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INEQUALITIES FOR A CLASS OF FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRIC POINTS

DEEPAK BANSAL

ABSTRACT. The purpose of the present paper is to investigate a subordination theorem, boundedness properties associated with partial sums and an integral mean inequality for a class of functions starlike with respect to symmetric points.

1. INTRODUCTION

Let $\mathcal{S}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$, analytic and univalent in the open unit disk $\mathbb{U} = \{z; z \in \mathbb{C} : |z| < 1\}$, then $f(z)$ can be expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Consider the subclass $\mathcal{T}$ of the class $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

If the functions $g(z)$ and $h(z)$ belonging to the class $\mathcal{S}$ are, respectively, given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ and $h(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n$ then the Hadamard product (or convolution) denoted by $(g * h)(z)$ of the two functions $g(z)$ and $h(z)$ is defined by

$$(g * h)(z) = z + \sum_{n=2}^{\infty} b_n c_n z^n = (h * g)(z). \tag{1.3}$$

A domain $D \subset \mathbb{C}$ is convex if the line segment joining any two points in D lies entirely in D, while a domain is starlike with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D. A function $f \in \mathcal{S}$ is starlike if $f(\mathbb{U})$ is a starlike domain with respect to origin, and convex if $f(\mathbb{U})$ is convex. Analytically, $f \in \mathcal{S}$ if and only if $\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$, whereas $f \in \mathcal{S}$ is convex if and

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only if $\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0$. The classes consisting of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively. The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α, $0 \le \alpha < 1$, are respectively characterized by $\text{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha$ and $\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha$.

Let $\mathcal{S}_s^*$ be the subclass of $\mathcal{S}$ consisting of functions given by (1.1), satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0\ (z\in\mathbb{U}). \tag{1.4}$$

Function $f(z)\in\mathcal{S}_s^*$ are called starlike with respect to symmetric points and were introduced by Sakaguchi [2]. A subclass $\mathcal{S}_s^*(\alpha,\beta)$ of $\mathcal{S}_s^*$ of functions $f(z)$, regular and univalent in $\mathbb{U}$ given by (1.1) and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}+1\right|\ (z\in\mathbb{U}, 0\le\alpha\le1, 1/2<\beta\le1) \tag{1.5}$$

was introduced in [4]. Further, we let

$$\mathcal{TS}_s^*(\alpha,\beta)=\mathcal{S}_s^*(\alpha,\beta)\cap\mathcal{T} \tag{1.6}$$

The objective of the present paper is to investigate the integral means inequality, a subordination theorem and partial sums for the class $\mathcal{S}_s^*(\alpha,\beta)$. For this we need the following results:

Lemma 1.1. *A function of the form* (1.1) *is in*

$$\sum_{n=2}^{\infty}\psi(n;\alpha,\beta)|a_n|\le1, \tag{1.7}$$

where

$$\psi(n;\alpha,\beta)=\frac{n(1+\alpha\beta)+(\beta-1)[1-(-1)^n]}{\beta(2+\alpha)-1}\ (0\le\alpha\le1, 1/2<\beta\le1), \tag{1.8}$$

then $f(z)\in\mathcal{S}_s^*(\alpha,\beta)$.

Lemma 1.2. *A function of the form* (1.2) *is in* $\mathcal{TS}_s^*(\alpha,\beta)$ $(0\le\alpha\le1, 1/2<\beta\le1)$ *if and only if*

$$\sum_{n=2}^{\infty}\psi(n;\alpha,\beta)|a_n|\le1, \tag{1.9}$$

where $\psi(n;\alpha,\beta)$ *is given by* (1.8).

Lemma 1.1 and Lemma 1.2 were earlier proved by Rosy *et al.* [4].

From (1.8) it is easy to check that

$$\psi(n+1;\alpha,\beta)-\psi(n;\alpha,\beta)=\begin{cases}\frac{\alpha\beta+2\beta-1}{\beta(2+\alpha)-1}, n\ even\\ \frac{1+\alpha\beta+2(1-\beta)}{\beta(2+\alpha)-1}, n\ odd\end{cases} \tag{1.10}$$

which is positive for $0\le\alpha\le1, 1/2<\beta\le1$. Hence sequence (1.8) is non-decreasing sequence. Again $\psi(2;\alpha,\beta)=\frac{2(1+\alpha\beta)}{(\beta(2+\alpha)-1)}$ which is positive for $0\le\alpha\le1, 1/2<\beta\le1$, hence all the terms of sequence $\psi(n;\alpha,\beta)$ are positive. Similarly

$$\psi(n;\alpha,\beta)-n=\begin{cases}\frac{2n(1-\beta)}{\beta(2+\alpha)-1}, n\ even\\ \frac{2(n-1)(1-\beta)}{\beta(2+\alpha)-1}, n\ odd\end{cases} \tag{1.11}$$

which is positive for $0 \leq \alpha \leq 1, 1/2 < \beta \leq 1$. Hence all the terms of the sequence $\langle \psi(n;\alpha,\beta) - n \rangle_{n=2}^{\infty}$ are positive.

2. Integral Means Inequalities

The following subordination result due to Littlewood [1] will be required in our investigation.

Lemma 2.1. *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| g(re^{i\theta}) \right|^{\mu} d\theta, \tag{2.1}$$

where $\mu > 0,\ z = re^{i\theta}$ $(0 < r < 1)$.

Theorem 2.1. *Let $\mu > 0$. If $f(z) \in \mathcal{TS}_s^*(\alpha,\beta)$ $(0 \leq \alpha \leq 1, 1/2 < \beta \leq 1)$ is given by (1.2) then for $z = re^{i\theta}$ $(0 < r < 1)$:*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| f_1(re^{i\theta}) \right|^{\mu} d\theta, \tag{2.2}$$

where

$$f_1(z) = z - \frac{\beta(2+\alpha) - 1}{2(1+\alpha\beta)} z^2. \tag{2.3}$$

The proof of the above theorem is simple so we leave it here.

3. Subordination Theorem

Before stating and proving our subordination theorem, we need the following definition and a lemma due to Wilf [6].

Definition 3.1. *If $f, g \in \mathcal{H}$ where $\mathcal{H}$ denote the class of all holomorphic functions, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:*

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 3.2. *An infinite sequence $\{b_n\}_1^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{3.1}$$

is analytic, univalent and convex in $\mathbb{U}$, then

$$\sum_{n=1}^{\infty} a_n b_n z^n \subseteq f(z)\ (z \in \mathbb{U}, a_1 = 0). \tag{3.2}$$

Lemma 3.1. *The sequence $\{b_n\}_1^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0\ (z \in \mathbb{U}). \tag{3.3}$$

Theorem 3.1. *Let $f(z)$ of the form (1.1) satisfy the coefficient inequality (1.7), then*

$$\frac{1+\alpha\beta}{1+3\alpha\beta+2\beta}\left(f*g\right)(z)\prec g(z), \tag{3.4}$$

for every function $g(z)\in\mathcal{K}$ (Class of convex functions). In particular:

$$\Re\left\{f(z)\right\}>-\frac{1+3\alpha\beta+2\beta}{2(1+\alpha\beta)}(z\in\mathbb{U}). \tag{3.5}$$

The constant factor $\frac{1+\alpha\beta}{1+3\alpha\beta+2\beta}$ in the subordination result (3.4) cannot be replaced by any larger one.

Proof. Let $f(z)$ defined by (1.1) satisfy the coefficient inequality (1.7). In view of Definition 3.2, the subordination (3.4) will hold true if the sequence

$$\left\{\frac{1+\alpha\beta}{1+3\alpha\beta+2\beta}a_n\right\}_{n=1}^{\infty}\ (a_1=1)$$

is a subordinating factor sequence which by virtue of Lemma 3.1 is equivalent to the inequality

$$\Re\left\{1+2\sum_{n=1}^{\infty}\frac{(1+\alpha\beta)}{1+3\alpha\beta+2\beta}a_nz^n\right\}>0\ (z\in\mathbb{U}). \tag{3.6}$$

Now for $|z|=r(0<r<1)$, we obtain

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{2(1+\alpha\beta)}{1+3\alpha\beta+2\beta}a_nz^n\right\}=\Re\left\{1+\frac{2(1+\alpha\beta)}{1+3\alpha\beta+2\beta}+\sum_{n=2}^{\infty}\frac{2(1+\alpha\beta)}{1+3\alpha\beta+2\beta}a_nz^n\right\}$$

$$\geq 1-\frac{2(1+\alpha\beta)}{1+3\alpha\beta+2\beta}r-\sum_{n=2}^{\infty}\frac{n(1+\alpha\beta)+(\beta-1)[1-(-1)^n]}{1+3\alpha\beta+2\beta}|a_n|r^n$$

$$\geq 1-\frac{2(1+\alpha\beta)}{1+3\alpha\beta+2\beta}r-\frac{\beta(\alpha+2)-1}{1+3\alpha\beta+2\beta}r.$$

This evidently establishes the inequality (3.6) and consequently the subordination result (3.4) of Theorem 3.1 is proved. The assertion (3.5) follows readily from (3.4) when the function $g(z)$ is selected as

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n. \tag{3.7}$$

The sharpness of the multiplying factor in (3.4) can be established by considering a functions $h(z)$ defined by

$$h(z)=z-\frac{\beta(\alpha+2)-1}{1+3\alpha\beta+2\beta}z^2, \tag{3.8}$$

which belongs to the class $\mathcal{TS}_s^*(\alpha,\beta)$. Using (3.4), we infer that

$$\frac{1+\alpha\beta}{1+3\alpha\beta+2\beta}h(z)\prec\frac{z}{1-z},$$

and it follows that

$$\min_{|z|\leq 1}\left\{\mathrm{Re}\left(\frac{1+\alpha\beta}{1+3\alpha\beta+2\beta}h(z)\right)\right\}=-\frac{1}{2}.$$

This completes the proof. ☐

4. Partial Sums

In this section we investigate the ratio of real parts of functions involving (1.1) and its sequence of partial sums defined by

$$f_1(z) = z \text{ and } f_N(z) = z - \sum_{n=2}^{N} a_n z^n \ (\ for\ all\ n \in \mathbb{N}\ \{1\}), \tag{4.1}$$

and determine sharp lower bounds for $\Re\{f(z)/f_N(z)\}$, $\Re\{f_N(z)/f(z)\}$, $\Re\{f'(z)/f'_N(z)\}$ and $\Re\{f'_N(z)/f'(z)\}$.

Theorem 4.1. *Let $f(z)$ of the form* (1.1) *satisfy the coefficient inequality* (1.7), *then*

$$\Re\left(\frac{f(z)}{f_N(z)}\right) \geq 1 - \frac{1}{\psi\left(N+1;\alpha,\beta\right)}, \tag{4.2}$$

and

$$\Re\left(\frac{f_N(z)}{f(z)}\right) \geq \frac{\psi\left(N+1;\alpha,\beta\right)}{\psi\left(N+1;\alpha,\beta+1\right)} \tag{4.3}$$

where $\psi\left(N+1;\alpha,\beta\right)$ is given by (1.8). *The results are sharp for every N, with the extremal functions given by*

$$f(z) = z + \frac{1}{\psi\left(N+1;\alpha,\beta\right)} z^{N+1}\left(N \in \mathbb{N}\backslash\left\{1\right\}\right) \tag{4.4}$$

Proof. We prove (4.2) by setting

$$\begin{aligned} g(z) &= \psi\left(N+1;\alpha,\beta\right)\left\{\frac{f(z)}{f_N(z)} - \left(1 - \frac{1}{\psi\left(N+1;\alpha,\beta\right)}\right)\right\} \\ &= 1 + \frac{\psi\left(N+1;\alpha,\beta\right)\sum\limits_{n=N+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{N} a_n z^{n-1}}, \end{aligned}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\psi\left(N+1;\alpha,\beta\right)\sum\limits_{n=N+1}^{\infty}|a_n|}{2 - 2\sum\limits_{n=2}^{N}|a_n| - \psi\left(N+1;\alpha,\beta\right)\sum\limits_{n=N+1}^{\infty}|a_n|}$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$, if

$$\sum_{n=2}^{N}|a_n| + \psi\left(N+1;\alpha,\beta\right)\sum_{n=N+1}^{\infty}|a_n| \leq 1$$

In view of (1.7), this is equivalent to showing that

$$\sum_{n=2}^{N}\left(\psi\left(n;\alpha,\beta\right)-1\right)|a_n| + \sum_{n=N+1}^{\infty}\left(\psi\left(n;\alpha,\beta\right) - \psi\left(N+1;\alpha,\beta\right)\right)|a_n| \geq 0$$

Which is true in view of (1.10) and (1.11). Finally it can be verified that equality in (4.2) is attained for the function given by (4.4), when $z = re^{i\pi/N}$ and $r \to 1^-$. The proof of (4.3) is similar hence omitted here. □

Theorem 4.2. *Let $f(z)$ of the form (1.1) satisfy the coefficient inequality (1.7), then*

$$\left(\frac{f'(z)}{f'_N(z)}\right) \geq 1 - \frac{N+1}{\psi\left(N+1;\alpha,\beta\right)}, \tag{4.5}$$

and

$$\Re\left(\frac{f'_N(z)}{f'(z)}\right) \geq \frac{\psi\left(N+1;\alpha,\beta\right)}{N+1+\psi\left(N+1;\alpha,\beta\right)} \tag{4.6}$$

where $\psi\left(N+1;\alpha,\beta\right)$ is given by (1.8) *. The results are sharp for every N, with the extremal functions given by* (4.4)*.*

Proof. We prove (4.5) by setting

$$g(z) = \frac{\psi\left(N+1;\alpha,\beta\right)}{N+1}\left\{\frac{f'(z)}{f'_N(z)} - \left(1 - \frac{N+1}{\psi\left(N+1;\alpha,\beta\right)}\right)\right\} \tag{4.7}$$

$$= 1 + \frac{\frac{\psi(N+1;\alpha,\beta)}{N+1}\sum\limits_{n=N+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{N} na_n z^{n-1}}, \tag{4.8}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{\psi(N+1;\alpha,\beta)}{N+1}\sum\limits_{n=N+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{N} n\left|a_n\right| - \frac{\psi(N+1;\alpha,\beta)}{N+1}\sum\limits_{n=N+1}^{\infty} n\left|a_n\right|}. \tag{4.9}$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$, if

$$\sum_{n=2}^{N} n\left|a_n\right| + \frac{\psi\left(N+1;\alpha,\beta\right)}{N+1}\sum_{n=N+1}^{\infty} n\left|a_n\right| \leq 1$$

In view of (1.7), this is equivalent to showing that

$$\sum_{n=2}^{N}\left(\psi\left(n;\alpha,\beta\right)-n\right)\left|a_n\right| + \sum_{n=N+1}^{\infty}\left(\psi\left(n;\alpha,\beta\right) - \frac{\psi\left(N+1;\alpha,\beta\right)}{N+1}n\right)\left|a_n\right| \geq 0 \tag{4.10}$$

Which is true in view of (1.10) and (1.11). This completes the proof of (4.5). The proof of (4.6) is similar, hence omitted. □

References

[1] J.E. LITTLEWOOD, On in inequalities in the theory of functions, Proc. London Math. Soc., 23(1925), 481-519.

[2] K. SAKAGUCHI, On certain univalent mapping, J. Math. Soc. Japan, 11(1959), 72-75.

[3] H. SILVERMAN, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975), 109-116.

[4] THOMAS ROSY, K.G. SUBRAMANIAN AND T.V. SUDHARSAN, On fixed coefficients for a class of functions starlike with respect to symmetric points, Proceedings of the International Conference on Analysis and its Applications.

[5] T.V. SUDHARSAN, P. BALASUBRAHMANYAM AND K.G. SUBRAMANIAN, On functions starlike with respect to symmetric and conjugate points, Taiwanese J. Math., 2(1)(1998), 57-68.

[6] H. S., WILF, Subordinating factor sequences for convex maps of the unit circle, Proc. Amer. Math. Soc., 12(1961), 689-693.

Deepak Bansal
Department of Mathematics, Govt. College of Engineering and Technology, Bikaner-334004, Rajasthan, India
E-mail address: deepakbansal_79@yahoo.com