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ON THE GENERALIZED RELATIVE TYPE AND GENERALIZED RELATIVE WEAK TYPE RELATED GROWTH ANALYSIS OF ENTIRE FUNCTIONS

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Abstract. In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis generalized relative type and generalized relative weak type.

1. Introduction and preliminaries

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire functions which are available in [19]. In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \log^{[0]} x &= x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \exp^{[0]} x &= x. \end{aligned}$$

Taking this into account, Juneja, Kapoor and Bajpai [12] defined the *(p, q)-th order* and *(p, q)-th lower order* of an entire function f respectively as follows:

$$\rho_f(p, q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p, q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$.

These definitions extended the definitions of *order* ρ_f and *lower order* λ_f of an entire function f which are classical in complex analysis for integers $p = 2$ and $q = 1$ since these correspond to the particular case $\rho_f(2, 1) = \rho_f$ and $\lambda_f(2, 1) = \lambda_f$.

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In this connection we just recall the following definitions :

Definition 1. [12] *An entire function f is said to have* index-pair (p,q), $p \geq q \geq 1$ *if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1)$ is not a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ if $p>q$. Moreover if $0<\rho_f(p,q)<\infty$, then*

$$\rho_f(p-n,q) = \infty \text{ for } n<p,\ \rho_f(p,q-n) = 0 \text{ for } n<q \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n=1,2,.... .$$

Similarly for $0<\lambda_f(p,q)<\infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n<p,\ \lambda_f(p,q-n) = 0 \text{ for } n<q \text{ and}$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n=1,2,.... .$$

To compare the growth of entire functions having the same (p,q)-*th order*, Juneja, Kapoor and Bajpai [13] also introduced the concepts of (p,q)-*th type* and (p,q)-*th lower type* in the following manner :

Definition 2. [13] *The* (p,q) th type *and* the (p,q) th lower type *of entire function f having finite positive* (p,q) th order $\rho_f(p,q)$ $(b<\rho_f(p,q)<\infty)$ *are defined as :*

$$\begin{aligned}\sigma_f(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_f(p,q)}} \text{ and}\\ \overline{\sigma}_f(p,q) &= \liminf_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_f(p,q)}},\ \ 0\leq \overline{\sigma}_f \leq \sigma_f \leq \infty,\end{aligned}$$

where p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$.

Similarly, extending the notion of *weak type* as introduced by Datta and Jha [4], one can define (p,q) *-th weak type* to determine the relative growth of two entire functions having same non zero finite (p,q)-*th lower order* in the following manner:

Definition 3. *The* (p,q) -th weak type $\tau_f(p,q)$ *of an entire function f having finite positive* (p,q)-th lower order $\lambda_f(p,q)$ $(b<\lambda_f(p,q)<\infty)$ *is defined by*

$$\tau_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_f(p,q)}}$$

where p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$.

Also one may define the growth indicator $\overline{\tau}_f(p,q)$ of an entire function f in the following way :

$$\overline{\tau}_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_f(p,q)}}\ ,\ \ 0<\lambda_f(p,q)<\infty.$$

For $p=l$ and $q=1$, the above definitions reduces to *generalized order* $\rho_f^{[l]}$ [17] (respectively, *generalized lower order* $\lambda_f^{[l]}$) which are as follows:

Definition 4. [17] *The* generalized order $\rho_f^{[l]}$ *(respectively,* generalized lower order $\lambda_f^{[l]}$*) of an entire function f is defined as*

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}$$

$$\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \right)$$

where $l \geq 1$.

Definition 5. *The* generalized type $\sigma_f^{[l]}$ *and* generalized lower type $\overline{\sigma}_f^{[l]}$ *of an entire function f are defined as*

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}},\ 0 < \rho_f^{[l]} < \infty.$$

where $l \geq 1$. Moreover, when $l = 2$ then $\sigma_f^{[2]}$ and $\overline{\sigma}_f^{[2]}$ are correspondingly denoted as σ_f and $\overline{\sigma}_f$ which are respectively known as type *and* lower type *of entire f.*

Definition 6. *The* generalized weak type $\tau_f^{[l]}$ *for $l \geq 1$ of an entire function f of finite positive* generalized lower order $\lambda_f^{[l]}$ *are defined by*

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f^{[l]} < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^{[l]}$ of an entire function f in the following way :

$$\overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}}\ ,\ 0 < \lambda_f^{[l]} < \infty.$$

For $l = 2$, the above definition reduces to the classical definition as established by Datta and Jha [4]*. Also τ_f and $\overline{\tau}_f$ are stand for $\tau_f^{[2]}$ and $\overline{\tau}_f^{[2]}$.*

For any two entire functions f and g, Bernal {[1], [2]} initiated the definition of relative order of f with respect to g, indicated by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}, \end{aligned}$$

which keeps away from comparing growth just with $\exp z$ to find out *order* of entire functions as we see in the earlier and of course this definition corresponds with the classical one [18] for $g = \exp z$.

Analogously, one may define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}\ .$$

To compare the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [16] recently introduced the notion of relative type of two entire functions in the following manner:

Definition 7. [16] *Let f and g be any two entire functions such that $0<\rho_g(f)<\infty$. Then the* relative type $\sigma_g(f)$ *of f with respect to g is defined as :*

$$\begin{aligned} & \sigma_g(f) \\ = & \inf\left\{k>0 : M_f(r)<M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\} \\ = & \limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} . \end{aligned}$$

Likewise one can define the relative lower type *of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :*

$$\overline{\sigma}_g(f)=\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}},\ 0<\rho_g(f)<\infty .$$

Analogously to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, Datta and Biswas [9] introduced the definition of relative weak type of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g(f)$ in the following way:

Definition 8. [9] *The relative weak type $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive relative lower order $\lambda_g(f)$ is defined as:*

$$\tau_g(f)=\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}} .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f)=\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}},\ 0<\lambda_g(f)<\infty .$$

Lahiri and Banerjee [14] gave a more generalized concept of relative order in the following way:

Definition 9. [14] *If $l\geq 1$ is a positive integer, then the l- th* generalized relative order *of f with respect to g, denoted by $\rho_g^{[l]}(f)$ is defined by*

$$\begin{aligned} \rho_g^{[l]}(f) &= \inf\left\{\mu>0 : M_f(r)<M_g\left(\exp^{[l-1]}r^{\mu}\right) \text{ for all } r>r_0(\mu)>0\right\} \\ &= \limsup_{r\to\infty}\frac{\log^{[l]}M_g^{-1}M_f(r)}{\log r} . \end{aligned}$$

Clearly $\rho_g^{[1]}(f)=\rho_g(f)$ and $\rho_{\exp z}^{[1]}(f)=\rho_f$.

Likewise one can define the generalized relative lower *order of f with respect to g denoted by $\lambda_g^{[l]}(f)$ as*

$$\lambda_g^{[l]}(f)=\liminf_{r\to\infty}\frac{\log^{[l]}M_g^{-1}M_f(r)}{\log r} .$$

Further to compare the relative growth of two entire functions having same non zero finite *generalized relative order* with respect to another entire function, Datta et al [10] introduced the definition of *generalized relative type* and *generalized relative lower type* of an entire function with respect to another entire function which are as follows :

Definition 10. [10] *The* generalized relative type $\sigma_f^{[l]}$ *and* generalized relative lower type $\overline{\sigma}_f^{[l]}$ *of an entire function f are defined as*

$$\sigma_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \quad \text{and}$$
$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}},\ 0 < \rho_g^{[l]}(f) < \infty.$$

For $l = 2$, Definition 10 reduces to Definition 7.

Similarly to determine the relative growth of two entire functions having same non zero finite *generalized relative lower order* with respect to another entire function, Datta et al [12] also introduced the concepts of *generalized relative weak type* of an entire function with respect to another entire function in the following manner:

Definition 11. [10] *The* generalized relative weak type $\tau_g^{[l]}(f)$ *of an entire function f with respect to another entire function g having finite positive* generalized relative lower order $\lambda_g^{[l]}(f)$ *is defined as:*

$$\tau_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} .$$

Further one may define the growth indicator $\overline{\tau}_g^{[l]}(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}},\ 0 < \lambda_g^{[l]}(f) < \infty .$$

Definition 11 also reduces to Definition 8 for particular $l = 2$.

For entire functions, the notions of the growth indicators such as *order* and *type (weak type)* are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative order (generalized relative orders) relative type (generalized relative type)* and *relative weak type* (*generalized relative weak type)* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their *relative order (generalized relative orders) relative type (generalized relative type)* and *relative weak type* (*generalized relative weak type)* are the prime concern of this paper. In fact some light has already been thrown on such type of works by Datta et al. in [3], [5], [6], [7], [8], [9] and [10]. Actually in this paper we study some relative growth properties of entire functions with respect to another entire function on the basis of *generalized relative type* and *generalized relative weak type* which infact extend some results of [11]. In this connection we recall one related known property which will be needed in order to prove our results, as we see in the following theorem:

Theorem 1. [15] *Let f be an entire function with $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive* *integers such that $m \geq p$. Then*

$$\frac{\lambda_f^{[m]}}{\rho_g(m,p)} \leq \lambda_g^{[p]}(f) \leq \min\left\{\frac{\lambda_f^{[m]}}{\lambda_g(m,p)}, \frac{\rho_f^{[m]}}{\rho_g(m,p)}\right\}$$
$$\leq \max\left\{\frac{\lambda_f^{[m]}}{\lambda_g(m,p)}, \frac{\rho_f^{[m]}}{\rho_g(m,p)}\right\} \leq \rho_g^{[p]}(f) \leq \frac{\rho_f^{[m]}}{\lambda_g(m,p)} .$$

2. Results

In this section we present the main results of the paper.

Theorem 2. *Let f be an entire function with $0 < \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\max\left\{\left[\frac{\overline{\sigma}_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\sigma_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq \sigma_g^{[p]}(f) \leq \left[\frac{\sigma_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}} .$$

Proof. From the definitions of $\sigma_f^{[m]}$ and $\overline{\sigma}_f^{[m]}$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp^{[m-1]}\left\{\left(\sigma_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\}, \tag{1}$$
$$M_f(r) \geq \exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\} \tag{2}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp^{[m-1]}\left\{\left(\sigma_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\}, \tag{3}$$
$$M_f(r) \leq \exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\} . \tag{4}$$

Similarly from the definitions of $\sigma_g(m,p)$ and $\overline{\sigma}_g(m,p)$, it follows for all sufficiently large values of r that

$$M_g(r) \leq \exp^{[m-1]}\left\{(\sigma_g(m,p)+\varepsilon)\left(\log^{[p-1]} r\right)^{\frac{1}{\rho_g(m,p)}}\right\}$$
$$i.e., \ r \leq M_g^{-1}\left[\exp^{[m-1]}\left\{(\sigma_g(m,p)+\varepsilon)\left(\log^{[p-1]} r\right)^{\frac{1}{\rho_g(m,p)}}\right\}\right]$$
$$i.e., \ M_g^{-1}(r) \geq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\sigma_g(m,p)+\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right] \text{ and} \tag{5}$$

$$M_g^{-1}(r) \leq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\overline{\sigma}_g(m,p)-\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right] . \tag{6}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\sigma_g(m,p)-\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right] \text{ and} \tag{7}$$

$$M_g^{-1}(r) \geq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\overline{\sigma}_g(m,p)+\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right]. \quad (8)$$

From the definitions of $\overline{\tau}_f^{[m]}$ and $\tau_f^{[m]}$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp^{[m-1]}\left\{\left(\overline{\tau}_f^{[m]}+\varepsilon\right) r^{\lambda_f^{[m]}}\right\}, \quad (9)$$

$$M_f(r) \geq \exp^{[m-1]}\left\{\left(\tau_f^{[m]}-\varepsilon\right) r^{\lambda_f^{[m]}}\right\} \quad (10)$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp^{[m-1]}\left\{\left(\overline{\tau}_f^{[m]}-\varepsilon\right) r^{\lambda_f^{[m]}}\right\}, \quad (11)$$

$$M_f(r) \leq \exp^{[m-1]}\left\{\left(\tau_f^{[m]}+\varepsilon\right) r^{\lambda_f^{[m]}}\right\}. \quad (12)$$

Similarly from the definitions of $\overline{\tau}_g(m,p)$ and $\tau_g(m,p)$, it follows for all sufficiently large values of r that

$$M_g(r) \leq \exp^{[m-1]}\left\{(\overline{\tau}_g(m,p)+\varepsilon)\left(\log^{[p-1]} r\right)^{\lambda_g(m,p)}\right\}$$

$$i.e.,\ r \leq M_g^{-1}\left[\exp^{[m-1]}\left\{(\overline{\tau}_g(m,p)+\varepsilon)\left(\log^{[p-1]} r\right)^{\lambda_g(m,p)}\right\}\right]$$

$$i.e.,\ M_g^{-1}(r) \geq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\overline{\tau}_g(m,p)+\varepsilon)}\right)^{\frac{1}{\lambda_g(m,p)}}\right] \text{ and} \quad (13)$$

$$M_g^{-1}(r) \leq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\tau_g(m,p)-\varepsilon)}\right)^{\frac{1}{\lambda_g(m,p)}}\right]. \quad (14)$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\overline{\tau}_g(m,p)-\varepsilon)}\right)^{\frac{1}{\lambda_g(m,p)}}\right] \text{ and} \quad (15)$$

$$M_g^{-1}(r) \geq \exp^{[p-1]}\left[\left(\frac{\log^{[m-1]} r}{(\tau_g(m,p)+\varepsilon)}\right)^{\frac{1}{\lambda_g(m,p)}}\right]. \quad (16)$$

Now from (3) and in view of (13), we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_g^{-1} M_f(r) \geq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\sigma_f^{[m]}-\varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r)$$

$$\geq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]} \exp^{[m-1]}\left\{\left(\sigma_f^{[m]}-\varepsilon\right) r^{\rho_f^{[m]}}\right\}}{(\overline{\tau}_g(m,p)+\varepsilon)}\right)^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \geq \left[\frac{\left(\sigma_f^{[m]} - \varepsilon\right)}{\left(\overline{\tau}_g(m,p) + \varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\lambda_g(m,p)}} .$$

Since in view of Theorem 1, $\frac{\rho_f^{[m]}}{\lambda_g(m,p)} \geq \rho_g^{[p]}(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \geq \left[\frac{\sigma_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \sigma_g^{[p]}(f) \geq \left[\frac{\sigma_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} . \tag{17}$$

Similarly from (2) and in view of (16), it follows for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_g^{-1} M_f(r) \geq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r)$$

$$\geq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]} \exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\}}{\left(\tau_g(m,p) - \varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \geq \left[\frac{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right)}{\left(\tau_g(m,p) + \varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\lambda_g(m,p)}} .$$

Since in view of Theorem 1, it follows that $\frac{\rho_f^{[m]}}{\lambda_g(m,p)} \geq \rho_g^{[p]}(f)$ and $\varepsilon\,(>0)$ is arbitrary. Therefore we get from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \geq \left[\frac{\overline{\sigma}_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \sigma_g^{[p]}(f) \geq \left[\frac{\overline{\sigma}_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} . \tag{18}$$

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$\log^{[p-1]} M_g^{-1} M_f(r) \leq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\sigma_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r)$$

$$\leq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]} \exp^{[m-1]}\left\{\left(\sigma_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\}}{\left(\overline{\sigma}_g(m,p) - \varepsilon\right)}\right)^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \leq \left[\frac{\left(\sigma_f^{[m]} + \varepsilon\right)}{\left(\overline{\sigma}_g(m,p) - \varepsilon\right)} \right]^{\frac{1}{\rho_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\rho_g(m,p)}} . \qquad (19)$$

As in view of Theorem 1, it follows that $\frac{\rho_f^{[m]}}{\rho_g(m,p)} \leq \rho_g^{[p]}(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from (19) that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \leq \left[\frac{\sigma_f^{[m]}}{\overline{\sigma}_g(m,p)} \right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \sigma_g^{[p]}(f) \leq \left[\frac{\sigma_f^{[p]}}{\overline{\sigma}_g(m,p)} \right]^{\frac{1}{\rho_g(m,p)}} . \qquad (20)$$

Thus the theorem follows from (17), (18) and (20). □

The conclusion of the following corollary can be carried out from (6) and (9); (9) and (14) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Corollary 1. *Let f be an entire function with $0 < \lambda_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\sigma_g^{[p]}(f) \leq \min\left\{ \left[\frac{\overline{\tau}_f^{[m]}}{\tau_g(m,p)} \right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_f^{[m]}}{\overline{\sigma}(m,p)} \right]^{\frac{1}{\rho_g(m,p)}} \right\} .$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carried out the following theorem from pairwise inequalities numbers (10) and (13); (7) and (9); (6) and (12) respectively and therefore its proofs is omitted:

Theorem 3. *Let f be an entire function with $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\left[\frac{\tau_f^{[m]}}{\overline{\tau}_g(m,p)} \right]^{\frac{1}{\lambda_g(m,p)}} \leq \tau_g^{[p]}(f) \leq \min\left\{ \left[\frac{\tau_f^{[m]}}{\overline{\sigma}_g(m,p)} \right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\overline{\tau}_f^{[m]}}{\sigma_g(m,p)} \right]^{\frac{1}{\rho_g(m,p)}} \right\} .$$

Corollary 2. *Let f be an entire function with $0 < \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\tau_g^{[p]}(f) \geq \max\left\{ \left[\frac{\overline{\sigma}_f^{[m]}}{\sigma_g(m,p)} \right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\tau}_g(m,p)} \right]^{\frac{1}{\lambda_g(m,p)}} \right\} .$$

With the help of Theorem 1, the conclusion of the above corollary can be carry out from (2), (5) and (2), (13) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Theorem 4. *Let f be an entire function with $0 < \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} \leq \overline{\sigma}_g^{[p]}(f) \leq \min\left\{\left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\sigma_f^{[m]}}{\sigma_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}.$$

Proof. From (2) and in view of (13), we get for all sufficiently large values of r that

$$\log^{[p-1]} M_g^{-1} M_f(r) \geq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$\begin{aligned} &i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \\ &\geq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right) r^{\rho_f^{[m]}}\right\}}{\left(\overline{\tau}_g(m,p) + \varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}} \end{aligned}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \geq \left[\frac{\left(\overline{\sigma}_f^{[m]} - \varepsilon\right)}{\left(\overline{\tau}_g(m,p) + \varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\lambda_g(m,p)}}.$$

Now in view of Theorem 1, it follows that $\frac{\rho_f^{[m]}}{\lambda_g(m,p)} \geq \rho_g^{[p]}(f)$. Since $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \geq \left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \overline{\sigma}_g^{[p]}(f) \geq \left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}. \tag{21}$$

Further in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_g^{-1} M_f(r) \leq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\sigma_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$\begin{aligned} &i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \\ &\leq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\sigma_f^{[m]} + \varepsilon\right) r^{\rho_f^{[m]}}\right\}}{\left(\sigma_g(m,p) - \varepsilon\right)}\right)^{\frac{1}{\rho_g(m,p)}} \end{aligned}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \leq \left[\frac{\left(\sigma_f^{[m]} + \varepsilon\right)}{\left(\sigma_g(m,p) - \varepsilon\right)}\right]^{\frac{1}{\rho_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\rho_g(m,p)}}. \tag{22}$$

Again as in view of Theorem 1, $\frac{\rho_f^{[m]}}{\rho_g(m,p)} \leq \rho_g^{[p]}(f)$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from (22) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \leq \left[\frac{\sigma_f^{[m]}}{\sigma_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \overline{\sigma}_g^{[p]}(f) \leq \left[\frac{\sigma_f^{[p]}}{\sigma_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}. \tag{23}$$

Likewise from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_g^{-1} M_f(r) \leq \log^{[p-1]} M_g^{-1}\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]}+\varepsilon\right) r^{\rho_f^{[m]}}\right\}\right]$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r)$$

$$\leq \log^{[p-1]} \exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\overline{\sigma}_f^{[m]}+\varepsilon\right) r^{\rho_f^{[m]}}\right\}}{\left(\overline{\sigma}_g(m,p)-\varepsilon\right)}\right)^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]} M_g^{-1} M_f(r) \leq \left[\frac{\left(\overline{\sigma}_f^{[m]}+\varepsilon\right)}{\left(\overline{\sigma}_g(m,p)-\varepsilon\right)}\right]^{\frac{1}{\rho_g(m,p)}} \cdot r^{\frac{\rho_f^{[m]}}{\rho_g(m,p)}}. \tag{24}$$

Analogously, we get from (24) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[p]}(f)}} \leq \left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \overline{\sigma}_g^{[p]}(f) \leq \left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \tag{25}$$

since in view of Theorem 1, $\frac{\rho_f^{[m]}}{\rho_g(m,p)} \leq \rho_g^{[p]}(f)$ and $\varepsilon\,(>0)$ is arbitrary.

Thus the theorem follows from (21), (23) and (25). □

Corollary 3. *Let f be an entire function with $0 < \lambda_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\overline{\sigma}_g^{[p]}(f) \leq \min\left\{\begin{array}{c}\left[\frac{\tau_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \\ \left[\frac{\overline{\tau}_f^{[m]}}{\sigma_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\tau_f^{[m]}}{\overline{\sigma}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\end{array}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (6) and (12); (7) and (9); (12) and (14); (9) and (15) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carried out the following theorem from pairwise inequalities no (11) and (13) ; (10) and (16); (6) and (9) respectively and therefore its proofs is omitted:

Theorem 5. *Let f be an entire function with $0 < \lambda_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\max\left\{\left[\frac{\overline{\tau}_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\tau_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq \overline{\tau}_g^{[p]}(f) \leq \left[\frac{\overline{\tau}_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}.$$

Corollary 4. *Let f be an entire function with $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and g be an entire function with $0 < \lambda_g(m,p) \leq \rho_g(m,p) < \infty$ where p and m are positive integers such that $m \geq p$. Then*

$$\overline{\tau}_g^{[p]}(f) \geq \max\left\{\begin{array}{c}\left[\frac{\overline{\sigma}_f^{[m]}}{\overline{\sigma}_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\sigma_f^{[m]}}{\sigma_g(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \\ \left[\frac{\sigma_f^{[m]}}{\overline{\tau}_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\sigma}_f^{[m]}}{\tau_g(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\end{array}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (3) and (5) ; (2) and (8) ; (3) and (13); (2) and (16) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

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