_g(f)$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *with respect to another entire function* $g(z_1, z_2, ..., z_n)$ *having finite positive* relative lower order ${}_{v_n}\lambda_g(f)$ *is defined as:*

$$ {}_{v_n}\tau_g(f) = \liminf_{r_1,r_2,...,r_n \to \infty} \frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g(f)}} \ . $$

Also one may define the growth indicator ${}_{v_n}\overline{\tau}_g(f)$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *with respect to an entire function* $g(z_1, z_2, ..., z_n)$ *in the following way :*

$$ {}_{v_n}\overline{\tau}_g(f) = \limsup_{r_1,r_2,...,r_n \to \infty} \frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g(f)}}, \ 0 < {}_{v_n}\lambda_g(f) < \infty \ . $$

If we consider $g(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)$, then Definition 5, Definition 6 and Definition 7 reduces to the following classical definition of order (lower order), type (lower type) and weak type in connection with several complex variables:

Definition 8. *The order* ${}_{v_n}\rho_f$ *and the lower order* ${}_{v_n}\lambda_f$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *are defined as*

$$ \begin{aligned} {}_{v_n}\rho_f &= \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[2]} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \ \text{and} \\ {}_{v_n}\lambda_f &= \liminf_{r_1,r_2,...,r_n \to \infty} \frac{\log^{[2]} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \ . \end{aligned} $$

Definition 9. *Let* $f(z_1, z_2, ..., z_n)$ *be an entire function such that* $0 < {}_{v_n}\rho_f < \infty$. *Then the* type ${}_{v_n}\sigma_f$ *and the* lower type ${}_{v_n}\overline{\sigma}_f$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *are defined as follows:*

$$ \begin{aligned} {}_{v_n}\sigma_f &= \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\rho_f}} \ \text{and} \\ {}_{v_n}\overline{\sigma}_f &= \liminf_{r_1,r_2,...,r_n \to \infty} \frac{\log M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\rho_f}}, \ 0 < {}_{v_n}\rho_f < \infty \ . \end{aligned} $$

Definition 10. *Let* $f(z_1, z_2, ..., z_n)$ *be an entire function such that* $0 < {}_{v_n}\lambda_f < \infty$. *Then the* weak type ${}_{v_n}\tau_f$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *is defined as:*

$$ {}_{v_n}\tau_f = \liminf_{r_1,r_2,...,r_n \to \infty} \frac{\log M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}} \ . $$

Also one may define the growth indicator ${}_{v_n}\overline{\tau}_f$ *of an entire function* $f(z_1, z_2, ..., z_n)$ *in the following way :*

$$ {}_{v_n}\overline{\tau}_f = \limsup_{r_1,r_2,...,r_n \to \infty} \frac{\log M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}}, \ 0 < {}_{v_n}\lambda_f < \infty \ . $$

In the paper we study some relative growth properties of entire functions of several complex variables with respect to another entire function of several complex variables on the basis of *relative type* and *relative weak type* of several complex variables. We do not explain the standard definitions and notations in the theory of entire function of two complex variables as those are available in [8].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [6] *Let $f(z_1, z_2, ..., z_n)$ be an entire function with $0 \leq {}_{v_n}\lambda_f \leq {}_{v_n}\rho_f < \infty$ and $g(z_1, z_2, ..., z_n)$ be entire of regular growth . Then*

$$_{v_n}\lambda_g(f) = \frac{_{v_n}\lambda_f}{_{v_n}\lambda_g} \quad \text{and} \quad {}_{v_n}\rho_g(f) = \frac{_{v_n}\rho_f}{_{v_n}\rho_g} \ .$$

Lemma 2. [6] *Let $f(z_1, z_2, ..., z_n)$ be an entire function with regular growth and $g(z_1, z_2, ..., z_n)$ be entire with $0 \leq {}_{v_n}\lambda_g \leq {}_{v_n}\rho_g < \infty$. Then*

$$_{v_n}\lambda_g(f) = \frac{_{v_n}\rho_f}{_{v_n}\rho_g} \quad \text{and} \quad {}_{v_n}\rho_g(f) = \frac{_{v_n}\lambda_f}{_{v_n}\lambda_g} \ .$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions with finite non-zero order. Also let $g(z_1, z_2, ..., z_n)$ be of regular growth. Then*

$$\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\sigma_g}\right]^{\frac{1}{v_2\rho_g}} \leq {}_{v_n}\overline{\sigma}_g(f) \leq \min\left\{\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}, \left[\frac{_{v_n}\sigma_f}{_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}\right\}$$
$$\leq \max\left\{\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}, \left[\frac{_{v_n}\sigma_f}{_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}\right\} \leq {}_{v_n}\sigma_g(f) \leq \left[\frac{_{v_n}\sigma_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}} \ .$$

Proof. From the definitions of $_{v_n}\sigma_f$ and $_{v_n}\overline{\sigma}_f$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) \leq \exp\left\{\left(_{v_n}\sigma_f + \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_f}\right\}, \tag{1}$$
$$M_f(r_1, r_2, ..., r_n) \geq \exp\left\{\left(_{v_n}\overline{\sigma}_f - \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_f}\right\} \tag{2}$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$M_f(r_1, r_2) \geq \exp\left\{\left(_{v_n}\sigma_f - \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_f}\right\}, \tag{3}$$
$$M_f(r_1, r_2) \leq \exp\left\{\left(_{v_n}\overline{\sigma}_f + \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_f}\right\} \ . \tag{4}$$

Similarly from the definitions of $_{v_n}\sigma_g$ and $_{v_n}\overline{\sigma}_f$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g(r_1, r_2, ..., r_n) \leq \exp\left\{\left(_{v_n}\sigma_g + \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_g}\right\}$$
$$\text{i.e., } [r_1 r_2 ... r_n] \leq M_g^{-1}\left[\exp\left\{\left(_{v_n}\sigma_g + \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_g}\right\}\right]$$
$$\text{i.e., } M_g^{-1}(r_1, r_2, ..., r_n) \geq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{\left(_{v_n}\sigma_g + \varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right], \tag{5}$$

$$M_g(r_1, r_2, ..., r_n) \geq \exp\left\{\left(_{v_n}\overline{\sigma} - \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_g}\right\}$$
$$\text{i.e., } [r_1 r_2 ... r_n] \geq M_g^{-1}\left[\exp\left\{\left(_{v_n}\overline{\sigma} - \varepsilon\right)[r_1 r_2 ... r_n]^{v_n\rho_g}\right\}\right]$$
$$\text{i.e., } M_g^{-1}(r_1, r_2, ..., r_n) \leq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{\left(_{v_n}\overline{\sigma}_g - \varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right] \tag{6}$$

and for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we obtain that

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\geq \exp\{({}_{v_n}\sigma_g - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_g}\} \\ i.e.,\ [r_1 r_2 ... r_n] &\geq M_g^{-1}[\exp\{({}_{v_n}\sigma_g - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_g}\}] \\ i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) &\leq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\sigma_g - \varepsilon)}\right)^{\frac{1}{v_n \rho_g}}\right], \end{aligned} \tag{7}$$

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\leq \exp\{({}_{v_n}\overline{\sigma}_g + \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_g}\} \\ i.e.,\ [r_1 r_2 ... r_n] &\leq M_g^{-1}[\exp\{({}_{v_n}\overline{\sigma}_g + \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_g}\}] \\ i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) &\geq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{v_n \rho_g}}\right]. \end{aligned} \tag{8}$$

Now from (3) and in view of (5), we get for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq M_g^{-1}[\exp\{({}_{v_n}\sigma_f - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_f}\}]$$

$$i.e.,\ M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \left[\left(\frac{\log \exp\{({}_{v_n}\sigma_f - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_f}\}}{({}_{v_n}\sigma_g + \varepsilon)}\right)^{\frac{1}{v_n \rho_g}}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1} M_f(r_1, r_2, ..., r_n) &\geq \left[\frac{({}_{v_n}\sigma_f - \varepsilon)}{({}_{v_n}\sigma_g + \varepsilon)}\right]^{\frac{1}{v_n \rho_g}} \cdot [r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \rho_g}} \\ i.e.,\ \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \rho_g}}} &\geq \left[\frac{({}_{v_n}\sigma_f - \varepsilon)}{({}_{v_n}\sigma_g + \varepsilon)}\right]^{\frac{1}{v_n \rho_g}}. \end{aligned}$$

As $\varepsilon (> 0)$ is arbitrary, in view of Lemma 1 it follows that

$$\begin{aligned} \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{v_n \rho_g(f)}} &\geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n \rho_g}} \\ i.e.,\ {}_{v_n}\sigma_g(f) &\geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n \rho_g}}. \end{aligned} \tag{9}$$

Analogously from (2) and in view of (8), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq M_g^{-1}\left[\exp\left\{({}_{v_n}\overline{\sigma}_f - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \left[\left(\frac{\log \exp\{({}_{v_n}\overline{\sigma}_f - \varepsilon)[r_1 r_2 ... r_n]^{v_n \rho_f}\}}{({}_{v_n}\overline{\sigma}_g + \varepsilon)}\right)^{\frac{1}{v_n \rho_g}}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1} M_f(r_1, r_2, ..., r_n) &\geq \left[\frac{({}_{v_n}\overline{\sigma}_f - \varepsilon)}{({}_{v_n}\overline{\sigma}_g + \varepsilon)}\right]^{\frac{1}{v_n \rho_g}} \cdot [r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \rho_g}} \\ i.e.,\ \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \rho_g}}} &\geq \left[\frac{({}_{v_n}\overline{\sigma}_f - \varepsilon)}{({}_{v_n}\overline{\sigma}_g + \varepsilon)}\right]^{\frac{1}{v_n \rho_g}}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above and Lemma 1 that

$$\limsup_{r_1,r_2,...,r_n\to\infty}\frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{v_n\rho_g(f)}} \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}$$

$$i.e.,\ {}_{v_n}\sigma_g\left(f\right) \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}. \quad (10)$$

Again in view of (6) we have from (1) , for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\le M_g^{-1}\left[\exp\left\{\left({}_{v_n}\sigma_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\le\left[\left(\frac{\log\exp\left\{\left({}_{v_n}\sigma_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) \leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}\cdot\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}$$

$$i.e.,\ \frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}} \leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain in view of Lemma 1 that

$$\limsup_{r_1,r_2,...,r_n\to\infty}\frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{v_n\rho_g(f)}} \leq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}$$

$$i.e.,\ {}_{v_n}\sigma_g\left(f\right) \leq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}. \quad (11)$$

Again from (2) and in view of (5) , we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\ge M_g^{-1}\left[\exp\left\{\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\ge\left[\left(\frac{\log\exp\left\{\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}}{\left({}_{v_n}\sigma_g+\varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) \geq \left[\frac{\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)}{\left({}_{v_n}\sigma_g+\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}\cdot\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}$$

$$i.e.,\ \frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}} \geq \left[\frac{\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)}{\left({}_{v_n}\sigma_g+\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above and Lemma 1 that

$$\liminf_{r_1,r_2,...,r_n\to\infty}\frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{v_n\rho_g(f)}} \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}$$

$$i.e.,\ {}_{v_n}\overline{\sigma}_g\left(f\right) \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}. \quad (12)$$

Also in view of (7) , we get from (1) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\le M_g^{-1}\left[\exp\left\{\left({}_{v_n}\sigma_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\leq\left[\left(\frac{\log\exp\left\{\left({}_{v_n}\sigma_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}}{\left({}_{v_n}\sigma_g-\varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) &\leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}\cdot\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}} \\ i.e.,\ \frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}} &\leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from Lemma 1 and above that

$$\begin{aligned} \liminf_{r_1,r_2,...,r_n\to\infty}\frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{v_n\rho_g(f)}} &\leq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}} \\ i.e.,\ {}_{v_n}\overline{\sigma}_g\left(f\right) &\leq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}. \end{aligned} \tag{13}$$

Similarly from (4) and in view of (6) , it follows for a sequence of values of $r_1,r_2,...,r_n$ tending to infinity that

$$M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\leq M_g^{-1}\left[\exp\left\{\left({}_{v_n}\overline{\sigma}_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\leq\left[\left(\frac{\log\exp\left\{\left({}_{v_n}\overline{\sigma}_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{v_n\rho_f}\right\}}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{v_n\rho_g}}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) &\leq \left[\frac{\left({}_{v_n}\overline{\sigma}_f+\varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}\cdot\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}} \\ i.e.,\ \frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{\frac{v_n\rho_f}{v_n\rho_g}}} &\leq \left[\frac{\left({}_{v_n}\overline{\sigma}_f+\varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from Lemma 1 and above that

$$\begin{aligned} \liminf_{r_1,r_2,...,r_n\to\infty}\frac{M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\left[r_1r_2...r_n\right]^{v_n\rho_g(f)}} &\leq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}} \\ i.e.,\ {}_{v_n}\overline{\sigma}_g\left(f\right) &\leq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}. \end{aligned} \tag{14}$$

Thus the theorem follows from (9) , (10) , (11) , (12) , (13) and (14) . □

Theorem 2. *Let $f(z_1,z_2,...,z_n)$ and $g(z_1,z_2,...,z_n)$ be any two entire functions with finite non-zero order. Also let $f(z_1,z_2,...,z_n)$ be of regular growth. Then*

$$\begin{aligned} \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_2\lambda_g}} \leq\ & {}_{v_n}\overline{\sigma}_g\left(f\right)\leq\min\left\{\left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}},\left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}}\right\} \\ & \leq\max\left\{\left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}},\left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}}\right\}\leq {}_{v_n}\sigma_g\left(f\right)\leq\left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}}. \end{aligned}$$

Proof. From the definitions of ${}_{v_n}\overline{\tau}_f$ and ${}_{v_n}\tau_f$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\begin{aligned} M_f(r_1, r_2, ..., r_n) &\leq \exp\left\{({}_{v_n}\overline{\tau}_f + \varepsilon) \cdot [r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}\right\}, \\ M_f(r_1, r_2, ..., r_n) &\geq \exp\left\{({}_{v_n}\tau_f - \varepsilon) \cdot [r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}\right\} \end{aligned}$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$\begin{aligned} M_f(r_1, r_2, ..., r_n) &\geq \exp\left\{({}_{v_n}\overline{\tau}_f - \varepsilon) \cdot [r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}\right\}, \\ M_f(r_1, r_2, ..., r_n) &\leq \exp\left\{({}_{v_n}\tau_f + \varepsilon) \cdot [r_1 r_2 ... r_n]^{{}_{v_n}\lambda_f}\right\}. \end{aligned}$$

Similarly from the definitions of ${}_{v_{\backslash n}}\overline{\tau}_g$ and ${}_{v_n}\tau_g$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\leq \exp\left\{({}_{v_n}\overline{\tau}_g + \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\} \\ i.e., \ (r_1 r_2 ... r_n) &\leq M_g^{-1}\left[\exp\left\{({}_{v_n}\overline{\tau}_g + \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\}\right] \\ i.e., \ M_g^{-1}(r_1, r_2, ..., r_n) &\geq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right], \end{aligned}$$

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\geq \exp\left\{({}_{v_n}\tau_g - \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\} \\ i.e., \ (r_1 r_2 ... r_n) &\geq M_g^{-1}\left[\exp\left\{({}_{v_n}\tau_g - \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\}\right] \\ i.e., \ M_g^{-1}(r_1, r_2, ..., r_n) &\leq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\tau_g - \varepsilon)}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right] \end{aligned}$$

and for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we obtain that

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\geq \exp\left\{({}_{v_n}\overline{\tau}_g - \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\} \\ i.e., \ (r_1 r_2 ... r_n) &\geq M_g^{-1}\left[\exp\left\{({}_{v_n}\overline{\tau}_g - \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\}\right] \\ i.e., \ M_g^{-1}(r_1, r_2, ..., r_n) &\leq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\overline{\tau}_g - \varepsilon)}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right], \end{aligned}$$

$$\begin{aligned} M_g(r_1, r_2, ..., r_n) &\leq \exp\left\{({}_{v_n}\tau_g + \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\} \\ i.e., \ (r_1 r_2 ... r_n) &\leq M_g^{-1}\left[\exp\left\{({}_{v_n}\tau_g + \varepsilon)[r_1 r_2 ... r_n]^{{}_{v_n}\lambda_g}\right\}\right] \\ i.e., \ M_g^{-1}(r_1, r_2, ..., r_n) &\geq \left[\left(\frac{\log(r_1 r_2 ... r_n)}{({}_{v_n}\tau_g - \varepsilon)}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right]. \end{aligned}$$

□

Now using the same technique of Theorem 1, one can easily prove the conclusion of the present theorem by the help of Lemma 2 and the above inequalities. Therefore the remaining part of the proof of the present theorem is omitted.

Similarly in the line of Theorem 1 and Theorem 2 and with the help of Lemma 1 and Lemma 2, one may easily prove the following two theorems and therefore their proofs are omitted:

Theorem 3. *Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions with finite non-zero lower order. Also let $g(z_1, z_2, ..., z_n)$ be of regular growth. Then*

$$\left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}} \leq {}_{v_n}\tau_g(f) \leq \min\left\{\left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}}, \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}}\right\}$$
$$\leq \max\left\{\left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}}, \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}}\right\} \leq {}_{v_n}\overline{\tau}_g(f) \leq \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n\lambda_g}}.$$

Theorem 4. *Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions with finite non-zero order. Also let $f(z_1, z_2, ..., z_n)$ be of regular growth. Then*

$$\left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}} \leq {}_{v_n}\tau_g(f) \leq \min\left\{\left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}, \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}\right\}$$
$$\leq \max\left\{\left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}, \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}\right\} \leq {}_{v_n}\overline{\tau}_g(f) \leq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}}.$$

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S. K. Datta

Department of Mathematics, University of Kalyani, P.O.- Kalyani, Dist-Nadia,PIN-741235, West Bengal, India

E-mail address: sanjib_kr_datta@yahoo.co.in

T. Biswas
Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India
E-mail address: tanmaybiswas_math@rediffmail