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SOLUTION FOR COUPLED FRACTIONAL PDES WITH NON - CONSTANT COEFFICIENTS

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Abstract. In this article, it is shown that the combined use of exponential operators and integral transforms provides a powerful tool to solve a certain system of fractional PDEs. A system of space fractional partial differential equations is solved. It may be concluded that the integral transforms and exponential operators are effective methods for solving certain fractional linear equations with non-constant coefficients.

1. INTRODUCTION

Until now, two methods, have been more extensively used for solving PDEs, Laplace and Fourier transforms on the one hand and separation of variables on the other hand. Let us mention also solution in the form of a series of functions. We present a general method of operational nature to obtain exact solutions for several kinds of fractional partial differential equations.

Definition 1. The Laplace transform of function $f(t)$ is defined as [2]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt := F(s). \tag{1}$$

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{2}$$

where $F(s)$ is analytic in the region $\mathrm{R}e(s) > c$.

Definition 2. If the function $\Phi(t)$ belongs to $C[a,b]$ and $a < t < b$, the left Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I_a^{RL,\alpha}\{\Phi(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\Phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi. \tag{3}$$

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Definition 3. The left Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as follows [6]

$$D_a^{RL,\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^t \frac{\Phi(\xi)}{(t-\xi)^\alpha}d\xi, \tag{4}$$

it follows that $D_a^{RL,\alpha}\phi(x)$ exists for all $\Phi(t)$ belongs to $C[a,b]$ and $a < t < b$.
Note: A very useful fact about the R- L operators is that, they satisfy semi group properties of fractional integrals.
The special case of fractional derivative when $\alpha = 0.5$, is called semi - derivative.
Definition 4. The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\phi(t)$, is defined as[6]

$$D_a^{c,\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{5}$$

Let us recall some important properties of the Laplace transform, useful Lemmas, that will be considered in the next part of this article.
Lemma 1. Let $\mathcal{L}\{f(t)\} = F(s)$ then, the following identities hold true.

(1) $\mathcal{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})}\int_0^\infty e^{-t\xi-\frac{k^2}{4\xi}}d\xi,$
(2) $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega cos\beta\pi)}sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau-r\tau}d\tau)dr,$
(3) $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha cos\alpha\pi}sin(ur^\alpha sin\alpha\pi)drdu,$
(4) $\mathcal{L}^{-1}(F(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}f(u)du.$

Proof. See [1,2].
Lemma 2. The following exponential identities hold true.

(1) $\exp(\pm\lambda\frac{d}{dt})\Phi(t) = \Phi(t\pm\lambda),$
(2) $\exp(\pm\lambda t\frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda}),$
(3) $\exp(\lambda q(t)\frac{d}{dt})\Phi(t) = \Phi(Q(F(t)+\lambda)).$

Where $F(t)$ is primitive of $(q(t))^{-1}$, and $Q(t)$ is inverse of $F(t)$.
Proof. See [3,4,5].
The most important use of the Caputo fractional derivative is treated in initial value problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this type of derivative

$$\mathcal{L}\{D_a^{c,\alpha}f(t)\} = sF(s) - f(0+), 0 < \alpha < 1,$$

and generally [6]

$$\mathcal{L}\{D_a^{c,\alpha}f(t)\} = s^{\alpha-1}F(s) - \sum_{k=0}^{k=m-1-k} s^{\alpha-1-k}f^k(0+), m-1 < \alpha < m.$$

The Laplace transform provides a useful technique for the solution of fractional singular integro-differential equations.

Example 1. Let us solve the following fractional Volterra equation of convolution type

$$\lambda\int_k^{t+k}\sinh(a(t-\xi+k))D^\alpha\phi(\xi-k)d\xi = (\tfrac{t}{b})^{\frac{\mu}{2}}I_\mu(2\sqrt{bt}), \qquad \phi(k) = 0.$$

Solution. Upon taking the Laplace transform of the given integral equation, we obtain

$$s^{\alpha}\Phi(s)\frac{a\lambda}{(s^2-a^2)}=\frac{e^{\frac{b}{s}}}{s^{1+\mu}},$$

solving the above equation, leads to

$$\Phi(s)=\frac{(s^2-a^2)e^{\frac{b}{s}}}{(a\lambda)s^{1+\alpha+\mu}},$$

or equivalently

$$\Phi(s)=\frac{(s^2e^{\frac{b}{s}}-a^2e^{\frac{b}{s}})}{(a\lambda)s^{1+\alpha+\mu}},$$

at this point, taking the inverse Laplace transform term wise, after simplifying we obtain

$$\phi(t)=\frac{1}{a\lambda}(\frac{t-k}{b})^{\frac{\alpha+\nu-2}{2}}I_{\alpha+\mu-2}(2\sqrt{a(t-k)})-\frac{a}{\lambda}(\frac{t-k}{b})^{\frac{(\alpha+\mu)}{2}}I_{\alpha+\mu}(2\sqrt{b(t-k)}).$$

Where $I_{\eta}(.)$, stands for the modified Bessel's function of the first kind of order η.

Example 2. Let us solve the following impulsive fractional differential equation

$$D^{R.L,\alpha}y(t)+\lambda y(t)=\delta(t-\xi),0<\alpha<1.$$

Solution. The above fractional differential equation can be written as follows

$$y(t)=\frac{1}{\lambda+D^{R.L,\alpha}}\delta(t-\xi),$$

let us recall the following welll-known identity from Laplace transform of the exponential function

$$\frac{1}{\lambda+s^{\alpha}}=\int_0^{+\infty}e^{-\lambda u-s^{\alpha}u}du,$$

by choosing $s=D_t$, and using integral representation for exponential fraction, we get

$$y(t)=\int_0^{+\infty}du(e^{-\lambda u-uD_t^{\alpha}}\delta(t-\xi)),$$

at this point, in order to obtain the result of the action of the exponential operator on Dirac delta function, we may use part 2 of Lemma 1.1, to obtain

$$y(t)=\int_0^{+\infty}e^{-\lambda u}\frac{1}{\pi}\int_0^{\infty}e^{-r^{\alpha}(ucos\alpha\pi)}sin(ur^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau-\tau D_t}\delta(t-\xi))d\tau)dr)du,$$

after simplifying the inner integral, we arrive at

$$y(t)=\frac{1}{\pi}\int_0^{+\infty}e^{-\lambda u}(\int_0^{\infty}e^{-r(t-\xi)-r^{\alpha}(u\cos\alpha\pi)}\sin(ur^{\alpha}sin\alpha\pi)dr)du.$$

Let us consider the special case $\alpha=0.5$, the result after simplifying is

$$y(t)=\frac{1}{\pi}\int_0^{+\infty}e^{-\lambda u}(\int_0^{\infty}e^{-r(t-\xi)}\sin(u\sqrt{r})dr)du,$$

by changing the order of integration, we get

$$y(t)=\frac{1}{\pi}\int_0^{+\infty}\frac{\sqrt{r}e^{-(t-\xi)r}}{r+\lambda^2}dr,$$

hence, we deduce that

$$y(t)=e^{\lambda^2(t-\xi)}(1+\sqrt{2}Erf(\lambda(t-\xi))).$$

2. EVALUATION OF CERTAIN INTEGRALS VIA LAPLACE TRANSFORM

The Fourier and Laplace transforms are by far the most widely used of all integral transforms. The Laplace transform is especially well-suited for evaluation of the integrals and in the solution of certain boundary - value problems and in other applications.

Lemma 3. Let us assume that

$$\mathcal{L}\{\frac{I_\mu(\lambda t)}{t}\} = \int_0^\infty e^{-st}\frac{I_\mu(\lambda t)}{t}dt = \frac{((\sqrt{s^2-\lambda^2})+s)^{-\mu}}{\lambda^{-\mu}}, \tag{6}$$

then, we have the following integral identity

$$\int_0^\infty \frac{K_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}}{t}dt = -\frac{\pi\cosh\phi\mu}{\mu\sin\pi\mu}. \tag{7}$$

Where$K_\nu(.)$ stands for the modified Bessel's function of the second kind of order ν or Mac donald's function.

Proof. By definition of the Laplace transform of a modified Bessel's function of the second kind of order ν, we have

$$\mathcal{L}\{\frac{I_\mu(\lambda t)}{t}\} = \int_0^\infty \frac{I_\mu(\lambda t)e^{-st}}{t}dt = \frac{((\sqrt{s^2-\lambda^2})+s)^{-\mu}}{\lambda^{-\mu}}. \tag{8}$$

Let us introduce a change of parameter $s = \lambda\cosh\phi$ in the above integral and simplifying, we obtain

$$\int_0^\infty \frac{I_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}}{t}dt = \frac{e^{-\phi\mu}}{\mu}. \tag{9}$$

Using the well-known identity for the modified Bessel functions of the first and second kind as below

$$\frac{K_\mu(\lambda t)}{t} = (\frac{\pi}{2})\frac{I_{-\mu}(\lambda t)-I_\mu(\lambda t)}{tsin\mu\pi}. \tag{10}$$

Therefore, we have

$$\int_0^\infty \frac{K_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}}{t}dt = -\pi\frac{e^{\phi\mu}+e^{-\phi\mu}}{2\mu\sin\mu\pi} = -\frac{\pi\cosh\phi\mu}{\mu\sin\pi\mu}. \tag{11}$$

In relation (11), let us choose $\phi = 0$ to obtain

$$\int_0^\infty \frac{K_\mu(\lambda t)e^{-(\lambda)t}}{t}dt = -\pi lim_{\phi->0}\frac{e^{\phi\mu}+e^{-\phi\mu}}{2\mu\sin\mu\pi}, \tag{12}$$

finally, after simplifying we get

$$\int_0^\infty \frac{K_\mu(\lambda t)e^{-\lambda t}}{t}dt = -\frac{\pi}{\mu\sin\pi\mu}. \tag{13}$$

Corollary 1. The following identity holds true

$$\int_0^\infty \frac{K_\mu(\lambda t)}{t}dt = \frac{-2\pi}{\mu\sin(\frac{\pi\mu}{2})}.$$

Proof.

In relation (11), let us take the limit as $\phi -> \frac{i\pi}{2}$, we arrive at

$$\int_0^\infty \frac{K_\mu(\lambda t)}{t}dt = lim_{\phi -> \frac{i\pi}{2}}(\frac{\pi\cosh\mu\phi}{\mu\sin\mu\pi}) = \frac{-2\pi}{\mu\sin(\frac{\pi\mu}{2})}. \tag{14}$$

Corollary 2. The following identity holds true

$$\int_0^\infty \frac{Ai((1.5t)^{\frac{2}{3}})}{t^{\frac{4}{3}}}dt = -4\sqrt{3}\sqrt[3]{1.5}.$$

Proof. In the relation (14), let us put $\lambda = 1$, $\mu = \frac{1}{3}$ and using the following well - known identity for the modified Bessel's functions of the second kind and Airy function of first kind as below [7]

$$K_{\frac{1}{3}}(\xi) = \frac{\sqrt{3}\pi}{(1.5\xi)^{\frac{1}{3}}}Ai((1.5\xi)^{\frac{2}{3}}).$$

After substituting in (14), and simplifying we arrive at

$$\int_0^\infty \frac{Ai((1.5t)^{\frac{2}{3}})}{t^{\frac{4}{3}}}dt = -4\sqrt{3}\sqrt[3]{1.5}.$$

Lemma 4. Let us assume that

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\} = \int_0^\infty e^{-st}\frac{J_\mu(\lambda t)}{t}dt := \frac{((\sqrt{s^2+\lambda^2})+s)^{-\mu}}{\mu\lambda^{-\mu}}, \tag{15}$$

then, we have the following integral identities

$$\int_0^\infty \frac{Y_\mu(\lambda t)}{t}dt = \frac{cos\mu\pi - 1}{\mu\sin\mu\pi}. \tag{16}$$

$$\int_0^\infty \frac{Y_{\pm 0.5}(\lambda t)}{t}dt := -2. \tag{17}$$

Where$Y_\nu(.)$ stands for the Bessel's function of the second kind of order ν or Weber's function.

Proof. By definition of the Laplace transform, we have

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\} = \int_0^\infty \frac{J_\mu(\lambda t)e^{-st}}{t}dt = \frac{((\sqrt{s^2+\lambda^2})+s)^{-\mu}}{\mu\lambda^{-\mu}}. \tag{18}$$

Let us introduce a change of parameter $s = \lambda\sinh\phi$ in the above integral and simplifying, we obtain

$$\int_0^\infty \frac{J_\mu(\lambda t)e^{-(\lambda\sinh\phi)t}}{t}dt = \frac{e^{-\phi\mu}}{\mu}. \tag{19}$$

Using the well-known identity for the Besse'sl functions of the first and second kind as below

$$\frac{Y_\mu(\lambda t)}{t} = \frac{J_\mu(\lambda t)cos\mu\pi - J_{-\mu}(\lambda t)}{t\sin\pi\mu}. \tag{20}$$

Therefore, we have

$$\int_0^\infty \frac{Y_\mu(\lambda t)e^{-(\lambda \sinh \phi)t}}{t}dt = \frac{e^{-\phi\mu}\cos\mu\pi - e^{\phi\mu}}{\mu\sin\mu\pi}. \tag{21}$$

In relation (21), let us choose $\phi = 0$, to obtain

$$\int_0^\infty \frac{Y_\mu(\lambda t)}{t}dt = \frac{cos\mu\pi - 1}{\mu\sin\mu\pi}, \tag{22}$$

by setting $\mu = \pm 0.5$ in (22), we have

$$\int_0^\infty \frac{Y_{\pm.5}(\lambda t)}{t}dt = -2. \tag{23}$$

3. SOLUTION TO TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

In this section, the results which has been introduced are used to solve certain time fractional systems of equations. The author implemented the integral transform technique for solving partial fractional differential equations, where the fractional semi-derivative is in the Caputo sense.

Problem 1. Let us consider the following time fractional PDE, with the initial condition

$$\frac{\partial^{0.5}u(x,t)}{\partial t^{0.5}} + k\frac{\partial u(x,t)}{\partial x} = 0, \tag{24}$$

where $-\infty < x < \infty$, $t > 0$ and subject to the initial condition $u(x,0) = \phi(x)$.

Note: Fractional derivative is in the Caputo sense.

Solution: Let us define the joint Laplace - Fourier transform as following

$$\mathcal{F}\{\mathcal{L}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{i\omega x}\int_0^\infty e^{-st}u(x,t)dt)dx := U(\omega,s),$$

application of the joint Laplace - Fourier transform to (24) leads to the transformed problem

$$U(\omega,s) = \tfrac{s^{-0.5}\Phi(\omega)}{s^{0.5}+ik\omega},$$

upon inverting the joint Laplace - Fourier transform leads to

$$\mathcal{F}^{-1}\{\mathcal{L}^{-1}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}(\int_{c-i\infty}^{c+i\infty} \tfrac{s^{-0.5}\Phi(\omega)e^{st}}{s^{0.5}+ik\omega}ds)d\omega := u(x,t),$$

or, equivalently

$$u(x,t) = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)(\int_{c-i\infty}^{c+i\infty} \tfrac{e^{st}}{\sqrt{s}(\sqrt{s}+ik\omega)}ds)d\omega,$$

after calculation of inner integral we get the following formal solution

$$u(x,t) = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x-k^2\omega^2 t}Erfc(ik\omega\sqrt{t})\Phi(\omega)d\omega,$$

obviously, we have

$$u(x,0) = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)d\omega = \phi(x).$$

4. SOLUTION TO SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND NON - HOMOGENOUS FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

Fractional calculus has been used to model physical and engineering processes which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer order derivatives, including nonlinear models do not work adequately in many cases. In this section, the author implemented the exponential operational method for solving certain systems of space fractional partial differential equations with non-constant coefficients.
We also used the integral transform technique for solving non-homogeneous fractional differential equations, where the fractional derivative is in the Caputo sense.
Problem 2. Let us consider the following system of FDEs.

$$\begin{bmatrix} D^{c,\alpha}x(t) \\ D^{c,\alpha}y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \lambda \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}. \tag{25}$$

with the initial condition $\vec{X(0)} = \binom{e}{f}$.
Solution: Taking one dimensional Laplace transform of equation (25) with respect to t, and letting

$$\vec{F(s)} = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}, \tag{26}$$

we obtain

$$s^{\alpha}\vec{F(s)} - s^{\alpha-1}\vec{X(0)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \lambda \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}, \tag{27}$$

after simplifying, we get

$$\begin{bmatrix} s^{\alpha}x(s) \\ s^{\alpha}y(s) \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \lambda \begin{bmatrix} es^{\alpha-1} + G(s) \\ fs^{\alpha-1} + H(s) \end{bmatrix}, \tag{28}$$

or

$$\begin{bmatrix} s^{\alpha} - a & -b \\ -c & s^{\alpha} - d \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \lambda \begin{bmatrix} s^{\alpha-1}e + G(s) \\ s^{\alpha-1}f + H(s) \end{bmatrix}, \tag{29}$$

from the above relation, we get

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \frac{\lambda}{\Delta} \begin{bmatrix} s^{\alpha} - d & b \\ c & s^{\alpha} - a \end{bmatrix} \begin{bmatrix} s^{\alpha-1}e + G(s) \\ s^{\alpha-1}f + H(s) \end{bmatrix}, \tag{30}$$

we have also, $\Delta = s^{2\alpha} - (a+d)s^{\alpha} + (ad - bc)$ from relationship (30), we obtain the solution to the system

$$\mathcal{L}^{-1}\{x(s)\} = x(t) = \lambda\mathcal{L}^{-1}\left(\frac{es^{2\alpha-1} + (s^{\alpha} - d)G(s) + (bf - ed)s^{\alpha-1} + bH(s)}{s^{2\alpha} - (a+d)s^{\alpha} + (ad - bc)}\right), \tag{31}$$

$$\mathcal{L}^{-1}\{y(s)\} = y(t) = \lambda\mathcal{L}^{-1}\left(\frac{fs^{2\alpha-1} + (s^{\alpha} - a)H(s) + (ce - af)s^{\alpha-1} + cG(s)}{s^{2\alpha} - (a+d)s^{\alpha} + (ad - bc)}\right). \tag{32}$$

Example 3.
Let us consider the following case

$$\begin{bmatrix} D^{c,0.5}x(t) \\ D^{c,0.5}y(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \lambda \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}, \tag{33}$$

by setting the numerical values in relations (31), (32), we get the solution to the system as below

$$\mathcal{L}^{-1}\{x(s)\} = x(t) = \lambda\mathcal{L}^{-1}(\frac{1+(\sqrt{s}-2)G(s)-\frac{2}{\sqrt{s}}+H(s)}{s-6\sqrt{s}+6}), \tag{34}$$

$$\mathcal{L}^{-1}\{y(s)\} = y(t) = \lambda\mathcal{L}^{-1}(\frac{(\sqrt{s}-4)H(s)}{s-6\sqrt{s}+6}). \tag{35}$$

Let us start with the evaluation of (35), in order to find $y(t)$, we may re write (35) as follows

$$\mathcal{L}^{-1}\{y(s)\} = y(t) = \lambda\mathcal{L}^{-1}((\frac{(\sqrt{s}-4)}{s-6\sqrt{s}+6})H(s)) = \mathcal{L}^{-1}(\frac{(\sqrt{s}-4)}{s-6\sqrt{s}+6}) * \mathcal{L}^{-1}(H(s)). \tag{36}$$

At this point, we should evaluate the first term in the right hand side as below

$$\mathcal{L}^{-1}\{F(\sqrt{s})\} = \lambda\mathcal{L}^{-1}(\frac{(\sqrt{s}-4)}{s-6\sqrt{s}+6}), \tag{37}$$

from relation (37), we get the following

$$\mathcal{L}^{-1}\{F(s)\} = \lambda\mathcal{L}^{-1}(\frac{s-4}{s^2-6s+6}) = \lambda\mathcal{L}^{-1}(\frac{s-4}{(s-3)^2-(\sqrt{3})^2}), \tag{38}$$

from which we deduce that

$$\mathcal{L}^{-1}\{F(s)\} = \lambda\mathcal{L}^{-1}(\frac{s-4}{(s-3)^2-(\sqrt{3})^2}) = \lambda(e^{3t}\cosh(\sqrt{3}t) - 4e^{3t}sinh(\sqrt{3}t)), \tag{39}$$

in view of part 4 of Lemma (1.1), we have the following

$$\mathcal{L}^{-1}\{F(\sqrt{s})\} = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}\lambda(\exp(3u)cosh(\sqrt{3}u) - 4\exp(3u)sinh(\sqrt{3}u))du, \tag{40}$$

from relation (36), we get the solution

$$\mathcal{L}^{-1}\{y(s)\} = y(t) = \lambda h(t) * \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{12tu+u^2}{4t}}(e^{(\sqrt{3}u)} - 3sinh(\sqrt{3}u))du, \tag{41}$$

or,

$$\mathcal{L}^{-1}\{y(s)\} = \lambda\int_0^t \frac{h(t-\xi)}{2\xi\sqrt{\pi\xi}}(\int_0^\infty ue^{-\frac{12tu+u^2}{4\xi}}(e^{(\sqrt{3}u)} - 3sinh(\sqrt{3}u))du)d\xi. \tag{42}$$

Note: By following the same procedure as above, we can find $\mathcal{L}^{-1}x(s) = x(t)$.

Problem 3. Let us solve the following coupled space fractional PDE with non-constant coefficients, where the fractional derivative is in the Riemann-Liouville sense

$$t^{-b}\frac{\partial u(x,t)}{\partial t} - \beta t^k v(x,t) + \lambda(b+1)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = 0, \tag{43}$$

$$t^{-b}\frac{\partial v(x,t)}{\partial t} + \beta t^k u(x,t) + \lambda(b+1)\frac{\partial^\alpha v(x,t)}{\partial x^\alpha} = 0, \tag{44}$$

where $-\infty < x < \infty$, $t > 0$ and subject to the boundary conditions and the initial condition

$$u(x,0) = \phi(x), v(x,0) = \psi(x), -\infty < x < \infty.$$

Solution: Let us define the function $w(x,t)=u(x,t)+iv(x,t)$ and the initial condition $w(x,0)=\theta(x)$ we get the following space fractional partial differential equation

$$t^{-b}\frac{\partial w(x,t)}{\partial t}+it^{k}\beta w(x,t)+\lambda(b+1)\frac{\partial^{\alpha}w(x,t)}{\partial x^{\alpha}}=0, \tag{45}$$

with initial condition $w(x,0)=\theta(x)$. At this point, in order to solve the above linear space fractional PDE, we may rewrite the equation in the following exponential operator form

$$\frac{\partial w(x,t)}{\partial t}=-(i\beta t^{b+k}+\lambda(b+1)t^{b}\frac{\partial^{\alpha}}{\partial x^{\alpha}})w(x,t). \tag{46}$$

In order to obtain a solution for equation (4), first by solving the first order PDE with respect to t, and using the initial condition, we get the following

$$w(x,t)=\exp(\tfrac{-i\beta t^{b+k+1}}{b+k+1})\exp(-\lambda t^{b+1}\tfrac{\partial^{\alpha}}{\partial x^{\alpha}})\theta(x),$$

by virtue of Lemma 1.1, we have

$$w(x,t)=\tfrac{1}{\pi}\exp(\tfrac{-i\beta t^{b+k+1}}{b+k+1})\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}\sin(\lambda t^{b+1}r^{\alpha}\sin\alpha\pi)(\int_0^{\infty}(e^{-r\tau-\tau D_x}\theta(x))d\tau)dr,$$

finally, we obtain the solution to the system as below

$$w(x,t)=\tfrac{1}{\pi}\exp(\tfrac{-i\beta t^{b+k+1}}{b+k+1})\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau}\theta(x-\tau))d\tau)dr,$$

from which we obtain

$$u(x,t)=\cos(\tfrac{\beta t^{b+k+1}}{b+k+1})\tfrac{1}{\pi}\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau}\phi(x-\tau))d\tau)dr+$$

$$\sin(\tfrac{\beta t^{b+k+1}}{b+k+1})\tfrac{1}{\pi}\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}\sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau}\psi(x-\tau))d\tau)dr,$$

and

$$v(x,t)=\cos(\tfrac{\beta t^{b+k+1}}{b+k+1})\tfrac{1}{\pi}\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau}\psi(x-\tau))d\tau)dr-$$

$$\sin(\tfrac{\beta t^{b+k+1}}{b+k+1})\tfrac{1}{\pi}\int_0^{\infty}e^{-r^{\alpha}(\lambda t^{b+1}\cos\alpha\pi)}sin(\lambda t^{b+1}r^{\alpha}sin\alpha\pi)(\int_0^{\infty}e^{-r\tau}\phi(x-\tau))d\tau)dr.$$

Note: It is easy to verify that $u(x,0+)=\phi(x)$, $v(x,0+)=\psi(x)$.

5. Conclusions

Operational methods provide fast and universal mathematical tool for obtaining the solution of PDEs or even FPDEs. Combination of integral transforms, operational methods and special functions give more powerful analytical instrument for solving a wide range of engineering and physical problems. The paper is devoted to study the Laplace transform, exponential operators and their applications in solving certain systems of boundary value problems. The procedure as described above should be generally applicable to most partial fractional differential equations.

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