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NEW LAPLACE TRANSFORMS FOR THE $_2F_2$ GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we aim to show how one can obtain so far unknown Laplace transforms of three rather general cases of generalized hypergeometric function $_2F_2(a,b;c,d;x)$ by employing generalizations of Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem obtained earlier by Lavoie, Grondin and Rathie. The results established here may be useful in theoretical physics, engineering and mathematics.

1. Introduction

The Generalized hypergeometric function $_pF_q$ which is a natural generalization of Gauss's hypergeometric function $_2F_1$ with p numerator parameters and q denominator parameters is defined by (Rainville [1])

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right) = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty}\frac{(a_1)_n,\ldots,(a_p)_n}{(b_1)_n,\ldots,(b_q)_n}\frac{x^n}{n!}, \tag{1} $$

where $(a)_n$ denotes the well-known Pochhammer symbol (or the shifted or the raised factorial, since $(1)_n = n!$) defined for any complex number a by (see, for details, [2]; see also [3])

$$ \begin{aligned} (a)_n &:= \frac{\Gamma(a+n)}{\Gamma(a)} \quad \left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right) \\ &= \begin{cases} 1 & (n=0;\ a\in\mathbb{C}\setminus\{0\}) \\ a(a+1)\cdots(a+n-1) & (n\in\mathbb{N};\ a\in\mathbb{C}). \end{cases} \end{aligned} \tag{2} $$

The series (1) converges for all $|z|<\infty$ if $p\le q$ and for $|z|<1$ if $p=q+1$ while it is divergent for all z, $z\neq 0$ if $p>q+1$. When $|z|=1$ with $p=q+1$, the series (1) converges absolutely if $Re(\sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{p} a_j) > 0$ conditionally convergent if $-1 < Re(\sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{p} a_j) \le 0, z\neq 1$ and divergent if $Re(\sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{p} a_j) \le -1$.

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It should be remarked here that whenever hypergeometric function ${}_2F_1$ or generalized hypergeometric function ${}_pF_q$ reduce to gamma function, the results are very important from the applications point of view. Thus the classical summation theorems such as those of Gauss, Gauss second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play important role. Recently some authors established certain integral transforms and fractional integral Formulas for the extended hypergeometric functions (see [13] and [14]).

During 1992-96, in a series of three research papers, Lavoie *et al.* ([4],[5],[6]) have generalized the above mentioned classical summation theorems and obtained a large number of very interesting special as well as limiting cases of their results.

However, in our present investigations, we shall mention below the generalizations of Gauss's second summation theorem,Bailey's summation theorem and Kummer's summation theorem.

Theorem 1. *Generalization of Gauss's second summation theorem ([6])*

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; \frac{1}{2}\right) &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2})\Gamma(\frac{a}{2}-\frac{b}{2}-\frac{i}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}-\frac{b}{2}+\frac{|i|}{2}+\frac{1}{2})} \\ &\times \left\{ \frac{A_i(a,b)}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{i}{2}+\frac{1}{2}-\left[\frac{1+i}{2}\right])} + \frac{B_i(a,b)}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2}+\frac{i}{2}-\left[\frac{i}{2}\right])} \right\} \end{aligned} \tag{3}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

Theorem 2. *Generalization of Bailey's summation theorem ([6])*

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, 1-a+i \\ b \end{matrix}; \frac{1}{2}\right) &= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{|i|}{2}+\frac{i}{2})} \\ &\times \left\{ \frac{C_i(a,b)}{\Gamma(\frac{b}{2}-\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{a}{2}-\left[\frac{1+i}{2}\right])} + \frac{D_i(a,b)}{\Gamma(\frac{b}{2}-\frac{a}{2})\Gamma(\frac{b}{2}+\frac{a}{2}-\frac{1}{2}-\left[\frac{i}{2}\right])} \right\} \end{aligned} \tag{4}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

Theorem 3. *Generalization of Kummer's summation theorem ([6])*

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ 1+a-b+i \end{matrix}; -1\right) &= \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{|i|}{2}+\frac{i}{2})} \\ &\times \left\{ \frac{E_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+1)\Gamma(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+\frac{i}{2}-\lfloor\frac{i}{2}\rfloor)} \right\} \end{aligned} \tag{5}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

In all these results, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficient are given in Tables 1-3.

On the other hand , we define the (direct) Laplace transform of a function $f(t)$ of a real variable t as the integral $g(s)$ over a range of complex parameters s, whenever this integral exists in the Lebesgue sense by the following integral

$$g(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)\, dt \tag{6}$$

For more details, we refer [7] and [8].

We next mention the following Laplace transform of a generalized hypergeometric function ${}_pF_q$ ([9])

$$\int_0^\infty e^{-st}t^{\nu-1}{}_pF_q\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array};wt\right)\,dt=\Gamma(\nu)s^{-\nu}{}_{p+1}F_q\left(\begin{array}{c}\nu,a_1\ldots,a_p\\ b_1,\ldots,b_q\end{array};\frac{w}{s}\right) \tag{7}$$

for $p\le q$, provided $Re(\nu)>0$ and $Re(s)>\max\{Re(w),0\}$.

The result (7) can be derived with the help of the well-known formula for Gamma function

$$\int_0^\infty e^{-st}t^{\alpha-1}\,dt=\Gamma(\alpha)s^{-\alpha} \tag{8}$$

which is valid when $Re(s)>0$ and $Re(\alpha)>0$.

so we prefer to omit the detail. However, here we would like to mention that the interchange of order of integration and summation easily seen to be justified due to the uniform convergence of the series (1).

Now in particular when $p=q=2$, for generalized hypergeometric function we say that its Laplace transform would be

$$\int_0^\infty e^{-st}t^{d-1}{}_2F_2\left(\begin{array}{c}a,b\\ c,d\end{array};wt\right)\,dt=\Gamma(d)s^{-d}{}_2F_1\left(\begin{array}{c}a,b\\ c\end{array};\frac{w}{s}\right) \tag{9}$$

provided $Re(d)>0$ and $Re(s)>\max\{Re(w),0\}$, which is recorded in [9].

It is not out of place to mention here that the result (9) is the most general case, so it would be of some interest to find, as far as possible, less general cases involving various particular values of the parameters a,b, c and d.Several special cases can be seen in the standard texts in [9], [10] and [11].

We conclude this section by remarking that in the next section we shall mention three new and interesting Laplace transforms which we believe that are not recorded in any standard tables of Laplace transforms.

2. New Laplace Transforms of Special ${}_2F_2(a,b;c,d;x)$

In this section the following three new general Laplace transforms of special ${}_2F_2(a,b;c,d;x)$ will be established. These are each for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

$$\int_0^\infty e^{-st}t^{d-1}{}_2F_2\left(\begin{array}{c}a,b\\ d,\frac{1}{2}(a+b+i+1)\end{array};\frac{ts}{2}\right)\,dt=s^{-d}\frac{\Gamma(\frac{1}{2})\Gamma(d)\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2})\Gamma(\frac{a}{2}-\frac{b}{2}-\frac{i}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}-\frac{b}{2}+\frac{|i|}{2}+\frac{1}{2})}$$
$$\times\left\{\frac{A_i(a,b)}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{i}{2}+\frac{1}{2}-\left[\frac{1+i}{2}\right])}+\frac{B_i(a,b)}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2}+\frac{i}{2}-\left[\frac{i}{2}\right])}\right\} \tag{10}$$

provided $Re(d) > 0$ and $Re(s) > 0$ and the coefficient $A_i(a,b)$ and $B_i(a,b)$ are given in table-1.

$$\int_0^\infty e^{-st}t^{d-1}{}_2F_2\left(\begin{matrix} a, 1-a+i \\ d, c \end{matrix}; \frac{ts}{2}\right) dt = s^{-d}\frac{\Gamma(\frac{1}{2})\Gamma(d)\Gamma(c)\Gamma(1-a)}{2^{c-i-1}\Gamma(1-a+\frac{i}{2}+\frac{|i|}{2})}$$
$$\times\left\{\frac{C_i(a,c)}{\Gamma(\frac{c}{2}-\frac{a}{2}+\frac{1}{2})\Gamma(\frac{c}{2}+\frac{a}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{D_i(a,c)}{\Gamma(\frac{c}{2}-\frac{a}{2})\Gamma(\frac{c}{2}+\frac{a}{2}-\frac{1}{2}-\lfloor\frac{i}{2}\rfloor)}\right\} \tag{11}$$

provided $Re(d) > 0$ and $Re(s) > 0$ and the coefficient $C_i(a,c)$ and $D_i(a,c)$ can be easily be obtained from the table-2 by simply changing b to c.

$$\int_0^\infty e^{-st}t^{d-1}{}_2F_2\left(\begin{matrix} a, b \\ d, 1+a-b+i \end{matrix}; -st\right) dt = s^{-d}\frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(d)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{i}{2}+\frac{|i|}{2})}$$
$$\times\left\{\frac{E_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+1)\Gamma(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+\frac{i}{2}-\lfloor\frac{i}{2}\rfloor)}\right\} \tag{12}$$

provided $Re(d) > 0$ and $Re(s) > 0$ and the coefficient $E_i(a,b)$ and $F_i(a,b)$ are given in table-3.

Proofs: The proofs of the results (10) to (12) are quite straight forward. For this, if we set $w = \frac{s}{s}$, $c = \frac{1}{2}(a+b+i+1)$ and $b = 1-a+i$ in (9) each for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, then the series ${}_2F_2(\frac{1}{2})$ on the right-hand side of (9) can be summed by using the summation formulas (3) and (4) respectively. We get after some simplification, the results (10) and (11) respectively.

Similarly if we set in $w = -s$ and $c = 1+a-b+i$ for $i = 0, 1, 2, 3, 4, 5$ in (9), then the resulting series ${}_2F_2(-1)$ on the right-hand side of (9) can be summed by using the summation formula (5) to get the result (12).

2.1. **Special Cases.** In the results (10), (11) and (12), if we put $d = b$, $d = a+i$ and $d = b$, then for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ respectively, we get three general classes of Laplace transforms of Kummer's confluent hypergeometric function ${}_1F_1(a;b;x)$ obtained recently by Kim *et al.* [12].

In the result (12), if we put $d = a$, we get the following special case of our result which is also believed to be new.

$$\int_0^\infty e^{-st}t^{a-1}{}_1F_2\left(\begin{matrix} b \\ 1+a-b+i \end{matrix}; -st\right) dt = s^{-a}\frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(a)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{i}{2}+\frac{|i|}{2})}$$
$$\times\left\{\frac{E_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+1)\Gamma(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{a}{2}-b+\frac{i}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+\frac{i}{2}-\lfloor\frac{i}{2}\rfloor)}\right\} \tag{13}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$
provided $Re(d) > 0$ and $Re(s) > 0$.
Similarly other results can be obtained.

Table 1

i	$A_i(a,b)$	$B_i(a,b)$
5	$-(a+b+6)^2+\frac{1}{2}(b-a+6)(b+a+6)+\frac{1}{4}(b-a+6)^2+11(b+a+6)-\frac{13}{2}(b-a+6)+20$	$(a+b+6)^2+\frac{1}{2}(b-a+6)(b+a+6)-\frac{1}{4}(b-a+6)^2-17(b+a+6)-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(a+b+1)(a+b-3)-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a-1)$
3	$\frac{1}{2}(b-a+4)-(b+a+4)+3$	$\frac{1}{2}(b-a+4)+(b+a+4)-7$
2	$\frac{1}{2}(b+a+3)-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
-4	$\frac{1}{2}(a+b-3)(a+b+1)-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$
-5	$(b+a-4)^2-\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+4(b+a-4)-\frac{7}{2}(b-a-4)$	$(b+a-4)^2+\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+8(b+a-4)-\frac{1}{2}(b-a-4)+12$

Table 2

i	$C_i(a,b)$	$D_i(a,b)$
5	$-(4b^2-2ab-a^2-22b+13a+20)$	$4b^2+2ab-a^2-34b-a+62$
4	$2(b-2)(b-4)-(a-1)(a-4)$	$-4(b-3)$
3	$a-2b+3$	$a+2b-7$
2	$b-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	b	2
-3	$2b-a$	$a+2b+2$
-4	$2b(b+2)-a(a+3)$	$4(b+1)$
-5	$4b^2-2ab-a^2+8b-7a$	$4b^2+2ab-a^2+16b-a+12$

Table 3

i	$E_i(a,b)$	$F_i(a,b)$
5	$-4(6+a-b)^2+2b(6+a-b)+b^2-22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2-34(6+a-b)-b+62$
4	$2(a+b-3)(a-b+1)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2+16(a-b-4)-b+12$

References

[1] E. D. Rainville, Special Functions, The Macmillan company, New York, 1960.

[2] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[3] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam-London-New York, 2012.

[4] J. L. Lavoie, F. Grondin and A. K. Rathie, Generalizations of Watson's theorem on the sum of a $_3F_2$, Indian J.Math, Vol. 34, 23-32, 1992.

[5] J. L. Lavoie, F. Grondin, A. K. Rathie and K. Arora, Generalizations of Dixon's theorem on the sum of a $_3F_2$, Math. Comput., Vol. 62, 267-276, 1994.

[6] J. L. Lavoie, F. Grondin and A. K. Rathie, Generalizations of Whipple's theorem on the sum of a $_3F_2$, J. Comput. Appl. Math, Vol. 72, 293-300, 1996.

[7] B. Davis, Integral Transforms and their applications Third ed, Springer, New York, 2002.

[8] G.Doetsch, Introduction to theory and applications of the Laplace transforms, Springer, New York, 1974.

[9] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, Integrals and Series:Direct Laplace Transforms Vol.4, Grodon and Breach Science Publishers, New York, 1992.

[10] A Erdelyi *et al.*, Tables of Integral Transforms ,Vol 1 and 2, McGraw Hills, New York, 1954.

[11] F. Oberhettinger and L. Badi, Tables of Laplace Transforms, Springer- Berlin, 1973.

[12] Y. S. Kim, A. K. Rathie and D.Cvijovic, New Laplace transforms of Kummer,s confluent hypergeometric functions, Math. Comput. Modelling, Vol. 50, 1068-1071, 2012.

[13] J. Choi and P. Agarwal, Certain Integral Transform and Fractional Integral Formulas for the Generalized Gauss Hypergeometric Functions, Abstract and Applied Analysis, vol. 2014, Article ID 735946, 7 pages, 2014. doi:10.1155/2014/735946.

[14] P. Agarwal, J. Choi, K. B. Kachhia, J. C. Prajapati and H. Zhou, Some Integral Transforms and Fractional Integral Formulas for the Extended Hypergeometric Functions, Communications of Korean Mathematical Society, 31(3), 591–601, 2016.

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