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COMPARATIVE DYNAMICS OF FRACTIONAL HALF-LINEAR BOUNDARY VALUE PROBLEMS VIA LIAPUNOV INEQUALITIES

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Abstract. In this paper, we study two classes of fractional half-linear boundary value problems subject to the Dirichlet boundary conditions. The golden aims of this paper can be summarized as follows. First, we introduce extended theory of the conformable fractional calculus and its basic analysis. In the next level using the Green function technique, we obtain Liapunov inequalities of the under study fractional order boundary value problems. In the light of the obtained Liapunov inequalities, qualitative behavior of the mentioned problems such as disconjugacy, solvability, upper bound estimation for number of zeros of the non-trivial solutions and distance between consecutive zeros of the oscillatory solutions will be presented in a comparative manner.

1. Introduction

Consider the following fractional half-linear boundary value problems:

$$\begin{cases} \Theta_{\beta_2}\bigg(G_{a^+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\bigg) + \Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(p(t)u\right) = 0, \ a<t<b, \\ u(a)=0, \quad u(b)=0, \end{cases} \tag{1.1}$$

$$\begin{cases} \Theta_{\beta_2}\bigg(D_{a^+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\bigg) + \Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(q(t)u\right) = 0, \ a<t<b, \\ u(a)=0, \quad u(b)=0, \end{cases} \tag{1.2}$$

subject to the following general assumptions:

- (A_1) $\alpha \in (1,2)$ and $\beta_1, \beta_2 \in (0,+\infty)$.
- (A_2) $G_{a^+}^{\alpha}$ and $D_{a^+}^{\alpha}$ denote extended conformable fractional differentiation operator and the Riemann-Liouville fractional differentiation operator of order α, respectively, that will be defined a little later.
- (A_3) $\Theta_z(u) = |u|^{z-1}u$ with $z \in (0,+\infty)$.
- (A_4) $p, q : (a,b) \to \mathbb{R}$ stand for continuous and non-zero functions.

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This is well known that the concept of the Liapunov inequality turns to the deep studies of the Russian mathematician A. M. Liapunov on stability of motion, in the late XIX century,[13]. The cornerstone of the Liapunov inequalities can be stated as follows:

Theorem 1.1. *(cf. [5]) If the boundary value problem*

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0 = y(b), \end{cases} \tag{1.3}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.4}$$

Since then, the literature has been developed by many generalizations and refinements of the Liapunov inequality (1.4) by now. In this way, one may suggest for instance the pioneering papers [4],[5]-[7],[8],[16] for detailed consultation. One of the greatest advantages of the Liapunov inequalities in comparison with other ones turns to their ability in establishing dynamics of the related differential or difference equations in frame of their qualitative behavior such as stability, disconjugacy, solvability and spectral properties. Further more, applying these inequalities one may estimate maximum number of zeros for nontrivial solutions of considered problems and distance between consecutive zeros of the oscillatory solutions. In addition, the Liapunov inequalities have been played a crucial role in the literature for studying linear and quasi-linear partial differential equations. For instance, Liapunov inequalities can be applied to find minimizers of the given minimization problem and one may derive a lower bound for the first eigenvalue of mentioned problems beside on some another applications. See [2],[3],[9],[10].
So, now it clears that why we interested in the study of these inequalities for fractional order differential equations. To the best of our knowledge, investigation about Liapunov inequalities for differential and difference equations of fractional order introduced for fist time in literature by the Portuguese mathematician R. A. C. Ferreira. We briefly state his works as follows.
The author in [5], in the late 2013, studied the following two-point fractional boundary value problem

$$\begin{cases} \left({}_aD^{\alpha}y\right)(t) + q(t)y(t) = 0, & a < t < b,\ 1 < \alpha \leq 2, \\ y(a) = 0,\ y(b) = 0, \end{cases} \tag{1.5}$$

where ${}_aD^{\alpha}$ stands for the left sided Riemann-Liouville fractional derivative of order α and $q : [a, b] \to \mathbb{R}$ is a continuous function. The author using properties of the corresponding Green function to the (1.5) obtained a Liapunov inequality of the form

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.6}$$

The Liapunov inequality (1.6) generalizes the classic Liapunov inequality (1.4), (excepted $\alpha = 2$). He used then the Liapunov inequality (1.6) to prove that Mittag-Leffler function of the fractional eigenvalue problem corresponding to the (1.3) has no real zeros on a determined interval.
Also, the author in [7], in the 2015, considered the following two-point fractional

difference boundary value problem

$$\begin{cases} (\Delta^{\alpha}y)(t) + q(t+\alpha-1)y(t+\alpha-1) = 0, \quad t \in \mathbb{N}_0^{b+1}, \\ y(\alpha-2) = 0 = y(\alpha+b+1), \ or \ y(\alpha-2) = 0 = \Delta y(\alpha+b), \end{cases} \tag{1.7}$$

and using Green function technique, achieved to the following discrete fractional Liapunov inequality corresponding to the fist pair of boundary conditions:

$$\begin{aligned} &\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b}{2}+2\right)}{(b+2\alpha)(b+2)\Gamma^2\left(\frac{b}{2}+\alpha\right)\Gamma(b+3)}, \quad b: \text{ even}, \\ &\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b+3}{2}\right)}{\Gamma^2\left(\frac{b+1}{2}+\alpha\right)\Gamma(b+3)}, \quad b: \text{ odd}, \end{aligned} \tag{1.8}$$

In the second step, the author obtained for the second pair of boundary conditions, the following Liapunov inequality:

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{1}{(b+2)\Gamma(\alpha-1)}. \tag{1.9}$$

In this paper, we are interested in study Liapunov inequalities of the continuous fractional order boundary value problems (1.1) and (1.2), and their abilities to establish dynamics of the corresponding boundary value problems.
At the end of this section, we summarize the organization of the paper. In section 2, we introduce extended theory of the conformable fractional calculus. This calculus, acts on the extended conformable fractional differentiation operators that we will define them a little later. Section 3, includes strategies to extract Liapunov inequalities of the fractional boundary value problems (1.1) and (1.2) based on their Green functions and applicability of these inequalities to establish qualitative behavior of the corresponding boundary value problems as we stated above.

2. Extended Conformable Fractional Calculus

Philosophy of the conformable fractional calculus, is rooted in the some algebraic properties of the Riemann-Liouville based fractional operators. More precisely, lack of the well known Leibniz and Chain rules in R-L based fractional operators inappropriately can be observed. See [18] and [19], for instance. As addressed in these references fractional order Leibniz and Chain rules in the R-L sense, appear as infinite series involving the R-L fractional differentiation and integration operators having various orders. Overcoming these problems, more recently the idea of conformable fractional calculus that suggests the limit approach of fractional order differentiation operators has been presented. Here, we recall the basic elements of the conformable fractional calculus from the basic references [11] and [1], as follows.

Definition 2.1. (**Basic Conformable Fractional Differentiation Operators**). *Assume that* $0 < \alpha \leq 1$ *and* $f : [a,b] \to \mathbb{R}$. *Then the left and right sided conformable fractional differentiation operators are given by*

$$T_a^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon}, \tag{2.1}$$

$${}_bT^{\alpha} f(t) = -\lim_{\epsilon \to 0} \frac{f(t+\epsilon(b-t)^{1-\alpha}) - f(t)}{\epsilon}. \tag{2.2}$$

Definition 2.2. (**Conformable Fractional Integration Operators**). *Assume that $0 < \alpha \leq 1$ and $f \in L([a,b];\mathbb{R})$. Then the left and right sided conformable fractional integration operators are given by*

$$I_a^\alpha f(t) = \int_a^t \frac{f(s)}{(s-a)^{1-\alpha}} ds, \tag{2.3}$$

$${}_bI^\alpha f(t) = \int_t^b \frac{f(s)}{(b-s)^{1-\alpha}} ds, \tag{2.4}$$

Now, taking $h = \epsilon(t-a)^{1-\alpha}$, immediately one may derive the following golden identity of the conformable fractional calculus:

$$\begin{aligned} T_a^\alpha f(t) &= \lim_{\epsilon \to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon} \\ &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h(t-a)^{\alpha-1}} \\ &= (t-a)^{1-\alpha} \lim_{h\to 0} \frac{f(t+h)-f(h)}{h} \\ &= (t-a)^{1-\alpha} \frac{d}{dt} f(t), \quad 0 < \alpha \leq 1. \end{aligned} \tag{2.5}$$

Relying on the golden identity $T_a^\alpha u(t) = (t-a)^{1-\alpha}\frac{d}{dt}u(t)$, in a straight forward manner, one can prove the following expected results:

$$T_a^\alpha (c_1 f + c_2 g) = c_1 T_a^\alpha(f) + c_2 T_a^\alpha(g), \quad \text{for all } c_1, c_2 \in \mathbb{R}; \tag{2.6}$$

$$T_a^\alpha\,(t-a)^p = p(t-a)^{p-\alpha}, \quad \text{for all } p \in \mathbb{R}; \tag{2.7}$$

$$T_a^\alpha(\lambda) = 0, \quad \text{for all constant function } f(t) = \lambda; \tag{2.8}$$

$$T_a^\alpha(fg) = fT_a^\alpha(g) + gT_a^\alpha(f), \quad \text{(Leibniz rule)}; \tag{2.9}$$

$$T_a^\alpha(f \circ g) = \left(T_a^\alpha f\right)(g).\left(T_a^\alpha g\right).(g(t)-g(a))^{\alpha-1}, \quad \text{(Chain rule)}; \tag{2.10}$$

$$T_a^\alpha\left(\frac{f}{g}\right) = \frac{gT_a^\alpha(f) - fT_a^\alpha(g)}{g^2}, \quad g \not\equiv 0; \tag{2.11}$$

$$T_a^\alpha\left(\frac{1}{\alpha}(t-a)^\alpha\right) = 1; \tag{2.12}$$

$$T_a^\alpha\left(e^{\frac{1}{\alpha}(t-a)^\alpha}\right) = e^{\frac{1}{\alpha}(t-a)^\alpha}; \tag{2.13}$$

$$T_a^\alpha\left(\sin\ \frac{1}{\alpha}(t-a)^\alpha\right) = \cos\left(\frac{1}{\alpha}(t-a)^\alpha\right); \tag{2.14}$$

$$T_a^\alpha\left(\cos\ \frac{1}{\alpha}(t-a)^\alpha\right) = -\sin\left(\frac{1}{\alpha}(t-a)^\alpha\right). \tag{2.15}$$

In the sequel, fractional composition rules, Role's theorem, mean value theorem, conformable fractional Taylor expansion theorem and conformable Laplace transforms are given.

Theorem 2.3. (**Conformable Fractional Composition Rules**). *Assume that $0 < \alpha \leq 1$.*

1. *If $f \in L([a,b];\mathbb{R})$, then*

$$T_a^\alpha I_a^\alpha f(t) = f(t), \quad {}_bT^\alpha\ {}_bI^\alpha f(t) = f(t). \tag{2.16}$$

2. *If f be α-differentiable on (a,b), then*

$$I_a^\alpha T_a^\alpha f(t) = f(t) - f(a), \quad {}_bI^\alpha\ {}_bT^\alpha f(t) = f(t) - f(b). \tag{2.17}$$

Theorem 2.4. *(cf. [11]).(Rolle's Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

(i) *f is continuous on $[a,b]$,*
(ii) *f is α-differentiable for some $0 < \alpha < 1$,*
(iii) *$f(a) = f(b)$.*

Then, there exists $c \in (a,b)$, such that $(T^\alpha f)(c) = 0$.

Theorem 2.5. *(cf. [11]).(Mean Value Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

(i) *f is continuous on $[a,b]$,*
(ii) *f is α-differentiable for some $0 < \alpha < 1$.*

Then, there exists $c \in (a,b)$, such that $(T^\alpha f)(c) = \dfrac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Theorem 2.6. *(cf. [1]). Assume that f is a infinitely α-differentiable function in the basic sense (2.1), for some $0 < \alpha \le 1$ at a neighborhood of a point t_0. Then f has the fractional power series expansion:*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left(T_{t_0}^\alpha f\right)^{(k)}(t_0)(t - t_0)^{k\alpha}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{\frac{1}{\alpha}},\ R > 0. \tag{2.18}$$

Definition 2.7. *(cf. [1]). Let $a \in \mathbb{R},\ 0 < \alpha \le 1$ and $f : [a,\infty) \to \mathbb{R}$ be a real valued function. Then fractional Laplace transform of order α starting from lower terminal a of f is defined by*

$$L_a^\alpha\{f(t)\}(s) = F_a^\alpha(s) = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} f(t)(t-a)^{\alpha-1}dt. \tag{2.19}$$

Applying the golden identity $T_a^\alpha u(t) = (t-a)^{1-\alpha}\dfrac{d}{dt}u(t)$, one may derive the following theorem.

Theorem 2.8. *Let $a \in \mathbb{R},\ 0 < \alpha \le 1$ and $f : (a,\infty) \to \mathbb{R}$ be a differentiable real valued function. Then*

$$L_a^\alpha\left\{\left(T_a^\alpha f\right)(t)\right\}(s) = sF_a^\alpha(a) - f(a). \tag{2.20}$$

In this position, using the basic theory of the conformable fractional calculus we introduce extended theory of the conformable fractional calculus.

Definition 2.9. (Extended Conformable Fractional Derivatives). *The left sided extended conformable fractional derivative of order $0 < \alpha \le 1$, starting from lower terminal a of a function $f : [a,\infty) \to \mathbb{R}$ is defined by:*

$$\left(G_{a^+}^\alpha f\right)(t) = \lim_{\epsilon \to 0} \frac{f_{R(t>a;\epsilon,\alpha)}(t) - f(t)}{\epsilon}, \tag{2.21}$$

where, the representation function $R(t > a; \epsilon, \alpha)$ satisfies the following properties:

(A_1) *$R(t > a; \epsilon, \alpha)$ has the basic conformable fractional Taylor expansion series, i.e.*

$$R(t > a; \epsilon, \alpha) = \sum_{k=0}^{\infty} \frac{\left(T_a^{1-\alpha} R(t > a; \epsilon, \alpha)\right)^{(k)} (a)(t-a)^{k(1-\alpha)}}{(1-\alpha)^k k!}.$$

(A_2) $f_{R(t>a;0,\alpha)}(t) = f(t)$.

(A_3) *Based on the basic conformable fractional Taylor expansion of the representation function $R(t > a; \epsilon, \alpha)$,*

$$f_{R(t>a;\epsilon,\alpha)}(t) = f\left(t + \epsilon(t-a)^{1-\alpha}(1 + O(\epsilon^k))\right), \quad k \in \mathbb{Z}_+. \tag{2.22}$$

The right sided extended conformable fractional derivative of order $0 < \alpha \leq 1$, starting from upper terminal b of a function $f : (-\infty, b] \to \mathbb{R}$ is defined by:

$$\left(G_{b_-}^{\alpha} f\right)(t) = -\lim_{\epsilon \to 0} \frac{f_{R(t<b;\epsilon,\alpha)}(t) - f(t)}{\epsilon}, \tag{2.23}$$

where, the representation function $R(t < b; \epsilon, \alpha)$ satisfies the following properties:

(B_1) *$R(t < b; \epsilon, \alpha)$ has the basic conformable fractional Taylor expansion series, i.e.*

$$R(t < b; \epsilon, \alpha) = \sum_{k=0}^{\infty} \frac{\left({}_bT^{1-\alpha} R(t < b; \epsilon, \alpha)\right)^{(k)} (b)(b-t)^{k(1-\alpha)}}{(1-\alpha)^k k!}.$$

(B_2) $f_{R(t<b;0,\alpha)}(t) = f(t)$.

(B_3) *Based on the basic conformable fractional Taylor expansion of the representation function $R(t > a; \epsilon, \alpha)$,*

$$f_{R(t<b;\epsilon,\alpha)}(t) = f\left(t + \epsilon(b-t)^{1-\alpha}(1 + O(\epsilon^k))\right), \quad k \in \mathbb{Z}_+. \tag{2.24}$$

An interesting and straightaway outcome of the extended conformable fractional derivatives can be stated as follows: extended conformable fractional derivative of a given function $f:[a,b]\to\mathbb{R}$, has infinitely many representations such as

$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\ln\left(1+\epsilon(t-a)^{1-\alpha}\right)\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\Bigg\{\exp\left(\epsilon(t-a)^{1-\alpha}\right)-1\Bigg\}\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\sin\left(\epsilon(t-a)^{1-\alpha}\right)\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\sinh\left(\epsilon(t-a)^{1-\alpha}\right)\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\epsilon(t-a)^{1-\alpha}+\cos\left(\epsilon(t-a)^{1-\alpha}\right)-1\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\epsilon(t-a)^{1-\alpha}+\cosh\left(\epsilon(t-a)^{1-\alpha}\right)-1\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\tan\left(\epsilon(t-a)^{1-\alpha}\right)\Bigg)-f(t)}{\epsilon},$$
$$\left(G^{\alpha}_{a^+}f\right)(t)=\lim_{\epsilon\to 0}\frac{f\Bigg(t+\arctan\left(\epsilon(t-a)^{1-\alpha}\right)\Bigg)-f(t)}{\epsilon},$$
$$\vdots$$

Because, using the conformable fractional Taylor expansion series all of the functions

$$\begin{aligned}&\ln\left(1+\epsilon(t-a)^{1-\alpha}\right),\quad \exp\left(\epsilon(t-a)^{1-\alpha}\right)-1,\\ &\sin\left(\epsilon(t-a)^{1-\alpha}\right),\quad \sinh\left(\epsilon(t-a)^{1-\alpha}\right),\\ &\epsilon(t-a)^{1-\alpha}+\cos\left(\epsilon(t-a)^{1-\alpha}\right)-1,\quad \epsilon(t-a)^{1-\alpha}+\cosh\left(\epsilon(t-a)^{1-\alpha}\right)-1,\\ &\tan\left(\epsilon(t-a)^{1-\alpha}\right),\quad \arctan\left(\epsilon(t-a)^{1-\alpha}\right),\end{aligned}$$

can be stated as $\epsilon(t-a)^{1-\alpha}(1+O(\epsilon^k)),\ k\in\mathbb{Z}_+$. Therefore, taking $h=\epsilon(t-a)^{1-\alpha}(1+O(\epsilon^k))$ instead $h=\epsilon(t-a)^{1-\alpha}$, with a direct calculation we can reobtain all of basic analysis review above for the extended conformable fractional calculus. To this aim, it suffices to prove extended golden identity corresponding to the (2.5).

In this way, note that

$$\begin{aligned}
\left(G_{a^+}^{\alpha} f\right)(t) &= \lim_{\epsilon\to 0} \frac{f_{R(t>a;\epsilon,\alpha)}(t) - f(t)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{f\left(t+\epsilon(t-a)^{1-\alpha}(1+O(\epsilon^k))\right) - f(t)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{f\,(t+h) - f(t)}{\dfrac{h(t-a)^{\alpha-1}}{1+O(\epsilon^k)}} \\
&= (t-a)^{1-\alpha} \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \\
&= (t-a)^{1-\alpha} \frac{d}{dt} f(t), \quad 0<\alpha\le 1.
\end{aligned} \tag{2.25}$$

Now, making use of the identity (2.25), clearly one can prove all of the formulas (2.6)-(2.15). For instance, we just prove the Leibniz and Chain rules (2.9) and (2.10), and leave proofs of the other formulas. To prove the fractional Leibniz rule (2.9), note that

$$\begin{aligned}
\left(G_{a^+}^{\alpha}(fg)\right)(t) &= (t-a)^{1-\alpha}\frac{d}{dt}(fg)(t) \\
&= (t-a)^{1-\alpha}\left(f(t)\frac{d}{dt}g(t) + g(t)\frac{d}{dt}f(t)\right) \\
&= f(t)\left((t-a)^{1-\alpha}g^{'}(t)\right) + g(t)\left((t-a)^{1-\alpha}f^{'}(t)\right) \\
&= f(t)\left(G_{a^+}^{\alpha} g\right)(t) + g(t)\left(G_{a^+}^{\alpha} f\right)(t).
\end{aligned}$$

Also, we prove the fractional Chain rule (2.10) as follows

$$\begin{aligned}
\left(G_{a^+}^{\alpha}(f\circ g)\right)(t) &= (t-a)^{1-\alpha}\frac{d}{dt}(f\circ g)(t) \\
&= (t-a)^{1-\alpha}\left(f^{'}(g(t)).g^{'}(t)\right) \\
&= (g(t)-g(a))^{1-\alpha} f^{'}(g(t)).(t-a)^{1-\alpha}g'(t).(g(t)-g(a))^{\alpha-1} \\
&= \left(G_{a^+}^{\alpha} f\right)(g(t)).\left(G_{a^+}^{\alpha} g\right)(t).(g(t)-g(a))^{\alpha-1}.
\end{aligned}$$

It is time to examine the extended conformable fractional operators $G_{a^+}^{\alpha}$ and $G_{b^-}^{\alpha}$ to verify the composition rules. So, we have the following theorem.

Theorem 2.10. *Assume* $0<\alpha\le 1$.

(C_1) *If* $f\in L([a,b];\mathbb{R})$, *then*

$$\left(G_{a^+}^{\alpha} I_{a^+}^{\alpha} f\right)(t) = f(t), \quad \left(G_{b^-}^{\alpha} I_{b^-}^{\alpha} f\right)(t) = f(t).$$

(C_2) *If* f *be* α*-differentiable in the sense (2.21) and (2.23), at terminals* $t=a$ *and* $t=b$*, respectively, then*

$$\left(I_{a^+}^{\alpha} G_{a^+}^{\alpha} f\right)(t) = f(t) - f(a), \quad \left(I_{b^-}^{\alpha} G_{b^-}^{\alpha} f\right)(t) = f(t) - f(b).$$

Proof. We just prove the left sided assertions and as a result of similarity, omit proof of the right sided ones. Also, we point out this fact that keeping harmony we represent the right sided conformable fractional integral ${}_bI^{\alpha}$ with $I_{b^-}^{\alpha}$. As we stated above using extended golden identity (2.25), all of these assertions can be proved. Thus we begin as follows

$$\begin{aligned}
\left(G^{\alpha}_{a^+} I^{\alpha}_{a^+} f\right) &= (t-a)^{1-\alpha} \frac{d}{dt} \left(I^{\alpha}_{a^+} f\right)(t) \\
&= (t-a)^{1-\alpha} \frac{d}{dt} \left(\int_a^t \frac{f(s)}{(s-a)^{1-\alpha}} ds \right) \\
&= (t-a)^{1-\alpha} . \frac{f(t)}{(t-a)^{1-\alpha}} \\
&= f(t).
\end{aligned}$$

So, (C_1) is satisfied.
To see that (C_2) holds, note that

$$\begin{aligned}
\left(I^{\alpha}_{a^+} G^{\alpha}_{a^+} f\right)(t) &= \int_a^t \frac{\left(G^{\alpha}_{a^+} f\right)(s)}{(s-a)^{1-\alpha}} ds \\
&= \int_a^t \frac{(s-a)^{1-\alpha} f^{'}(s)}{(s-a)^{1-\alpha}} ds \\
&= \int_a^t f^{'}(s) ds \\
&= f(t) - f(a).
\end{aligned}$$

The proof is completed now. □

Showing that under the extended conformable fractional operators, the Rolle and mean value theorems hold, we have the following theorems.

Theorem 2.11. (**Rolle's Theorem**). *Let $\rho_1 > 0$ and $f : [\rho_1, \rho_2] \subset (a, b) \to \mathbb{R}$ be a given function that satisfies:*

- (R_1) *f is continuous on $[\rho_1, \rho_2]$;*
- (R_2) *f is α-differentiable in the sense (2.21), for some $\alpha \in (0, 1)$;*
- (R_3) *$f(\rho_1) = f(\rho_2)$.*

Then, there exists an $c \in (\rho_1, \rho_2)$ such that $\left(G^{\alpha}_{a^+} f\right)(c) = 0$.

Proof. The continuity of f on $[\rho_1, \rho_2]$ together with the assumption $f(\rho_1) = f(\rho_2)$, ensures that there exists an $c \in (\rho_1, \rho_2)$ at which, f has a local extrema. Then, in accordance with the (2.22), we conclude that,

$$\left(G^{\alpha}_{a^+} f\right)(c) = \lim_{\epsilon \to 0^+} \frac{f\left(c + \epsilon (c-a)^{1-\alpha}\left(1 + O(\epsilon^k)\right)\right) - f(c)}{\epsilon} = \lim_{\epsilon \to 0^-} \frac{f\left(c + \epsilon (c-a)^{1-\alpha}\left(1 + O(\epsilon^k)\right)\right) - f(c)}{\epsilon}.$$

The opposite signs of the above limits, yields that $\left(G^{\alpha}_{a^+} f\right)(c) = 0$. This completes the proof. □

Theorem 2.12. (**Mean Value theorem**). *Let $\rho_1 > 0$ and $f : [\rho_1, \rho_2] \subset (a, b) \to \mathbb{R}$ be a given function that satisfies:*

- (M_1) *f is continuous on $[\rho_1, \rho_2]$;*
- (M_2) *f is α-differentiable in the sense (2.21), for some $\alpha \in (0, 1)$.*

Then, there exists an $c \in (\rho_1, \rho_2)$, $\left(G^{\alpha}_{a^+} f\right)(c) = \dfrac{f(\rho_2) - f(\rho_1)}{\frac{1}{\alpha}(\rho_2 - a)^{\alpha} - \frac{1}{\alpha}(\rho_1 - a)^{\alpha}}$.

Proof. We define the function $g(t) : [\rho_1, \rho_2] \subset (a, b) \to \mathbb{R}$ as below:

$$g(t) = f(t) - f(\rho_1) - \frac{f(\rho_2) - f(\rho_1)}{\frac{1}{\alpha}(\rho_2 - a)^\alpha - \frac{1}{\alpha}(\rho_1 - a)^\alpha} \left(\frac{1}{\alpha}(t - a)^\alpha - \frac{1}{\alpha}(\rho_1 - a)^\alpha \right).$$

It is easy to check that,

1. g is continuous on $[\rho_1, \rho_2]$;
2. g is α-differentiable in the sense (2.21), for some $\alpha \in (0, 1)$;
3. $g(\rho_1) = g(\rho_2) = 0$.

Therefore, the Rolle's Theorem 2.11, implies that there exists an $c \in (\rho_1, \rho_2)$ such that $\left(G^\alpha_{a^+} g\right)(c) = 0$. Equivalently, using the property $G^\alpha_{a^+} \left(\frac{1}{\alpha}(t - a)^\alpha\right) = 1$, one has

$$\left(G^\alpha_{a^+} f\right)(c) = \frac{f(\rho_2) - f(\rho_1)}{\frac{1}{\alpha}(\rho_2 - a)^\alpha - \frac{1}{\alpha}(\rho_1 - a)^\alpha}.$$

This completes the proof. □

As we observed above applying the identity (2.25), the basic analysis of the basic conformable fractional calculus, can be reobtained for extended conformable fractional calculus. At the final stage of this section we are going to present the higher order extended conformable fractional operators.

Definition 2.13. *The left sided extended conformable fractional differentiation operator of order $n < \alpha \leq n + 1$, $n \in \mathbb{Z}_+$, starting from lower terminal a of an n-differentiable function $f : [a, \infty) \to \mathbb{R}$ is defined by:*

$$\left(G^\alpha_{a^+} f\right)(t) = \lim_{\epsilon \to 0} \frac{f^{([\alpha]-1)}_{R(t>a;\epsilon,\alpha)}(t) - f^{([\alpha]-1)}(t)}{\epsilon}, \tag{2.26}$$

where, the representation function $R(t > a; \epsilon, \alpha)$ satisfies the following properties:

(HL_1) *$R(t > a; \epsilon, \alpha)$ has the basic conformable fractional Taylor expansion series, i.e.*

$$R(t > a; \epsilon, \alpha) = \sum_{k=0}^{\infty} \frac{\left(T^{[\alpha]-\alpha}_a R(t > a; \epsilon, \alpha)\right)^{(k)} (a)(t - a)^{k([\alpha]-\alpha)}}{([\alpha] - \alpha)^k k!}.$$

(HL_2) $f^{([\alpha]-1)}_{R(t>a;0,\alpha)}(t) = f^{([\alpha]-1)}(t)$.

(HL_3) *Based on the conformable fractional Taylor expansion of the representation function $R(t > a; \epsilon, \alpha)$,*

$$f^{([\alpha]-1)}_{R(t>a;\epsilon,\alpha)}(t) = f^{([\alpha]-1)} \left(t + \epsilon(t - a)^{[\alpha]-\alpha}(1 + O(\epsilon^k))\right), \quad k \in \mathbb{Z}_+.$$

The right sided extended conformable fractional differentiation operator of order $n < \alpha \leq n + 1$, $n \in \mathbb{Z}_+$, starting from upper terminal b of an n-differentiable function $f : (-\infty, b] \to \mathbb{R}$ is defined by:

$$\left(G^\alpha_{b_-} f\right)(t) = (-1)^{n+1} \lim_{\epsilon \to 0} \frac{f^{([\alpha]-1)}_{R(t<b;\epsilon,\alpha)}(t) - f^{([\alpha]-1)}(t)}{\epsilon}, \tag{2.27}$$

where, the representation function $R(t < b; \epsilon, \alpha)$ satisfies the following properties:

(HR_1) *$R(t<b;\epsilon,\alpha)$ has the basic conformable fractional Taylor expansion series, i.e.*

$$R(t<b;\epsilon,\alpha)=\sum_{k=0}^{\infty}\frac{\left({}_bT^{[\alpha]-\alpha}R(t<b;\epsilon,\alpha)\right)^{(k)}(b)(b-t)^{k([\alpha]-\alpha)}}{([\alpha]-\alpha)^k k!}.$$

(HR_2) $f^{([\alpha]-1)}_{R(t<b;0,\alpha)}(t)=f^{([\alpha]-1)}(t)$.

(HR_3) *Based on the conformable fractional Taylor expansion of the representation function $R(t>a;\epsilon,\alpha)$,*

$$f^{([\alpha]-1)}_{R(t<b;\epsilon,\alpha)}(t)=f^{([\alpha]-1)}\left(t+\epsilon(b-t)^{[\alpha]-\alpha}(1+O(\epsilon^k))\right),\quad k\in\mathbb{Z}_+.$$

Corresponding conformable fractional integration operators are given as follows.

Definition 2.14. ([1]) *Assume that $n<\alpha\leq n+1,\ n\in\mathbb{Z}_+$ and $f\in L[a,b]$. Then the left and right sided conformable fractional integration operators are defined as :*

$$I^{\alpha}_{a^+}f(t)=\frac{1}{n!}\int_a^t(t-s)^n(s-a)^{\alpha-n-1}f(s)ds, \tag{2.28}$$

$$I^{\alpha}_{b^-}f(t)=\frac{1}{n!}\int_t^b(s-t)^n(b-s)^{\alpha-n-1}f(s)ds. \tag{2.29}$$

Lemma 2.15. *Let $n<\alpha\leq n+1,\ n\in\mathbb{Z}_+$ and $f:[a,b]\to\mathbb{R}$ be an $(n+1)$-differentiable function on (a,b). Then for each $t\in(a,b)$,*

$$\left(I^{\alpha}_{a^+}G^{\alpha}_{a^+}f\right)(t)=f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)(t-a)^k}{k!}, \tag{2.30}$$

$$\left(I^{\alpha}_{b^-}G^{\alpha}_{b^-}f\right)(t)=f(t)-\sum_{k=0}^{n}(-1)^k\frac{f^{(k)}(b)(b-t)^k}{k!}. \tag{2.31}$$

Proof. We just prove the assertion (2.30) and omit proof of the (2.31). Considering definition (2.28) and taking $\beta=\alpha-n$, one has

$$\left(I^{\alpha}_{a^+}G^{\alpha}_{a^+}f\right)(t)=I^{n+1}_{a^+}\left((t-a)^{\beta-1}G^{\alpha}_{a^+}f^{(n)}(t)\right).$$

Making use of the golden identity (2.25) on the $G^{\alpha}_{a^+}f^{(n)}(t)$, it follows that

$$\left(I^{\alpha}_{a^+}G^{\alpha}_{a^+}f\right)(t)=I^{n+1}_{a^+}\left((t-a)^{\beta-1}(t-a)^{1-\beta}f^{(n+1)}(t)\right).$$

Equivalently, we have

$$\left(I^{\alpha}_{a^+}G^{\alpha}_{a^+}f\right)(t)=\left(I^{n+1}_{a^+}f^{(n+1)}\right)(t).$$

Imposing $n+1$ times the integration by parts on the recent integration formula gives us the desired result (2.30). This completes the proof. □

Now, we are ready to apply the extended conformable fractional differentiation operators $G^{\alpha}_{a^+}$ defined by (2.21) and $G^{\alpha}_{b^-}$ defined by (2.23) in theory of the fractional differential equations and examine their abilities to verify dynamics of the related fractional differential equations in comparison with other fractional order differential operators such as Riemann-Liouville ones.

3. Liapunov Inequalities and Applications

We begin the applied aspect of our work with recalling the conformable fractional half-linear boundary value problem

$$\begin{cases} \Theta_{\beta_2}\left(G_{a^+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\right) + \Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(p(t)u\right) = 0, \ a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$ and $G_{a^+}^{\alpha}$ denotes the left sided extended conformable fractional derivative of order α defined by (2.21). The first step to obtain Liapunov inequality of the boundary value problem (3.1), is to characterization of the Green function of the (3.1). Thereby, we have the following.

Lemma 3.1. *Assume that $1 < \alpha < 2$ and $h \in L([a,b];\mathbb{R})$. Then each nontrivial solution $u(t)$ of the conformable fractional half-linear boundary value problem*

$$\begin{cases} \Theta_{\beta_2}\left(G_{a^+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\right) + \left(\Theta_{\beta_2}(t-a)\right).h(t) = 0, \ a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{3.2}$$

uniquely solves the integral equation

$$u(t) = \Theta_{\beta_1^{-1}}\left(\int_a^b \mathcal{G}(t,s)\Theta_{\beta_2^{-1}}(h(s))ds\right), \tag{3.3}$$

where

$$\mathcal{G}(t,s) = \frac{1}{b-a}\begin{cases} [(t-a)(b-s)-(b-a)(t-s)](s-a)^{\alpha-1} \ ; & a < s \le t < b, \\ (t-a)(b-s)(s-a)^{\alpha-1} \ ; & a < s \le t < b. \end{cases} \tag{3.4}$$

Proof. First let us point out that the half-linear operator $\Theta_p(u)$ is invertible with the inversion $\Theta_{p^{-1}}(u)$. Therefore, taking the inversion $\Theta_{\beta_2^{-1}}$ and then conformable fractional integration on both sides of the governing equation

$$\Theta_{\beta_2}\left(G_{a^+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\right) + \left(\Theta_{\beta_2}(t-a)\right).h(t) = 0,$$

and then making use of the composition rule (2.30), it follows that

$$\left(\Theta_{\beta_1}u\right)(t) = c_0 + c_1(t-a) - \int_a^t (t-s)(s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds. \tag{3.5}$$

Imposing the first boundary condition $u(a) = 0$, it follows that $c_0 = 0$. On the other hand, the second boundary condition $u(b) = 0$, gives us the constant $c_1$as follows

$$c_1 = \frac{1}{b-a}\int_a^b (b-s)(s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds. \tag{3.6}$$

Now, using the substitution (3.6) into the (3.5), we conclude that

$$\begin{aligned}
\left(\Theta_{\beta_1} u\right)(t) &= \frac{t-a}{b-a}\int_a^b (b-s)(s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds \\
&\quad - \int_a^t (t-s)(s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds \\
&= \frac{1}{b-a}\int_a^t [(t-a)(b-s)-(b-a)(t-s)](s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds \\
&\quad + \frac{1}{b-a}\int_t^b (t-a)(b-s)(s-a)^{\alpha-1}\Theta_{\beta_2^{-1}}(h(s))ds \\
&= \int_a^b \mathcal{G}(t,s)\Theta_{\beta_2^{-1}}(h(s))ds.
\end{aligned}$$

At the end, applying the inversion $\Theta_{\beta_1^{-1}}$ on both sides of the resulting equation, it follows that

$$u(t) = \Theta_{\beta_1^{-1}}\left(\int_a^b \mathcal{G}(t,s)\Theta_{\beta_2^{-1}}(h(s))ds\right),$$

in which the Green function $\mathcal{G}(t,s)$ is defined by (3.4). This completes the proof. □

The second step to obtain the Liapunov inequality of the boundary value problem (3.1), turns to analyze of the Green function $\mathcal{G}(t,s)$ defined by (3.4). To this aim, we present the following technical lemma.

Lemma 3.2. *The Green function $\mathcal{G}(t,s)$ defined by (3.4), satisfies the following assertions:*

(i) *$\mathcal{G}(t,s)$ is continuous on $(a,b)\times(a,b)$;*

(ii) $\sup\limits_{t,s\in(a,b)} \mathcal{G}(t,s) = \mathcal{G}\left(b, \dfrac{a+(\alpha-1)b}{\alpha}\right) = \dfrac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha$.

Proof. The assertion (i) is immediate. So, we prove the property (ii). Let us recall once again the Green function $\mathcal{G}(t,s)$ as follows

$$\mathcal{G}(t,s) = \frac{1}{b-a}\begin{cases} \mathcal{G}_1(t,s); & a < s \le t < b, \\ \\ \mathcal{G}_2(t,s); & a < t \le s < b, \end{cases}$$

where

$$\mathcal{G}_1(t,s) = [(t-a)(b-s)-(b-a)(t-s)](s-a)^{\alpha-1}, \tag{3.7}$$

$$\mathcal{G}_2(t,s) = (t-a)(b-s)(s-a)^{\alpha-1}. \tag{3.8}$$

As can be observed,

$$\sup_{t,s\in(a,b)} \mathcal{G}(t,s) = \sup_{t,s\in(a,b)} \mathcal{G}_2(t,s).$$

In one hand

$$\frac{\partial}{\partial t}\mathcal{G}_2(t,s) = \frac{(b-s)(s-a)^{\alpha-1}}{b-a} > 0, \tag{3.9}$$

and on the other hand,

$$\frac{\partial}{\partial s}\mathcal{G}_2(t,s) = \frac{(t-a)(s-a)^{\alpha-2}}{b-a}\{a+(\alpha-1)b-\alpha s\}.$$

Therefore,

$$\frac{\partial}{\partial s}\mathcal{G}_2(t,s)\begin{cases} >0; & s<\dfrac{a+(\alpha-1)b}{\alpha}, \\ \\ <0; & s>\dfrac{a+(\alpha-1)b}{\alpha}. \end{cases}$$

Hence, we conclude that

$$\sup_{s\in(a,b)}\mathcal{G}_2(t,s)=\mathcal{G}\left(t,\frac{a+(\alpha-1)b}{\alpha}\right). \tag{3.10}$$

As a result, gathering (3.9) and (3.10) we conclude that

$$\sup_{t,s\in(a,b)}\mathcal{G}(t,s)=\mathcal{G}\left(b,\frac{a+(\alpha-1)b}{\alpha}\right)=\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}\,(b-a)^{\alpha}\,.$$

This completes the proof. □

In what follows everywhere needed by $\|.\|$, we mean the standard sup-norm. Now, we can extract Liapunov inequality of the conformable fractional half-linear boundary value problem (3.1). To this aim, we present the following theorem.

Theorem 3.3. *Suppose that $u(t)$ is a nontrivial solution of the conformable fractional half-linear boundary value problem*

$$\begin{cases} \Theta_{\beta_2}\left(G^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right)+\Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(p(t)u\right)=0, \;\; a<t<b, \\ u(a)=0, \quad u(b)=0, \end{cases} \tag{3.11}$$

Then the Liapunov inequality

$$\int_a^b |p(s)|^{\beta_1}ds>(\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}, \tag{3.12}$$

holds.

Proof. In the light of Lemma 3.1, we can transform the boundary value problem (3.11) into the integral equation

$$u(t)=\Theta_{\beta_1^{-1}}\left(\int_a^b \mathcal{G}(t,s)\left(\Theta_{\beta_1}pu\right)(s)ds\right),$$

where $\mathcal{G}(t,s)$ is defined by (3.4). Since $u(t)$ is a nontrivial solution of the boundary value problem (3.11), making use of the property (ii) in Lemma 3.2, it follows that

$$\|u\|<\|u\|\left(\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}\,(b-a)^{\alpha}\right)^{\dfrac{1}{\beta_1}}\left(\int_a^b |p(s)|^{\beta_1}ds\right)^{\dfrac{1}{\beta_1}}.$$

Equivalently, one has

$$\int_a^b |p(s)|^{\beta_1}ds>(\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}.$$

This completes the proof. □

In this position, we shall consider the fractional boundary value problem (1.2) and repeat the above steps to obtain its Liapunov inequality. To this aim, first we present some standard definitions and lemmas form fractional calculus. For more discussion and detailed consultation, we refer interested followers to the basic references [12],[14],[15],[17].

Definition 3.4. [12] *The left and right sided Riemann-Liouville fractional integrals of order $\rho \geq 0$ for function $x \in L^1(a,b)$ are defined as below:*

$$J^{\rho}_{a^+(b_-)}x(t) = \begin{cases} J^{\rho}_{a^+}x(t) = \dfrac{1}{\Gamma(\rho)}\displaystyle\int_a^t (t-s)^{\rho-1}x(s)ds; & \rho > 0, \\ J^{\rho}_{b_-}x(t) = \dfrac{1}{\Gamma(\rho)}\displaystyle\int_t^b (s-t)^{\rho-1}x(s)ds; & \rho > 0, \\ x(t) \qquad\qquad\qquad\qquad\qquad ; & \rho = 0. \end{cases} \tag{3.13}$$

Definition 3.5. [12] *The left and right sided Riemann-Liouville fractional derivatives of order $\rho \geq 0$ for function $x \in L^1(a,b)$ are defined as below:*

$$D^{\rho}_{a^+(b_-)}x(t) = \begin{cases} D^{\rho}_{a^+}x(t) = \dfrac{1}{\Gamma(n-\rho)}\left(\dfrac{d^n}{dt^n}\right)\displaystyle\int_a^t (t-s)^{n-\rho-1}x(s)ds; & \rho > 0, \\ D^{\rho}_{b_-}x(t) = \dfrac{(-1)^n}{\Gamma(n-\rho)}\left(\dfrac{d^n}{dt^n}\right)\displaystyle\int_t^b (s-t)^{n-\rho-1}x(s)ds; & \rho > 0, \\ x(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad ; & \rho = 0. \end{cases} \tag{3.14}$$

where $n = [\rho] + 1$.

Lemma 3.6. [12] *Let $x_{n-\rho,a}(t) = J^{n-\rho}_{a^+}x(t)$ and $x_{n-\rho,b}(t) = J^{n-\rho}_{b_-}x(t)$ be the left and right sided Riemann-Liouville fractional integrals of order $n-\rho$, respectively. If $x(t) \in L^1([a,b];\mathbb{R})$ and $x_{n-\rho,.}(t) \in AC^n([a,b];\mathbb{R})$, then the following equalities hold:*

$$D^{\rho}_{a^+}J^{\rho}_{a^+}x(t) = x(t), \quad J^{\rho}_{a^+}D^{\rho}_{a^+}x(t) = x(t) + \sum_{k=1}^{n}\frac{x^{(n-k)}_{n-\rho,a}(a)}{\Gamma(\rho-k+1)}(t-a)^{\rho-k}, \tag{3.15}$$

$$D^{\rho}_{b_-}J^{\rho}_{b_-}x(t) = x(t), \quad J^{\rho}_{b_-}D^{\rho}_{b_-}x(t) = x(t) + \sum_{k=1}^{n}\frac{(-1)^{(n-k)}x^{(n-k)}_{n-\rho,b}(b)}{\Gamma(\rho-k+1)}(b-t)^{\rho-k}, \tag{3.16}$$

where $\rho > 0,\ n = [\rho] + 1$.

Lemma 3.7. *Suppose that $1 < \alpha < 2$, and $h \in L([a,b];\mathbb{R})$. Then each nontrivial solution $u(t)$ of the fractional half-linear boundary value problem*

$$\begin{cases} \Theta_{\beta_2}\Big(D^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\Big) + \left(\Theta_{\beta_2}(t-a)\right).h(t) = 0, \ a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{3.17}$$

uniquely solves the integral equation

$$u(t) = \Theta_{\beta_1^{-1}}\left(\int_a^b \mathcal{K}(t,s)\left(\Theta_{\beta_2^{-1}}h\right)(s)ds\right), \tag{3.18}$$

where

$$\mathcal{K}(t,s)=\begin{cases}\dfrac{[(t-a)^{\alpha-1}(b-s)^{\alpha-1}-(t-s)^{\alpha-1}(b-a)^{\alpha-1}](s-a)}{(b-a)^{\alpha-1}\Gamma(\alpha)} & ;\ a<s\le t<b,\\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}(s-a)}{(b-a)^{\alpha-1}\Gamma(\alpha)} & ;\ a<s\le t<b.\end{cases} \tag{3.19}$$

Proof. Making use of the composition rule (3.15), an argument similar to that of the Lemma 3.1 gives us the desired result (3.19). So, we omit details of the proof here. □

Lemma 3.8. *The Green function $\mathcal{K}(t,s)$ defined by (3.19), satisfies the following properties:*

(i) *$\mathcal{K}(t,s)$ is continuous on $(a,b)\times(a,b)$;*

(ii) $\sup\limits_{t,s\in(a,b)}\mathcal{K}(t,s)=\mathcal{K}\left(b,\dfrac{b+(\alpha-1)a}{\alpha}\right)=\dfrac{1}{\Gamma(\alpha)}\dfrac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)^{\alpha}.$

Proof. The proof process is similar to the proof of Lemma 3.2. So, we omit it here. □

Now, we are ready to obtain the Liapunov inequality of the fractional half-linear boundary value problem (1.2). In the light of the technical lemmas 3.7 and 3.8, we give the following theorem without proof.

Theorem 3.9. *Suppose that $u(t)$ is a nontrivial solution of the fractional half-linear boundary value problem*

$$\begin{cases}\Theta_{\beta_2}\left(D^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right)+\Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(q(t)u\right)=0,\ a<t<b,\\ u(a)=0,\quad u(b)=0.\end{cases} \tag{3.20}$$

Then the Liapunov inequality

$$\int_a^b|q(s)|^{\beta_1}ds>(\alpha-1)\Gamma(\alpha)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}, \tag{3.21}$$

holds.

Remark 3.10. *Let*

$$A:=(\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}.$$

Hence, the Liapunov inequalities (3.12) and (3.21) imply that

$$\begin{aligned}\int_a^b|p(s)|^{\beta_1}ds>&A,\\ \int_a^b|q(s)|^{\beta_1}ds>&\Gamma(\alpha)A.\end{aligned}$$

On the other hand, since $1<\alpha<2$, then $0<\Gamma(\alpha)<1$. Therefore, in view point of optimality we conclude that if we take $p\equiv q$, the conformable Liapunov inequality (3.12) is better than the Riemann-Liouville Liapunov inequality (3.21).

Having the Liapunov inequalities (3.12) and (3.21) in hand, it is time to examine their abilities to establish dynamics of the fractional half-linear boundary value problems (1.1) and (1.2). To this aim, first we need the following definition.

Definition 3.11. *Let $1 < \alpha < 2$. Then for each $X \in \{G, D\}$, the fractional differential equation*

$$\Theta_{\beta_2}\left(X^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right) + \Theta_{\beta_2}(t-a)\Theta_{\beta_1\beta_2}\left(q(t)u\right) = 0, \quad a < t < b,$$

is said to be disconjugate, if and only if each nontrivial solution $u(t)$ has less than $[\alpha]+1$ zeros on interval $[a,b]$.

Here, we classify applications of the Liapunov inequalities (3.12) and (3.21) as follows.

- ▷ **Disconjugacy**.

  Theorem 3.12. *Assume that*
$$\int_a^b |p(s)|^{\beta_1} ds \leq (\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}. \tag{3.22}$$
  Then, the conformable fractional half-linear boundary value problem (1.1) is disconjugate on $[a,b]$.

  Proof. Suppose on the contrary that, the boundary value problem (1.1) is not disconjugate on $[a,b]$. So, in accordance with definition 3.11 there exists a nontrivial solution $u(t)$ having at least two zeros $t_1, t_2 \in [a,b]$. Thus, Theorem 3.3 implies that
$$\int_{t_1}^{t_2} |p(s)|^{\beta_1} ds > (\alpha-1)\left(\frac{\alpha}{(\alpha-1)(t_2-t_1)}\right)^{\alpha}.$$
  Consequently, we can deduce the Liapunov inequality
$$\int_a^b |p(s)|^{\beta_1} ds > (\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}.$$
  But this inequality contradicts the inequality (3.22). Thus, the fractional boundary value problem (1.1) is disconjugate on $[a,b]$. The proof is completed. □

  Lemma 3.13. *Assume that*
$$\int_a^b |q(s)|^{\beta_1} ds \leq (\alpha-1)\Gamma(\alpha)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}.$$
  Then, the fractional half-linear boundary value problem (1.2) is disconjugate on $[a,b]$.

- ▷ **Non-existence**. As we observed above, disconjugacy of the boundary value problems (1.1) and (1.2) can be considered as a nonexistence criterion for nontrivial solutions of these boundary value problems. This criterion is given by the following theorem.

  Theorem 3.14. *Assume that*
$$\int_a^b |p(s)|^{\beta_1} ds \leq (\alpha-1)\left(\frac{\alpha}{(\alpha-1)(b-a)}\right)^{\alpha}. \tag{3.23}$$
  Then, the conformable fractional half-linear boundary value problem (1.1) has no nontrivial solution.

Proof. Suppose on the contrary that, the boundary value problem (1.1) has at least one nontrivial solution $u(t)$. So, $u(t)$ satisfies the boundary conditions $u(a) = 0$ and $u(b) = 0$. Therefore, Theorem 3.3 implies that

$$\int_a^b |p(s)|^{\beta_1} ds > (\alpha - 1) \left(\frac{\alpha}{(\alpha-1)(b-a)} \right)^{\alpha}.$$

The recent Liapunov inequality makes contradiction with the assumption (3.23). Hence, the fractional boundary value problem (1.1) has no nontrivial solution. This completes the proof. □

Lemma 3.15. *Assume that*

$$\int_a^b |q(s)|^{\beta_1} ds \leq (\alpha - 1)\Gamma(\alpha) \left(\frac{\alpha}{(\alpha-1)(b-a)} \right)^{\alpha}.$$

Then, the fractional half-linear boundary value problem (1.2) has no nontrivial solution.

▷ **Upper bound estimation for maximum number of zeros of the nontrivial solutions**.

Theorem 3.16. *Let $u(t)$ be a nontrivial solution of the conformable fractional half-linear boundary value problem (1.1). If $\{t_k\}_{k=1}^{2N+1}$, $N \in \mathbb{N}$, be an increasing sequence of zeros of the $u(t)$ in a compact interval I with length $l(I)$, then*

$$N < \left\{ \frac{(\alpha-1)^{\alpha-1}}{\left(\dfrac{\alpha}{l(I)}\right)^{\alpha}} \sum_{k=1}^{N} \int_{t_{2k-1}}^{t_{2k+1}} |p(s)|^{\beta_1} ds \right\}^{\dfrac{1}{\alpha+1}}. \tag{3.24}$$

Proof. For each $k = 1, 2, ..., N$, one may apply Theorem 3.3 to the interval $[t_{2k-1}, t_{2k+1}] \subset I$. Hence the Liapunov inequality (3.12) implies that

$$\int_{t_{2k-1}}^{t_{2k+1}} |p(s)|^{\beta_1} ds > \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}} (t_{2k+1} - t_{2k-1})^{-\alpha}.$$

Taking the sum on both sides of the recent inequality for k, from 1 to N, one has

$$\sum_{k=1}^{N} \int_{t_{2k-1}}^{t_{2k+1}} |p(s)|^{\beta_1} ds > \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}} \sum_{k=1}^{N} (t_{2k+1} - t_{2k-1})^{-\alpha}. \tag{3.25}$$

It is clear by assumption that for each $k = 1, 2, ..., N$, $a_k = t_{2k+1} - t_{2k-1} > 0$. So, the concave-up function $\psi(x) := x^{-\alpha}$ for $\alpha > 0$ on $(0, \infty)$, yields the following inequality

$$\frac{1}{N} \sum_{k=1}^{N} \psi(a_k) \geq \psi\left(\frac{1}{N} \sum_{k=1}^{N} a_k \right), \quad \text{(see Theorem 2.3, [4])}.$$

Applying this inequality on the right-side of (3.25), we conclude that

$$\begin{aligned}\sum_{k=1}^{N}\int_{t_{2k-1}}^{t_{2k+1}}|p(s)|^{\beta_1}ds &> \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}N\left(\frac{1}{N}\sum_{k=1}^{N}(t_{2k+1}-t_{2k-1})\right)^{-\alpha}\\ &= \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}N^{\alpha+1}(t_{2N+1}-t_1)^{-\alpha}\\ &\geq \frac{\left(\frac{\alpha}{l(I)}\right)^\alpha}{(\alpha-1)^{\alpha-1}}N^{\alpha+1}.\end{aligned}$$

This completes the proof. □

Lemma 3.17. *Let $u(t)$ be a nontrivial solution of the fractional half-linear boundary value problem (1.2). If $\{t_k\}_{k=1}^{2N+1}$, $N\in\mathbb{N}$, be an increasing sequence of zeros of the $u(t)$ in a compact interval I with length $l(I)$, then*

$$N < \left\{\frac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\left(\frac{\alpha}{l(I)}\right)^\alpha}\sum_{k=1}^{N}\int_{t_{2k-1}}^{t_{2k+1}}|p(s)|^{\beta_1}ds\right\}^{\frac{1}{\alpha+1}}. \tag{3.26}$$

▷ **Distence between consecutive zeros of an oscillatory solution.**

Theorem 3.18. *Let $u(t)$ be an oscillatory solution of the conformable fractional half-linear boundary value problem (1.1) with $\{t_n\}_{n=1}^{\infty}$ its increasing sequence of zeros in $[0,\infty)$. Assume for any positive constant M, we have*

$$\int_t^{t+M}|p(s)|^{\beta_1}ds \to 0, \quad \text{as } t\to\infty. \tag{3.27}$$

Then, $t_{n+2}-t_n\to\infty$, as $n\to\infty$.

Proof. Suppose on the contrary that, there exists a positive constant M and a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ of the $\{t_n\}_{n=1}^{\infty}$ such that $t_{n_k+2}-t_{n_k}\leq M$ for all large k. So, in the light of the assumption (3.27), it follows that

$$\int_{t_{n_k}}^{t_{n_k+2}}|p(s)|^{\beta_1}ds \leq \int_{t_{n_k}}^{t_{n_k}+M}|p(s)|^{\beta_1}ds \to 0, \quad \text{as } k\to\infty.$$

Implementing Theorem 3.3 to the interval $[t_{n_k}, t_{n_k+2}]$, we reach to the following Liapunov inequality

$$\int_{t_{n_k}}^{t_{n_k+2}}|p(s)|^{\beta_1}ds > (\alpha-1)\left(\frac{\alpha}{(\alpha-1)(t_{n_k+2}-t_{n_k})}\right)^\alpha.$$

Equivalently, we have

$$\begin{aligned}1 &< \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(t_{n_k+2}-t_{n_k})^\alpha\int_{t_{n_k}}^{t_{n_k+2}}|p(s)|^{\beta_1}ds\\ &\leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}M^\alpha\underbrace{\int_{t_{n_k}}^{t_{n_k+2}}|p(s)|^{\beta_1}ds}_{\to 0} \to 0, \quad \text{as } k\to\infty.\end{aligned}$$

Resulting contradiction completes the proof. □

Lemma 3.19. *Let $u(t)$ be an oscillatory solution of the fractional half-linear boundary value problem (1.2) with $\{t_n\}_{n=1}^{\infty}$ its increasing sequence of zeros in $[0,\infty)$. Assume that for any positive constant M, we have*

$$\int_t^{t+M} |q(s)|^{\beta_1} ds \to 0, \quad \text{as } t \to \infty. \tag{3.28}$$

Then, $t_{n+2} - t_n \to \infty$, as $n \to \infty$.

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