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CERTAIN SUBCLASSES OF STARLIKE HARMONIC FUNCTIONS DEFINED BY SUBORDINATION

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ABSTRACT. We consider certain classes of harmonic univalent functions defined by subordination associated with the modified multiplier transformation. Coefficient bounds, distortion theorem, radii of starlikeness and convexity, compactness and extreme points for these classes of harmonic univalent functions are determined.

1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \mathbb{U} \to \mathbb{C}$ has the representation

$$f = h + \bar{g}, \tag{1}$$

where h and g are analytic in $\mathbb{U}$ and have the following power series expansion,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad z \in \mathbb{U},$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$. If $b_0 = 0$, then the representation (1) is unique in $\mathbb{U}$ and is called the canonical representation of f [8]. For the univalent and sense-preserving harmonic functions f in $\mathbb{U}$, it is convenient to make further normalization (without loss of generality), $h(0) = 0$ $(i.e., a_0 = 0)$ and $h'(0) = 1$ $(i.e., a_1 = 1)$. The family of such functions f is denoted by $\mathscr{S}_{\mathcal{H}}$ [3]. The family of all functions $f \in \mathscr{S}_{\mathcal{H}}$ with the additional property that $f_{\bar{z}}(0) = g'(0) = 0$ $(i.e., b_1 = 0)$ is denoted by $\mathscr{S}_{\mathcal{H}^0}$ [3]. Such harmonic and sense-preserving functions $f = h + \bar{g} \in \mathscr{S}_{\mathcal{H}^0}$ may be represented by the power series

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \qquad z \in \mathbb{U}. \tag{2}$$

Observe that the classical family of univalent functions $\mathscr{S}$ consists of all functions $f \in \mathscr{S}_{\mathcal{H}^0}$ such that $g(z) \equiv 0$. Thus it is clear that (see [3])

$$\mathscr{S} \subset \mathscr{S}_{\mathcal{H}^0} \subset \mathscr{S}_{\mathcal{H}}.$$

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The class $\mathscr{S}_{\mathcal{H}^0}$ was studied by Clunie and Sheil-Small [2] who proved that the class $\mathscr{S}_{\mathcal{H}^0}$ is a compact family (with respect to the topology of locally uniform convergence). In particular, they investigated harmonic starlike and harmonic convex functions which are defined as follows.

For $0 \leq \alpha < 1$, we let $\mathscr{S}^*_{\mathcal{H}^0}(\alpha)$ and $\mathscr{S}^c_{\mathcal{H}^0}(\alpha)$, respectively, denote the subclasses of $\mathscr{S}_{\mathcal{H}^0}$ consisting of harmonic starlike and harmonic convex functions of order α.

A function f of the form (2) is in $\mathscr{S}^*_{\mathcal{H}^0}(\alpha)$ if and only if (e.g. see Clunie and Sheil-small [2] or Duren [3])

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) > \alpha, \qquad |z| = r < 1.$$

Similarly, a function f of the form (2) is in $\mathscr{S}^c_{\mathcal{H}^0}(\alpha)$ if and only if

$$\frac{\partial}{\partial\theta}\left(\arg \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right) > \alpha, \qquad |z| = r < 1.$$

We note that (see Dziok [5], Dziok et al. [7]) a harmonic function $f \in \mathscr{S}^*_{\mathcal{H}^0}(\alpha)$ if and only if

$$\Re \frac{\mathscr{I}_{\mathcal{H}} f(z)}{f(z)} \geq \alpha, \qquad |z| = r < 1,$$

or,

$$\left| \frac{\mathscr{I}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathscr{I}_{\mathcal{H}} f(z) + (1-\alpha) f(z)} \right| < 1, \qquad |z| = r < 1,$$

where

$$\mathscr{I}_{\mathcal{H}} f(z) := z h'(z) - \overline{z g'(z)}.$$

For $0 \leq \alpha < 1$, it is easy to verify that

$$f \in \mathscr{S}^c_{\mathcal{H}^0}(\alpha) \Leftrightarrow \mathscr{I}_{\mathcal{H}} f \in \mathscr{S}^*_{\mathcal{H}^0}(\alpha).$$

For $\lambda \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\gamma \geq 0$ and $f = h + \bar{g} \in \mathscr{S}_{\mathcal{H}^0}$ of the form (2), Yasar and Yalcin [15] defined the modified multiplier transformation $\mathscr{I}^{\gamma,\lambda}_{\mathcal{H}} : \mathscr{S}_{\mathcal{H}^0} \to \mathscr{S}_{\mathcal{H}^0}$ by

$$\begin{aligned}
\mathscr{I}^{0,0}_{\mathcal{H}} f(z) &:= h(z) + \overline{g(z)} \\
\mathscr{I}^{0,1}_{\mathcal{H}} f(z) &:= z h'(z) - \overline{z g'(z)} \\
\mathscr{I}^{\gamma,\lambda}_{\mathcal{H}} f(z) &:= \mathscr{I}_{\mathcal{H}}\left(\mathscr{I}^{\gamma,\lambda-1}_{\mathcal{H}} f(z)\right) \\
&:= z + \sum_{n=2}^{\infty} \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} a_n z^n + (-1)^{\lambda} \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \bar{b}_n \bar{z}^n, \qquad z \in \mathbb{U}.
\end{aligned}$$

For the analytic definition of the above operator (see [1]), it is interesting to note that if $\gamma = 0$, then the operator $\mathscr{I}^{\gamma,\lambda}_{\mathcal{H}} f(z)$ reduces to the modified Sălăgean operator [11].

We say that a function $f : \mathbb{U} \to \mathbb{C}$ is subordinate to a function $g : \mathbb{U} \to \mathbb{C}$ and write $f(z) \prec g(z)$ (or simply $f \prec g$), if there exists a complex-valued function w which map $\mathbb{U}$ into itself with $w(0) = 0$, such that $f(z) = g(w(z))$; $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Motivated by the works of Dziok [4, 5] and Dziok et al. [6, 7], we define the following subclass of the function class $\mathscr{S}_{\mathcal{H}^0}$.

For $\lambda \in \mathbb{N}_0$, $\gamma \geq 0$ and $-B \leq A < B \leq 1$, we denote by $\mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$ the class of functions $f \in \mathscr{S}_{\mathcal{H}^0}$ such that

$$\frac{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z)}{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)} \prec \frac{1+Az}{1+Bz}.$$

We let $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ be the class of functions $f \in \mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^{\lambda} \sum_{n=2}^{\infty} |b_n| \overline{z^n}, \qquad z \in \mathbb{U}. \tag{3}$$

We remark that the class

(1) $\mathscr{S}_{\mathcal{H}^0}(\lambda, A, B) := \mathscr{S}_{\mathcal{H}^0}(0, \lambda, A, B)$ (Dziok et al. [7])
(2) $\mathscr{S}^*_{\mathcal{H}^0}(\alpha) := \mathscr{S}_{\mathcal{H}^0}(0, 0, 2\alpha - 1, 1)$ (Jahangiri [9, 10] and Silverman [14])
(3) $\mathscr{S}^c_{\mathcal{H}^0}(\alpha) := \mathscr{S}_{\mathcal{H}^0}(0, 1, 2\alpha - 1, 1)$ (Jahangiri [9, 10] and Silverman [14])
(4) $\mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, \alpha) := \mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, 2\alpha - 1, 1)$ (Yasar and Yalcin [15])
(5) $\mathscr{S}_{\mathcal{H}^0}(\lambda, \alpha) := \mathscr{S}_{\mathcal{H}^0}(0, \lambda, 2\alpha - 1, 1)$ (Jahangiri et al. [11]).
(6) $\mathscr{S}_{\mathcal{T}^0}(\lambda, A, B) := \mathscr{S}_{\mathcal{T}^0}(0, \lambda, A, B)$ (Dziok et al. [7])
(7) $\mathscr{S}^*_{\mathcal{T}^0}(\alpha) := \mathscr{S}_{\mathcal{T}^0}(0, 0, 2\alpha - 1, 1)$ (Jahangiri [9, 10] and Silverman [14])
(8) $\mathscr{S}^c_{\mathcal{T}^0}(\alpha) := \mathscr{S}_{\mathcal{T}^0}(0, 1, 2\alpha - 1, 1)$ (Jahangiri [9, 10] and Silverman [14])
(9) $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, \alpha) := \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, 2\alpha - 1, 1)$ (Yasar and Yalcin [15])
(10) $\mathscr{S}_{\mathcal{T}^0}(\lambda, \alpha) := \mathscr{S}_{\mathcal{T}^0}(0, \lambda, 2\alpha - 1, 1)$ (Jahangiri et al. [11]).

Making use of the techniques and methodology used by Dziok [5], Dziok et al. [6], in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $\mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$.

2. Main Results

Our first theorem provides the sufficient coefficient bound for functions in $\mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$ is provided in the following.

Theorem 1 For $z \in \mathbb{U}$, the harmonic function $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}$ is in $\mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$, if

$$\sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma} \right)^{\lambda} \left[\frac{B(n+\gamma) + (n-1)}{1+\gamma} \; - \; A \right] |a_n| \right. \tag{4}$$
$$\left. + \left(\frac{n-\gamma}{1+\gamma} \right)^{\lambda} \left[\frac{B(n-\gamma) + (n+1)}{1+\gamma} \; + \; A \right] |b_n| \right\} \leq B - A.$$

Proof. Clearly the theorem is true for $f(z) \equiv z$. So, we assume that $a_n \neq 0$ or $b_n \neq 0$ for some or all $n \geq 2$. For $n \geq 2$ note that

$$\left\{ \left(\frac{n+\gamma}{1+\gamma} \right)^{\lambda} \left[\frac{B(n+\gamma) + (n-1)}{1+\gamma} \; - \; A \right] \right\} \geq n(B - A)$$

and

$$\left\{ \left(\frac{n-\gamma}{1+\gamma} \right)^{\lambda} \left[\frac{B(n-\gamma) + (n+1)}{1+\gamma} \; + \; A \right] \right\} \geq n(B - A).$$

Next, according to the required condition (4), we have

$$\begin{aligned}
|h'(z)| - |g'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=2}^{\infty} n|b_n||z|^n \\
&\geq 1 - |z| \sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \\
&\geq 1 - \frac{|z|}{B-A} \sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |a_n| \right. \\
&\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |b_n| \right\} \\
&\geq 1 - |z| > 0, \qquad z \in \mathbb{U}.
\end{aligned}$$

Therefore f is sense preserving and locally univalent in $\mathbb{U}$. For the univalence condition, consider $z_1, z_2 \in \mathbb{U}$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| = \left| \sum_{m=1}^{n} z_1^{m-1} z_2^{n-m} \right| \leq \sum_{m=1}^{n} |z_1|^{m-1} |z_2|^{n-m} < n, \qquad n \geq 2.$$

Hence

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&\geq \left| z_1 - z_2 - \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n) \right| \; - \; \left| \sum_{n=2}^{\infty} \overline{b_n (z_1^n - z_2^n)} \right| \\
&\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| \, |z_1^n - z_2^n| \; - \; \sum_{n=2}^{\infty} |b_n| \, |z_1^n - z_2^n| \\
&\geq |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| \; - \; \sum_{n=2}^{\infty} |b_n| \left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| \right) \\
&> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n|a_n| \; - \; \sum_{n=2}^{\infty} n|b_n| \right) \geq 0.
\end{aligned}$$

This proves that f is univalent in $\mathbb{U}$, that is, $f \in \mathscr{S}_{\mathcal{H}^0}$.

On the other hand, by definition, we have $f \in \mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$ if and only if there exist a complex-valued function w, $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) such that

$$\frac{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z)}{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \qquad z \in \mathbb{U},$$

or equivalently

$$\left| \frac{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - \mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)}{B\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - A\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)} \right| < 1, \qquad z \in \mathbb{U}. \tag{5}$$

The above inequality (5) holds, since for $|z| = r$, $0 < r < 1$, we obtain

$$
\begin{aligned}
&\left|\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - \mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)\right| - \left|B\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - A\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)\right| \\
&\quad = \left|\sum_{n=2}^{\infty} \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{n-1}{1+\gamma}\right] a_n z^n - (-1)^{\lambda} \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{n+1}{1+\gamma}\right] \bar{b}_n \bar{z}^n\right| \\
&\qquad - \left|(B-A)z + \sum_{n=2}^{\infty} \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)}{1+\gamma} - A\right] a_n z^n \right. \\
&\qquad\qquad \left. + (-1)^{\lambda} \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)}{1+\gamma} + A\right] \bar{b}_n \bar{z}^n\right| \\
&\quad \leq \sum_{n=2}^{\infty} \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{n-1}{1+\gamma}\right] |a_n| r^n + \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{n+1}{1+\gamma}\right] |b|_n r^n \\
&\qquad -(B-A)r + \sum_{n=2}^{\infty} \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)}{1+\gamma} - A\right] |a_n| r^n \\
&\qquad\qquad + \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)}{1+\gamma} + A\right] |b_n| r^n \\
&\quad \leq r \left\{ \sum_{n=2}^{\infty} \left[\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right] |a_n| \right.\right. \\
&\qquad\qquad \left.\left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right] |b_n| \right] - (B-A) \right\} \\
&\quad < 0.
\end{aligned}
$$

Therefore $f \in \mathscr{S}_{\mathcal{H}^0}(\gamma, \lambda, A, B)$. □

In the next theorem, we show that the restriction placed in Theorem 1 on the moduli of the coefficients of the harmonic function $f(z) = h + \bar{g}$ cannot be improved.

Theorem 2 Let $f(z) = h + \bar{g}$ be defined by (3). Then $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ if and only if the condition

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right] |a_n| \right. \\
&\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right] |b_n| \right\} < B - A \qquad (6)
\end{aligned}
$$

holds.

Proof. The 'if' part follows from Theorem 1. For the 'only-if' part, assume that $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, then, we have

$$\left|\frac{\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - \mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)}{B\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda+1} f(z) - A\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(z)}\right| < 1, \qquad z \in \mathbb{U}.$$

Therefore, for $z \in \mathbb{U}$ we have

$$\left| \frac{\sum\limits_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[\left(\frac{n+\gamma}{1+\gamma}\right)\,-\,1\right]|a_n|z^{n-1}+\sum\limits_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\left(\frac{n-\gamma}{1+\gamma}\right)\,+\,1\right]|b_n|\bar{z}^{n-1}}{(B-A)-\sum\limits_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[B\left(\frac{n+\gamma}{1+\gamma}\right)\,-\,A\right]|a_n|z^{n-1}-\sum\limits_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[B\left(\frac{n-\gamma}{1+\gamma}\right)\,+\,A\right]|b_n|\bar{z}^{n-1}} \right| < 1.$$

For $|z| = r$ $(0 \leq r < 1)$ the above inequality reduces to

$$\frac{\sum\limits_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[\left(\frac{n+\gamma}{1+\gamma}\right)\,-\,1\right]|a_n|r^{n-1}+\sum\limits_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\left(\frac{n-\gamma}{1+\gamma}\right)\,+\,1\right]|b_n|r^{n-1}}{(B-A)-\sum\limits_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[B\left(\frac{n+\gamma}{1+\gamma}\right)\,-\,A\right]|a_n|r^{n-1}-\sum\limits_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[B\left(\frac{n-\gamma}{1+\gamma}\right)\,+\,A\right]|b_n|r^{n-1}} < 1. \tag{7}$$

It is clear that the denominator of the left-hand side of the above inequality (7) does not vanish for $r \in (0, 1)$. Moreover, since it is positive for $r = 0$, it is positive for $r \in \langle 0, 1)$. Now, a simple algebra yields

$$\sum_{n=2}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n+\gamma)+(n-1)}{1+\gamma}\;-\;A\right]|a_n|r^{n-1}$$
$$+\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n-\gamma)+(n+1)}{1+\gamma}\;+\;A\right]|b_n|r^{n-1} < B-A.$$

Upon letting $r \to 1^-$, we obtain

$$\sum_{n=2}^{\infty}\left\{\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n+\gamma)+(n-1)}{1+\gamma}\;-\;A\right]|a_n|\right.$$
$$\left.+\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n-\gamma)+(n+1)}{1+\gamma}\;+\;A\right]|b_n|\right\} < B-A.$$

□

In the following we show that the class of functions of the form (3) is convex and compact.

Theorem 3 The class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ is a convex and compact subset of $\mathscr{S}_{\mathcal{H}^0}$.

Proof. Let $f_k \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, where

$$f_k(z) = z - \sum_{n=2}^{\infty}|a_{k,n}|\,z^n + (-1)^{\lambda}\sum_{n=1}^{\infty}|b_{k,n}|\,\bar{z}^n, \qquad z \in \mathbb{U},\; k \in \mathbb{N}. \tag{8}$$

Then, for $0 \leq \mu \leq 1$, we write

$$\mu f_1(z)+(1-\mu)f_2(z) = z+\sum_{n=2}^{\infty}\{\mu a_{1,n}+(1-\mu)a_{2,n}\}z^n+\sum_{n=2}^{\infty}\overline{\{\mu b_{1,n}+(1-\mu)b_{2,n}\}}\bar{z}^n.$$

Since $f_k \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, by Theorem 2, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |\mu a_{1,n} + (1-\mu) a_{2,n}| \right. \\
&\qquad\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |\mu b_{1,n} + (1-\mu) b_{2,n}| \right\} \\
&\le \mu \sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |a_{1,n}| \right. \\
&\qquad\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |b_{1,n}| \right\} \\
&\qquad\qquad + (1-\mu) \sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |a_{2,n}| \right. \\
&\qquad\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |b_{2,n}| \right\} \\
&\le \mu(B-A) + (1-\mu)(B-A) \\
&= B - A.
\end{aligned}$$

Therefore, again by Theorem 2, the function $\mu f_1(z) + (1-\mu) f_2(z)$ belongs to the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$. Hence the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ is convex. Furthermore, for $f_k \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, $|z| \le r$, $0 < r < 1$, we have

$$\begin{aligned}
|f_k(z)| \;&\le\; r + \sum_{n=2}^{\infty} [|a_{k,n}| + |b_{k,n}|] r^n \\
&\le\; r + \sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |a_{k,n}| \right. \\
&\qquad\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |b_{k,n}| \right\} r^n \\
&\le\; r + (B-A) r^n.
\end{aligned}$$

Therefore, the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ is locally uniformly bounded. For the compactness, it suffices (see [13] and [5, Lemma 15, p.6]) to show that $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ is closed. That is, if $f_k \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, $k \in \mathbb{N}$ and $f_k \to f$, then $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$. Let f_k and f be given by (8) and (2), respectively. Using Theorem 2, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left\{ \left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} \; - \; A\right] |a_{k,n}| \right. \\
&\qquad \left. + \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} \; + \; A\right] |b_{k,n}| \right\} r^n \le B - A, \qquad k \in \mathbb{N}.
\end{aligned}$$

Since $f_k \to f$, we conclude that $|a_{k,n}| \to |a_n|$ and $|b_{k,n}| \to |b_n|$ as $k \to \infty$, $k \in \mathbb{N}$. This gives condition (4), and, in consequence, $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$, which completes the proof. $\square$

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$.

Theorem 4 Let $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$. Then f is starlike of order α $(0 \leq \alpha < 1)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{n \geq 2} \left\{ \frac{1-\alpha}{B-A} \min \left(\frac{\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]}{n-\alpha}, \frac{\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}{n+\alpha} \right) \right\}^{\frac{1}{n-1}}.$$

Proof. Let $f \in \mathscr{S}_{\mathcal{T}}(\gamma, \lambda, A, B)$ be of the form

$$f(z) = h + \bar{g} = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^{\lambda} \sum_{n=2}^{\infty} |b_n| \bar{z}^n, \qquad z \in \mathbb{U}.$$

Then, for $|z| = r < 1$, we have

$$\begin{aligned}
&\left| \frac{\mathscr{I}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathscr{I}_{\mathcal{H}} f(z) + (1-\alpha) f(z)} \right| \\
&\leq \left| \frac{-\alpha z - \sum_{n=2}^{\infty} [n-1-\alpha] |a_n| z^n - \sum_{n=2}^{\infty} [n+1+\alpha] |b_n| \bar{z}^n}{(2-\alpha) z - \sum_{n=2}^{\infty} [n+1-\alpha] |a_n| z^n - (-1)^{\lambda} \sum_{n=2}^{\infty} [n-1+\alpha] |b_n| \bar{z}^n} \right| \\
&\leq \frac{\alpha + \sum_{n=2}^{\infty} \{[n-1-\alpha]|a_n| + [n+1+\alpha]|b_n|\} r^{n-1}}{(2-\alpha) - \sum_{n=2}^{\infty} \{[n+1-\alpha]|a_n| + [n-1+\alpha]|b_n|\} r^{n-1}}.
\end{aligned}$$

Note (see Jahangiri [10, Theorem 2, p. 474], Dziok et al. [7, Theorem 7, p.11]) that f is starlike of order α in $\mathbb{U}(r)$ if and only if

$$\left| \frac{\mathscr{I}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathscr{I}_{\mathcal{H}} f(z) + (1-\alpha) f(z)} \right| < 1, \qquad z \in \mathbb{U}(r)$$

or

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) r^{n-1} \leq 1. \tag{9}$$

Also, by Theorem 2, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{B-A} \Bigg\{ &\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right] |a_n| \\
&+ \left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right] |b_n| \Bigg\} \leq 1.
\end{aligned}$$

The condition (9) is true if

$$\frac{n-\alpha}{1-\alpha} r^{n-1} \leq \frac{\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]}{B-A}, \qquad n \geq 2$$

and

$$\frac{n+\alpha}{1-\alpha} r^{n-1} \leq \frac{\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda} \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}{B-A}, \qquad n \geq 2.$$

Or for $n \geq 2$, if

$$r \leq \left\{ \frac{1-\alpha}{B-A} \min \left(\frac{\left(\frac{n+\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]}{n-\alpha}, \frac{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}{n+\alpha} \right) \right\}^{\frac{1}{n-1}}.$$

It follows that the function f is starlike of order α in the disk $\mathbb{U}(r^*)$ where

$$r^* \leq \inf_{n \geq 2} \left\{ \frac{1-\alpha}{B-A} \min \left(\frac{\left(\frac{n+\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]}{n-\alpha}, \frac{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}{n+\alpha} \right) \right\}^{\frac{1}{n-1}}.$$

The function

$$\begin{aligned} f_n(z) &= h_n(z) + \overline{g_n(z)} \\ &= z - \frac{B-A}{\left(\frac{n+\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]} z^n + (-1)^\lambda \frac{B-A}{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]} \bar{z}^n \end{aligned}$$

proves that the radius r^* cannot be improved. $\square$

Using a similar argument as above, we state the following

Theorem 5 Let $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$. Then f is convex of order α $(0 \leq \alpha < 1)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_{n \geq 2} \left\{ \frac{1-\alpha}{B-A} \min \left(\frac{\left(\frac{n+\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]}{n(n-\alpha)}, \frac{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}{n(n+\alpha)} \right) \right\}^{\frac{1}{n-1}}.$$

In the following theorem we obtain the extreme points of the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$.

Theorem 6 Extreme points of the class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ are the functions f of the form (2), where

$$\begin{aligned} h_1(z) = z, \quad h_n(z) &= z - \frac{B-A}{\left(\frac{n+\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n+\gamma)+(n-1)}{1+\gamma} - A\right]} z^n \\ g_n(z) &= (-1)^\lambda \frac{B-A}{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]} \bar{z}^n, \; n \in \{2, 3, \dots\}, \; z \in \mathbb{U}. \end{aligned} \tag{10}$$

Proof. Let $g_n(z) = \mu f_1(z) + (1-\mu) f_2(z)$, where $0 < \mu < 1$ and $f_1, f_2 \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ are functions of the form

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{k,n} z^n + \sum_{n=2}^{\infty} \overline{b_{k,n} z^n}, \qquad k \in \{1, 2\}, \quad z \in \mathbb{U}.$$

Then, by (6), we have

$$|b_{1,n}| = |b_{2,n}| = \frac{(B-A)}{\left(\frac{n-\gamma}{1+\gamma}\right)^\lambda \left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]},$$

and therefore $a_{1,k} = a_{2,k} = 0$ for $k \in \{2, 3, \dots\}$ and $b_{1,k} = b_{2,k} = 0$ for $k \in \{2, 3, \dots\} \setminus \{n\}$. It follows that $g_n(z) = f_1(z) = f_2(z)$ and g_n are in the class of extreme points of the function class $\mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$. Similarly, we can verify that the functions $h_n(z)$ of the form (10) are the extreme points of the class

$\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$. Now, suppose that a function f of the form (2) is in the family of extreme points of the class $\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$ and f is not of the form (10). Then there exists $m\in\{2,3,\dots\}$ such that

$$0<|a_m|<\frac{B-A}{\left(\frac{m+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m+\gamma)+(m-1)}{1+\gamma}\ -\ A\right]}$$

or

$$0<|b_m|<\frac{B-A}{\left(\frac{m-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m-\gamma)+(m+1)}{1+\gamma}\ +\ A\right]}\ .$$

If

$$0<|a_m|<\frac{B-A}{\left(\frac{m+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m+\gamma)+(m-1)}{1+\gamma}\ -\ A\right]},$$

then putting

$$\mu=\frac{|a_m|\left\{\left(\frac{m+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m+\gamma)+(m-1)}{1+\gamma}\ -\ A\right]\right\}}{B-A}$$

and

$$\phi=\frac{f-\mu h_m}{1-\mu},$$

we have $0<\mu<1$, $h_m\neq\phi$, and

$$f=\mu h_m+(1-\mu)\phi.$$

Therefore, f is not in the family of extreme points of the class $\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$. Similarly, if

$$0<|b_m|<\frac{B-A}{\left(\frac{m-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m-\gamma)+(m+1)}{1+\gamma}\ +\ A\right]},$$

then putting

$$\mu=\frac{|b_m|\left\{\left(\frac{m-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(m-\gamma)+(m+1)}{1+\gamma}\ +\ A\right]\right\}}{B-A},$$

and

$$\phi=\frac{f-\mu g_m}{1-\mu},$$

we have $0<\mu<1$, $g_m\neq\phi$, and

$$f=\mu g_m+(1-\mu)\phi.$$

It follows that f is not in the family of extreme points of the class $\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$ and so the proof is completed. □

It is clear that if the class $\mathscr{F}=\{f_n\in\mathscr{S}_{\mathcal{H}^0}:n\in\mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{co}\mathscr{F}=\left\{\sum_{n=1}^{\infty}\mu_n f_n:\sum_{n=1}^{\infty}\mu_n=1,\ \mu_n\geq 0,\ n\in\mathbb{N}\right\}.$$

Therefore, by Theorem 6, we have the following

Corollary 1 Let $h_n,\ g_n$ be defined by (10). Then

$$\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)=\left\{\sum_{n=1}^{\infty}(\mu_n h_n+\delta_n g_n):\sum_{n=1}^{\infty}(\mu_n+\delta_n)=1,\ \delta_1=0,\ \mu_n,\ \delta_n\geq 0,\ n\in\mathbb{N}\right\}.$$

For all fixed values of $n,\lambda\in\mathbb{N}$ and $z\in\mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathscr{S}_{\mathcal{H}^0}$:

$$\mathscr{I}(f)=|a_n|,$$

$$\mathscr{I}(f)=|b_n|,$$

$$\mathscr{I}(f)=|f(z)|,$$

$$\mathscr{I}(f)=|\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda}f(z)|,\ f\in\mathscr{S}_{\mathcal{H}^0}.$$

Moreover, for $\mu>0$ and $0<r<1$, the real-valued functional

$$\mathscr{I}(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\mu}d\theta\right)^{1/\mu},\qquad f\in\mathscr{S}_{\mathcal{H}^0}$$

is continuous on $\mathscr{S}_{\mathcal{H}^0}$. For $\mu\geq 1$, it is also convex on $\mathscr{S}_{\mathcal{H}^0}$. Therefore by [5, Lemma 2, p. 10] and Theorem 6, we have the following corollaries.

Corollary 2 Let $f\in\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$, be a function of the form (3). Then

$$|a_n|\leq\frac{B-A}{\left(\frac{n+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n+\gamma)+(n-1)}{1+\gamma}-A\right]}$$

and

$$|b_n|\leq\frac{B-A}{\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n-\gamma)+(n+1)}{1+\gamma}+A\right]},\qquad n\geq 2.$$

The result is sharp for the extremal functions h_n, g_n of the form (10).

Corollary 3 Let $f\in\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$, be a function of the form (3). Then

$$r-\frac{B-A}{\left(\frac{2+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(2+\gamma)+1}{1+\gamma}-A\right]}\leq|f(z)|\leq r+\frac{B-A}{\left(\frac{2+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(2+\gamma)+1}{1+\gamma}-A\right]}$$

and

$$r-\frac{B-A}{\frac{B(2+\gamma)+1}{1+\gamma}-A}\leq|\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda}f(z)|\leq r+\frac{B-A}{\frac{B(2+\gamma)+1}{1+\gamma}-A}.$$

The result is sharp for the extremal function h_2 of the form (10).

Corollary 4 Let $f\in\mathscr{S}_{\mathcal{T}^0}(\gamma,\lambda,A,B)$, be of the form (3). Then $\mathbb{U}(r)\subset f(\mathbb{U})$, where

$$r=1-\frac{B-A}{\left(\frac{2+\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(2+\gamma)+1}{1+\gamma}-A\right]}\ .$$

Let h_n and g_n be defined by (10) and $\widetilde{g_n}(z) = \frac{B+A}{\left(\frac{n-\gamma}{1+\gamma}\right)^{\lambda}\left[\frac{B(n-\gamma)+(n+1)}{1+\gamma} + A\right]}$, $n \in \mathbb{N}$. Since

$$\frac{h_n(z)}{z} \prec \frac{h_2(z)}{z} \qquad \text{and} \qquad \frac{\widetilde{g_n}(z)}{z} \prec \frac{h_2(z)}{z},$$

by the integral means inequalities (see Littlewood [12]), we have

$$\int_0^{2\pi} \left|\frac{h_n(z)}{z}\right|^{\mu} d\theta \le \int_0^{2\pi} \left|\frac{h_2(z)}{z}\right|^{\mu} d\theta, \qquad z = re^{i\theta}$$

and

$$\int_0^{2\pi} \left|\frac{g_n(z)}{z}\right|^{\mu} d\theta = \int_0^{2\pi} \left|\frac{\widetilde{g_n}(z)}{z}\right|^{\mu} d\theta \le \int_0^{2\pi} \left|\frac{h_2(z)}{z}\right|^{\mu} d\theta, \qquad z = re^{i\theta}.$$

Thus we have the following

Corollary 5 Let $0 < r < 1$ and $\mu \ge 1$. If $f \in \mathscr{S}_{\mathcal{T}^0}(\gamma, \lambda, A, B)$ is of the form (3), then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|h_2(re^{i\theta})\right|^{\mu} d\theta,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} f(re^{i\theta})\right|^{\mu} d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|\mathscr{I}_{\mathcal{H}}^{\gamma,\lambda} h_2(re^{i\theta})\right|^{\mu} d\theta, \qquad \mu = 1, 2, 3, \ldots.$$

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