Journal of Fractional Calculus and Applications
Vol. 8(2) July 2017, pp. 166-181.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

A FRACTIONAL BLOCK PULSE OPERATIONAL METHOD FOR SOLVING A CLASS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we present a solution method for fractional partial differential equations. The proposed technique utilizes Block Pulse Functions (BPFs) operational matrices method in conjunction with Tau technique. The error analysis includes the error in the approximation of BPFs, the estimation of the error bound and the estimation of the error function for the proposed method. Numerical examples are provided to illustrate the efficiency and accuracy of the applying technique.

1. Introduction

Fractional differential equations are generalized from integer ordered ones. The order of such equations is fractional. These equations are more accurate in the case of natural physical processes and dynamical systems [3, 12]. It is noticeable that many researchers in diverse fields of science and engineering study the fractional calculus and employ the fractional equations in order to tackle the problems of modeling and controlling of many dynamical systems [2, 24]. As a remarkable example, the fractional calculus is applied to the fluid-dynamic traffic, the continuum and statistical mechanics, the frequency dependent damping behavior of many viscoelastic materials, the colored noise, the economics, the control theory and the signal processing [28]. There are a wide variety of approaches for solving fractional differential equations. The most commonly used ones are Variational Iteration Method [21], Adomian Decomposition Method [5, 8], Generalized Differential Transform Method [14, 15, 16], Operational Matrix Method [19, 20], Finite Difference Method [27] and Wavelet Method [4, 6, 22, 23].
During this article, we study a class of fractional partial differential equations in the form of Eq. (1).

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial^\beta u}{\partial x^\beta} + \lambda u(x,t) + g(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{1}$$

2010 *Mathematics Subject Classification.* 26A33, 65M70.
Key words and phrases. Block pulse functions, Operational matrix, Fractional partial differential equations, Sylvester equation, Error analysis, Numerical solution, Upper bound.
Submitted Oct. 27, 2016.

subject to the initial conditions

$$u(0,t) = p(t), \quad u(x,0) = v(x), \tag{2}$$

where $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ and $\frac{\partial^\beta u(x,t)}{\partial t^\beta}$ are the fractional derivative in Caputo sense, $g(x,t)$ is the known continuous function, $u(x,t)$ is the unknown function, $0 < \alpha$ and $\beta \leq 1$.

In recent years, a wide range of basic functions have been employed so as to estimate the solutions of fractional partial differential equations such as orthogonal functions and wavelets. The orthogonal functions are classified as

(1) Piecewise Constant Orthogonal Functions (PCOF) such as Walsh, Block-Pulse and Haar,
(2) Orthogonal polynomials such as Legendre, Laguerre and Chebyshev,
(3) Sine-Cosin functions in the Fourier series.

For more information on this classification, see [1, 9]. Here, we solve Eq. (1) by using BPFs.

The present article is organized as follows. In section 2, we conduct a review of the fractional calculus theory that is fundamental to our work. In section 3, Block pulse functions and their properties are studied. Furthermore, the operational matrix of the fractional integration of the block pulse functions are presented. The mathematical formulation of a fractional partial differential equation is described in Section 4. In section 5, the error analysis is discussed. Additionally, the numerical solutions are studied in Section 6. Finally, the article is concluded in Section 7.

2. Fractional calculus

In this section, we undertake a review of requiste definitions and preliminaries of the fractional calculus theory that are essential to study the present subject. For further details, see [10, 13, 17].

Definition 2.1 The Riemann-Liouville fractional integral operator I^α of order α, $\alpha \geq 0$, for function $u(t)$ is given by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, \mathrm{d}s \qquad \alpha > 0, \tag{3}$$

$$I^0 u(t) := u(t). \tag{4}$$

Definition 2.2 The Caputo fractional derivative operator D^α of order α, $\alpha \geq 0$, for function $u(t)$ is defined as

$$D_\star^\alpha u(t) := \begin{cases} \frac{\mathrm{d}^r u(t)}{\mathrm{d}t^r} & \alpha = r \in \mathbb{N}, \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{u^r(s)}{(t-s)^{\alpha-r+1}} \mathrm{d}s & 0 \leq r-1 < \alpha < r. \end{cases} \tag{5}$$

The relation between the Riemann-Liouville operator and Caputo fractional derivative is given by the following expressions

$$D_\star^\alpha I^\alpha u(t) = u(t), \tag{6}$$

$$I^\alpha D_\star^\alpha u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{(t)^k}{k!} \qquad t > 0. \tag{7}$$

3. Block pulse functions (BPFs)

Definition 3.1 For a given positive integer m, BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \\ 0 & otherwise, \end{cases} \tag{8}$$

where $i = 1, 2, \cdots, m$ and $h = \frac{1}{m}$. Some properties of BPFs are given in Proposition 3.1.

Proposition 3.1([7]). For $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, m$ we have the following statements.
$supp\{b_i(x)\} = [\frac{i-1}{m}, \frac{i}{m}]$.
Disjointness:

$$b_i(t)b_j(t) = \begin{cases} b_i(t) & i = j, \\ 0 & i \neq j. \end{cases} \tag{9}$$

Orthogonality:

$$\int_0^1 b_i(t)b_j(t) = \begin{cases} h & i = j, \\ 0 & i \neq j. \end{cases} \tag{10}$$

Completeness: For every $f \in L^2([0,1))$ whenever m escapes to the infinity, Parseval's identity holds.

$$\int_0^1 f^2(x)\, \mathrm{d}x = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2, \tag{11}$$

where

$$f_i = \frac{1}{h}\int_0^1 f(x)b_i(x)\, \mathrm{d}x. \tag{12}$$

Every function $f(x) \in L^2([0,1))$ can be expressed as

$$f(x) \cong \sum_{i=1}^{m} f_i b_i(x) = f^T B_m(x), \tag{13}$$

where $f = [f_1, f_2, \cdots, f_m]^T$ and $B_m(x) = [b_1(x), b_2(x), \cdots, b_m(x)]^T$ such that f_i are defined as in (12) for $i = 1, 2, \cdots, m$.
Remark 3.1 Every two dimensional function $u(x,t) \in L^2([0,1) \times [0,1))$ can be expressed as

$$u(x,t) \cong \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} u_{i_1,i_2} b_{i_1}(x) b_{i_2}(t) = B^T(x) U B(t), \tag{14}$$

where $U = [u_{i_1,i_2}]$, $h_1 = \frac{1}{m_1}$, $h_2 = \frac{1}{m_2}$ and we have

$$u_{i_1,i_2} = \frac{1}{h_1 h_2}\int_0^1\int_0^1 u(x,t) b_{i_1}(x) b_{i_2}(t)\, \mathrm{d}x\, \mathrm{d}t, \tag{15}$$

$$B(x) = [b_1(x), \cdots, b_{m_1}(x)]^T \quad \text{and} \quad B(t) = [b_1(t), \cdots, b_{m_2}(t)]^T. \tag{16}$$

3.1. **BPFs-operational matrix of the fractional integration.**

In this part, we introduce the operational matrix of the fractional integration of the block pulse functions.

Definition 3.2([11]). α-Fractional integration order of the BPFs-vector can be expressed by themselve as

$$I^{\alpha}B(x) \cong P_{\alpha}B(x),$$

where

$$P_{\alpha} = \left(\frac{1}{m}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{m-1} \\ 0 & 1 & \epsilon_1 & \cdots & \epsilon_{m-2} \\ 0 & 0 & 1 & \cdots & \epsilon_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $\epsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Here, P_{α} is called the block pulse operational matrix of the fractional integration.

4. The solution of the fractional partial differential equation

In this section, we assume that $m_1 = m_2 = m$. Consider the fractional partial differential equation given by Eq. (1). We approximate the function $\frac{\partial^{\beta}u}{\partial x^{\beta}}$ by the BPFs as

$$\frac{\partial^{\beta}u}{\partial x^{\beta}} \cong B^T(x)UB(t). \tag{17}$$

By applying the operator I_x^{β} on Eq. (17) and using Eq. (7), we have

$$I_x^{\beta}(\frac{\partial^{\beta}u}{\partial x^{\beta}}) \cong I_x^{\beta}[B^T(x)UB(t)] = u(x,t) - u(0,t). \tag{18}$$

Making use of operational matrix P_{α}, we get

$$u(x,t) \cong p(t) + B^T(x)P_{\beta}^T UB(t). \tag{19}$$

Now, approximating $p(t)$ by $B^T(x)XB(t)$ results that

$$u(x,t) \cong B^T(x)[X + P_{\beta}^T U]B(t). \tag{20}$$

Hence, by substituting Eqs. (17) and (20) in Eq. (1), we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} \cong -B^T(x)UB(t) + \lambda B^T(x)[X + P_{\beta}^T U]B(t) + B^T(x)GB(t), \tag{21}$$

where

$$g(x,t) \cong B^T(x)GB(t).$$

By applying the operator I_t^{α} and considering Eq. (7), we have

$$u(x,t) \cong B^T(x)[-U + G + \lambda(X + P_{\beta}^T U)]P_{\alpha}B(t) - v(x). \tag{22}$$

Perceiving Eqs. (20) and (22) and utilizing the orthogonal property of BPFs, we have

$$\begin{aligned}[X + P_{\beta}^T U] &= [-U + G + \lambda(X + P_{\beta}^T U)]P_{\alpha} - V \\ &= [(-I + \lambda P_{\beta}^T)U + \lambda X + G]P_{\alpha} - V \\ &= (-I + \lambda P_{\beta}^T)UP_{\alpha} + (\lambda X + G)P_{\alpha} - V,\end{aligned} \tag{23}$$

where $v(x) \cong B^T(x)VB(t)$. Finally, from Eq. (23), we obtain

$$(I - \lambda P_{\beta}^T)^{-1}P_{\beta}^T U + UP_{\alpha} + (I - \lambda P_{\beta}^T)^{-1}[X + V - (G + \lambda\ X)P_{\alpha}] = 0, \tag{24}$$

that is a system of equations called Sylvester equation.

By solving Sylvester equation (24) for unknown matrix U, we have an approximate function as Eq. (19).

5. Error Analysis

5.1. **Error in BPFs approximation. Theorem 5.1** Let $D \subset R^2$ be an open convex set, $u : D \longrightarrow R$ be a differentiable function and there exists a real number M such that

$$\|\frac{\partial u(x,t)}{\partial x}\| \leq M.$$

then

$$|u(b,t) - u(a,t)| \leq M|b-a|, \quad \forall (a,t) \in D, (b,t) \in D.$$

Proof. See [18].

Here, we obtain the representation of error when a differentiable function $u(x,t)$ is represented in a series of 2D-BPFs over the region $D = [0,1) \times [0,1)$ as Eq. (14). We put $m_1 = m_2 = m$, so $h_1 = h_2 = \frac{1}{m}$. We define the representation of error between $u(x,t)$ and its 2D-BPFs expansion, $u_m(x,t)$, over every subregion D_{i_1,i_2} as follows

$$e_{i_1,i_2} = u_{i_1,i_2} b_{i_1,i_2}(x,t) - u(x,t) = u_{i_1,i_2} - u(x,t), \qquad (x,t) \in D_{i_1,i_2},$$

where

$$D_{i_1,i_2} = \{(x,t) \big| \frac{i_1 - 1}{m} \leq x < \frac{i_1}{m}, \frac{i_2 - 1}{m} \leq t < \frac{i_2}{m}\}.$$

Thus, we have

$$\|e_{i_1,i_2}\|^2 = \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}} \int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} e^2_{i_1,i_2}(x,t)\,\mathrm{d}x\mathrm{d}t = \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}} \int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} (u_{i_1,i_2} - u(x,t))^2\,\mathrm{d}x\mathrm{d}t.$$

By using integral mean value theorem, there exist η_1 and η_2 such that

$$\|e_{i_1,i_2}\|^2 = \frac{1}{m^2}(u_{i_1,i_2} - u(\eta_1,\eta_2))^2, \qquad (\eta_1,\eta_2) \in D_{i_1,i_2}. \tag{25}$$

Now, by Eq. (15) and using the mean value theorem, we have

$$u_{i_1,i_2} = m^2 \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}} \int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} u(x,t)\,\mathrm{d}x\mathrm{d}t = m^2 \frac{1}{m^2} u(\xi_1,\xi_2) = u(\xi_1,\xi_2), \qquad (\xi_1,\xi_2) \in D_{i_1,i_2}. \tag{26}$$

By substituting Eq. (26) into Eq. (25) and by Theorem 5.1, we have

$$\|e_{i_1,i_2}\|^2 = \frac{1}{m^2}(u(\xi_1,\xi_2) - u(\eta_1,\eta_2))^2 \leq \frac{1}{m^2} M^2((\xi_1 - \xi_2) - (\eta_1 - \eta_2))^2 \leq 2\frac{M^2}{m^2}.$$

Therefore,

$$\begin{aligned} \|e(x,t)\|^2 &= \int_0^1 \int_0^1 e^2(x,t)\,\mathrm{d}x\mathrm{d}t = \int_0^1 \int_0^1 (\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} e_{i_1,i_2}(x,t))^2\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^1 \int_0^1 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} e^2_{i_1,i_2}(x,t)\,\mathrm{d}x\mathrm{d}t + 2 \sum_{i_1<j_1} \sum_{i_2<j_2} \int_0^1 \int_0^1 e_{i_1,i_2}(x,t) e_{j_1,j_2}(x,t)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{27}$$

Since $i_1 < j_1, i_2 < j_2$, it follows that

$$D_{i_1,i_2} \cap D_{j_1,j_2} = \oslash,$$

so in Eq. (27) we can write

$$\|e(x,t)\|^2 = \int_0^1 \int_0^1 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} e_{i_1,i_2}^2(x,t)\,\mathrm{d}x\mathrm{d}t = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \|e_{i_1,i_2}^2\| \leq m^2(\frac{2M^2}{m^4}). \quad (28)$$

So $\|e(x,t)\| = O(\frac{1}{m})$, where $e(x,t) = u(x,t) - u_m(x,t)$.

5.2. **Estimation of the error bound.** In this section, we assume that $\frac{\partial u(x,t)}{\partial x}$ is continuous and bounded on $(0,1]$, this means that

$$\exists M > 0, \forall\, x,t \in (0,1], \qquad |\frac{\partial u(x,t)}{\partial x}| \leq M. \quad (29)$$

Theorem 5.2 Suppose that the function $D_x^\alpha u_m(x,t)$ is the approximation of $D_x^\alpha u(x,t) := \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ that is obtained by using 2D-BPFs, then we have an exact upper bound as follows:

$$\|D_x^\alpha u(x,t) - D_x^\alpha u_m(x,t)\|_2 \leq \frac{2M}{\Gamma(1-\alpha).(1-\alpha)} \frac{1}{m^{(1-\alpha)}},$$

where

$$\|u(x,t)\|_2 = \left(\int_0^1 \int_0^1 u^2(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}.$$

Proof. By using Caputo fractional definition and with Eq. (29), we have

$$|D_x^\alpha u(\xi_1,\xi_2) - D_x^\alpha u_m(\eta_1,\eta_2)| = \frac{1}{\Gamma(1-\alpha)} \mid \int_0^{\xi_1} \frac{\partial_\tau u(\tau,\xi_2)}{(\xi_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{\eta_1} \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau|$$

$$= \frac{1}{\Gamma(1-\alpha)} \mid \int_0^{\xi_1} \frac{\partial_\tau u(\tau,\xi_2)}{(\xi_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{\xi_1} \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau - \int_{\xi_1}^{\eta_1} \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau \mid$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \left(\mid \int_0^{\xi_1} \frac{\partial_\tau u(\tau,\xi_2)}{(\xi_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{\xi_1} \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau \mid + \mid \int_{\xi_1}^{\eta_1} \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau \mid\right)$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \left(\int_0^{\xi_1} \mid \frac{\partial_\tau u(\tau,\xi_2)}{(\xi_1-\tau)^\alpha} \mid \mathrm{d}\tau - \int_0^{\xi_1} \mid \frac{\partial u_\tau(\tau,\eta_2)}{(\eta_1-\tau)^\alpha} \mid \mathrm{d}\tau + \int_{\xi_1}^{\eta_1} \mid \frac{\partial_\tau u(\tau,\eta_2)}{(\eta_1-\tau)^\alpha} \mid \mathrm{d}\tau\right)$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \left(\int_0^{\xi_1} \frac{\mid \partial_\tau u(\tau,\xi_2) \mid}{(\xi_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{\xi_1} \frac{\mid \partial_\tau u(\tau,\eta_2) \mid}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau + \int_{\xi_1}^{\eta_1} \frac{\mid \partial_\tau u(\tau,\eta_2) \mid}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau\right)$$

$$\leq \frac{M}{\Gamma(1-\alpha)} \left(\int_0^{\xi_1} \frac{1}{(\xi_1-\tau)^\alpha}\mathrm{d}\tau - \int_0^{\xi_1} \frac{1}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau + \int_{\xi_1}^{\eta_1} \frac{1}{(\eta_1-\tau)^\alpha}\mathrm{d}\tau\right)$$

$$= \frac{M}{\Gamma(1-\alpha).(1-\alpha)} \left([\xi_1^{1-\alpha} + (\eta_1-\xi_1)^{1-\alpha} - \eta_1^{1-\alpha}] + (\eta_1-\xi_1)^{1-\alpha}\right).$$

Since $\xi_1 < \eta_1$, hence $\xi_1^{1-\alpha} - \eta_1^{1-\alpha} < 0$. Therefore

$$|D_x^\alpha u(\xi_1,\xi_2) - D_x^\alpha u_m(\eta_1,\eta_2)| \le \frac{2M}{\Gamma(1-\alpha).(1-\alpha)}.(\eta_1-\xi_1)^{1-\alpha} \le \frac{2M}{\Gamma(1-\alpha).(1-\alpha)}\frac{1}{m^{(1-\alpha)}}.$$

Thus we have

$$(D_x^\alpha u(\xi_1,\xi_2) - D_x^\alpha u_m(\eta_1,\eta_2))^2 \le \frac{4M^2}{\Gamma^2(1-\alpha).(1-\alpha)^2}\frac{1}{m^{2(1-\alpha)}}.$$

Similar to the method presented in Subsection 5.1 we have

$$\|(D_x^\alpha u(\xi_1,\xi_2) - D_x^\alpha u_m(\eta_1,\eta_2))\|_2^2 \le \frac{1}{m^2}\frac{4M^2}{\Gamma^2(1-\alpha).(1-\alpha)^2}\frac{1}{m^{2(1-\alpha)}}.$$

So that

$$\begin{aligned}\|(D_x^\alpha u(x,t) - D_x^\alpha u_m(x,t))\|_2^2 &= \int_0^1\int_0^1 (D_x^\alpha u(x,t) - D_x^\alpha u_m(x,t))^2\,\mathrm{d}x\mathrm{d}t\\ &= \sum_{i_1=1}^m\sum_{i_2=1}^m \int\int_{D_{i_1,i_2}} (D_x^\alpha u(x,t) - D_x^\alpha u_m(x,t))^2\,\mathrm{d}x\mathrm{d}t\\ &\le \sum_{i_1=1}^m\sum_{i_2=1}^m \int_{\frac{i_1-1}{m}}^{\frac{i_1}{m}}\int_{\frac{i_2-1}{m}}^{\frac{i_2}{m}} \frac{4M^2}{\Gamma^2(1-\alpha).(1-\alpha)^2}\frac{1}{m^{2(1-\alpha)}}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Hence we have

$$\|(D_x^\alpha u(x,t) - D_x^\alpha u_m(x,t))\|_2 \le \frac{2M}{\Gamma(1-\alpha).(1-\alpha)}\frac{1}{m^{(1-\alpha)}}.$$

The following process presents an approach for estimating the value of M. We know $\partial_x u(x,t)$ is continuous and bounded on $(0,1]$, therefore we can approximate $\partial_x u(x,t)$ by

$$\partial_x u(x,t) \cong \sum_{i_1=1}^m\sum_{i_2=1}^m u_{i_1,i_2} b_{i_1,i_2}(x,t) = B^T(x)UB(t). \tag{30}$$

By integrating Eq. (30), we have

$$u(x,t) = \int_0^x \partial_x u(s,t)\,\mathrm{d}s + u(0,t) = \int_0^x \partial_x u(s,t)\,\mathrm{d}s \cong B^T(x)P^TUB(t) \tag{31}$$

So we have

$$u(x,t) \cong B^T(x)(P_1)^TUB(t). \tag{32}$$

Now, we determine the points $x_{i_1} = \frac{i_1-1}{m}, t_{i_2} = \frac{i_2-1}{m}, i_1, i_2 = 1,\cdots,m$. By evaluating Eq. (32) at the points x_{i_1}, t_{i_2}, we have

$$u(x_{i_1},t_{i_2}) \cong B^T(x_{i_1})(P_1)^TUB(t_{i_2}). \tag{33}$$

We write Eq. (33) in matrix form, then we have

$$\mathbf{u} = B_x^T(P_1)^TUB_t. \tag{34}$$

where

$$\mathbf{u} = [u(x_{i_1},t_{i_2})], \quad B_x := [B(x_{i_1}),\cdots,B(x_{i_m})].$$

From Eq. (34), we can find U and from Eq. (30), we find the value of $\frac{\partial u(x,t)}{\partial x}$ for $x,t \in (0,1]$.

Theorem 5.3 Suppose that the function $D_x^{\alpha} u_m(x,t)$ obtained by using 2D-BPFs are the approximation of $D_x^{\alpha} u(x,t)$, then we have an exact upper bound as follows:

$$\|u(x,t) - u_m(x,t)\|_2 \leq \frac{2M}{\Gamma(\alpha)\Gamma(1-\alpha).\alpha(1-\alpha)} \frac{1}{m^{(1-\alpha)}} \tag{35}$$

where

$$\|u(x,t)\|_2 = \left(\int_0^1 \int_0^1 u^2(x,t) \mathrm{d}x \mathrm{d}t \right)^{\frac{1}{2}}.$$

We can prove Theorem 5.2 by using of Theorem 5.2.
From Eq. (35), we can see clearly that $\|u(x,t) - u_m(x,t)\|_2 \longrightarrow 0$ as $m \longrightarrow \infty$. We can conclude that BPFs method is convergent when it is used to solve the numerical solution of fractional differential equations.

5.3. **Estimation of the error function.** Consider Eq. (1). Let $e(x,t) = u(x,t) - u_m(x,t)$ be the error function, where $u_m(x,t)$ is the estimation of the exact solution $u(x,t)$. Then we consider

$$r_m(x,t) = -\frac{\partial^{\alpha} u_m}{\partial t^{\alpha}} - \frac{\partial^{\beta} u_m}{\partial x^{\beta}} + \lambda u_m(x,t) + g(x,t), \tag{36}$$

where $r_m(x,t)$ is the perturbation function that depends only on $u_m(x,t)$. By substituting Eqs. (1) and (36), we get

$$r_m(x,t) = \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^{\alpha} u_m}{\partial t^{\alpha}} \right) + \left(\frac{\partial^{\beta} u}{\partial x^{\beta}} - \frac{\partial^{\beta} u_m}{\partial x^{\beta}} \right) - \lambda e(x,t),$$

or

$$r_m(x,t) = \frac{\partial^{\alpha} e}{\partial t^{\alpha}} + \frac{\partial^{\beta} e}{\partial x^{\beta}} - \lambda e(x,t).$$

The above equation is a fractional order partial differential equation which is similar to the main Eq. (1). We can apply the mentioned method for solving the above equation to obtain an approximate solution for the error function.

6. The numerical solution

Example 6.1 Consider the following nonhomogeneous partial differential equation, where are the fractional orders α, β [25, 26]

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \sin(x+t), \quad 0 \leq x, t < 1,$$

with the initial conditions $u(0,t) = u(x,0) = 0$.

The exact solution of the above initial value problem for $\alpha = \beta = 1$ is

$$u(x,t) = \sin x \sin t.$$

The corresponding absolute errors for $m = 8, 16, 32, 64$ in some points are shown in Table 1.

Also, Figures 1 and 2, present graphs of the exact solution as well as the numerical solution for $m = 32$.

Figure 1. Numerical solution for $m = 32$, (Example (6.1)).

TABLE 1. Error values

(x,t)	$e_8(x,t)$	$e_{16}(x,t)$	$e_{32}(x,t)$	$e_{64}(x,t)$
$(0,0)$	0	0	0	0
$(1/8,1/8)$	$1.1600e-02$	$6.7000e-03$	$3.6000e-03$	$1.8000e-03$
$(2/8,2/8)$	$2.6500e-02$	$1.4100e-02$	$7.3000e-03$	$3.7000e-03$
$(3/8,3/8)$	$3.9900e-02$	$2.0700e-02$	$1.0500e-02$	$5.3000e-03$
$(4/8,4/8)$	$5.0800e-02$	$2.5800e-02$	$1.3000e-02$	$6.5000e-03$
$(5/8,5/8)$	$5.8600e-02$	$2.9500e-02$	$1.4800e-02$	$7.4000e-03$
$(6/8,6/8)$	$6.2900e-02$	$3.1300e-02$	$1.5600e-02$	$7.8000e-03$
$(7/8,7/8)$	$6.3400e-02$	$3.1200e-02$	$1.5500e-02$	$7.7000e-03$

FIGURE 2. Exact solution for $m = 32$, (Example (6.1)).

Figure 3 shows the numerical results for $m = 32, t = 1$ and various $0 < \alpha \leq 1$. The comparisons show that as $\alpha \longrightarrow 1$ the approximate solutions tend to the exact solution of the equation in the case of $\alpha = 1$.

FIGURE 3. The approximate solution for $m = 32, t = 1$ and some $0 < \alpha \leq 1$, (Example (6.1)).

Example 6.2 Consider the following nonhomogeneous partial differential equation[25, 26]

$$\frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^\alpha u}{\partial t^\alpha} = 1, \quad 0 \leq x, t < 1,$$

with the initial conditions $u(0,t) = u(x,0) = 0$.

The exact solution of the above initial value problem, for $\alpha = \beta = 1$, is

$$u(x,t) = \begin{cases} t & x \geq t, \\ x & x < t. \end{cases}$$

The corresponding absolute errors for $m = 8, 16, 32, 64$ in some points are shown in Table 2.

TABLE 2. Error values

(x,t)	$e_8(x,t)$	$e_{16}(x,t)$	$e_{32}(x,t)$	$e_{64}(x,t)$
$(0,0)$	0	0	0	0
$(1/8,1/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(2/8,2/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(3/8,3/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(4/8,4/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(5/8,5/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(6/8,6/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$
$(7/8,7/8)$	$9.3700e-02$	$4.6900e-02$	$2.3400e-02$	$1.1700e-02$

Also, Figures 4 and 5, present graphs of the exact solution as well as the numerical solution for $m = 16$.

FIGURE 4. Numerical solution for $m = 16$, (Example (6.2)).

FIGURE 5. Exact solution for $m = 16$, (Example (6.2)).

Figure 6 shows the numerical results for $m = 16, t = 0.9$ and various $0 < \alpha \leq 1$. The comparisons show that as $\alpha \longrightarrow 1$ the approximate solutions tend to the exact solution of the equation in the case of $\alpha = 1$.

FIGURE 6. The approximate solution for $m = 16, t = 0.9$ and some $0 < \alpha \leq 1$, (Example (6.2)).

Example 6.3 Consider the following nonhomogeneous partial differential equation[25, 26]

$$\frac{\partial^{1/4} u}{\partial x^{1/4}} + \frac{\partial^{1/4} u}{\partial t^{1/4}} = g(x,t), \quad 0 \leq x, t < 1,$$

with the initial conditions $u(0,t) = u(x,0) = 0$ and $g(x,t) = \frac{4(x^{3/4}t + xt^{3/4})}{3\Gamma(3/4)}$.

The exact solution of the above initial value problem is $u(x,t) = xt$.
The corresponding absolute errors for $m = 8, 16, 32, 64$ in some points are shown in Table 3. Also, the Figure 7, presents the graph of the error for $m = 16$.

TABLE 3. Error values

(x,t)	$e_8(x,t)$	$e_{16}(x,t)$	$e_{32}(x,t)$	$e_{64}(x,t)$
$(0,0)$	0	0	0	0
$(1/8,1/8)$	$1.1300e-02$	$6.8000e-03$	$3.6000e-03$	$1.9000e-03$
$(2/8,2/8)$	$2.7400e-02$	$1.4700e-02$	$7.6000e-03$	$3.8000e-03$
$(3/8,3/8)$	$4.3000e-02$	$2.2400e-02$	$1.1500e-02$	$5.8000e-03$
$(4/8,4/8)$	$5.8600e-02$	$3.0300e-02$	$1.5400e-02$	$7.8000e-03$
$(5/8,5/8)$	$7.4200e-02$	$3.8100e-02$	$1.9300e-02$	$9.7000e-03$
$(6/8,6/8)$	$8.9900e-02$	$4.5900e-02$	$2.3200e-02$	$1.1700e-02$
$(7/8,7/8)$	$1.0550e-01$	$5.3700e-02$	$2.7100e-02$	$1.3600e-02$

FIGURE 7. Error for $m = 16$, Example (6.3)).

Example 6.4 Consider the following nonhomogeneous partial differential equation[25, 26]

$$\frac{\partial^{1/3}u}{\partial x^{1/3}} + \frac{\partial^{1/2}u}{\partial t^{1/2}} = g(x,t), \quad 0 \le x, t < 1,$$

with the initial conditions $u(0,t) = u(x,0) = 0$ and $g(x,t) = \frac{9x^2t^{5/3}}{5\Gamma(2/3)} + \frac{8x^{3/2}t^2}{3\Gamma(1/2)}$.

The exact solution of the above initial value problem is $u(x,t) = x^2t^2$.
The corresponding absolute errors for $m = 16, 32, 64, 128$ in some points are shown in Table 4.

TABLE 4. Error values

(x,t)	$e_{16}(x,t)$	$e_{32}(x,t)$	$e_{64}(x,t)$	$e_{128}(x,t)$
$(0,0)$	0	0	0	0
$(1/8,1/8)$	$1.0003e+00$	$9.9850e-01$	$5.7440e-01$	$3.2090e-01$
$(2/8,2/8)$	$1.6000e-03$	$9.0000e-04$	$5.0000e-04$	$3.0000e-04$
$(3/8,3/8)$	$5.8000e-03$	$3.2000e-03$	$1.8000e-03$	$1.0000e-04$
$(4/8,4/8)$	$1.4400e-02$	$7.9000e-03$	$4.3000e-03$	$2.4000e-03$
$(5/8,5/8)$	$2.8900e-02$	$1.5700e-02$	$8.6000e-03$	$4.9000e-03$
$(6/8,6/8)$	$5.0900e-02$	$2.7400e-02$	$1.5100e-02$	$8.7000e-03$
$(7/8,7/8)$	$8.1900e-02$	$4.4100e-02$	$2.4300e-02$	$1.4100e-03$

Also, Figures 8 and 9, present graphs of the exact solution as well as the numerical solution for $m = 32$.

FIGURE 8. Numerical solution for$m = 32$, (Example (6.4)).

FIGURE 9. Exact solution for$m = 32$, (Example (6.4)).

Now we obtain error and upper bound of error for our examples.

Table 5. Error and upper bound of error for different values of m for Example (6.1).

m	$\|u(x,t) - u_m(x,t)\|$	upper bound of error
8	$7.2778e-04$	$8.3039e-04$
16	$1.5736e-04$	$2.0750e-04$
32	$3.6588e-05$	$5.1810e-05$
64	$8.8209e-06$	$1.2961e-05$

Table 6. Error and upper bound of error for different values of m for Example (6.2).

m	$\|u(x,t) - u_m(x,t)\|$	upper bound of error
8	$1.0900e-02$	$1.7200e-02$
16	$2.9000e-03$	$4.5000e-03$
32	$7.2696e-04$	$1.2000e-03$
64	$1.8257e-04$	$3.0044e-04$

Table 7. Error and upper bound of error for different values of m for Example (6.3).

m	$\|u(x,t) - u_m(x,t)\|$	upper bound of error
8	$2.0600e-02$	$5.3610e-01$
16	$5.6000e-03$	$3.1010e-01$
32	$1.4000e-03$	$1.8150e-01$
64	$3.7264e-04$	$1.0700e-01$

Table 8. Error and upper bound of error for different values of m for Example (6.4).

m	$\|u(x,t) - u_m(x,t)\|$	upper bound of error
16	$1.5050e-01$	$5.6730e-01$
32	$1.3840e-01$	$3.9470e-01$
64	$1.3270e-01$	$2.7590e-01$
128	$1.2980e-01$	$1.9380e-01$

7. Conclusion

This article uses block pulse operational matrix method to solve a class of fractional partial differential equation. Numerical examples show that the approximate solution has a good degree of accuracy.

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