Journal of Fractional Calculus and Applications
Vol. 8(2) July 2017, pp. 211-226.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

A STUDY OF MELLIN TRANSFORM OF FRACTIONAL OPERATORS IN BICOMPLEX SPACE AND APPLICATIONS

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Abstract. In this paper, we obtain the bicomplex Mellin transform of Riemann-Liouville fractional integral and Caputo fractional derivative of order $\alpha(\geq 0)$ of certain functions and some of their properties. Application of bicomplex Mellin transform has been illustrated to find the solution of differential equation involving fractional derivatives of bicomplex-valued functions. Bicomplex Mellin transform of fractional operators provide large class of frequency domain, which are useful in solving the fractional differential equations of bicomplex-valued functions. Also, the real world problems modelled via fractional order derivatives present better results when matching their mathematical representation with experimental data.

1. Introduction

In 1892, Segre Corrado [6] defined bicomplex numbers as

$$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 |\ x_0, x_1, x_2, x_3 \in C_0\},$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2z_2 |\ z_1, z_2 \in C_1\}.$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1,\ i_1i_2 = i_2i_1 = j$, $j^2 = 1$ and C_0, C_1 and C_2 are sets of real numbers, complex numbers and bicomplex numbers, respectively. The set of bicomplex numbers is a commutative ring with unit and zero divisors. Hence, contrary to quaternions, bicomplex numbers are commutative with some non-invertible elements situated on the null cone.

In 1928 and 1932, Futagawa Michiji originated the concept of holomorphic functions of a bicomplex variable in a series of papers [15], [16]. In 1934, Dragoni [11] gave some basic results in the theory of bicomplex holomorphic functions while Price G.B. [9] and Rönn S. [29] have developed the bicomplex algebra and function theory.

In recent developments, authors have done efforts to extend Polygamma function [25], inverse Laplace transform, it's convolution theorem [21], Stieltjes transform [19], Tauberian Theorem of Laplace-Stieltjes transform [23] and Bochner Theorem

2010 *Mathematics Subject Classification.* Primary 30G35; Secondary 42B10.
Key words and phrases. Fractional calculus, Riemann-Liouville fractional integral, Caputo fractional derivative, Bicomplex numbers and Bicomplex Mellin transform.
Submitted Jan. 18, 2017.

of Fourier-Stieltjes transform [20] in the bicomplex variable from their complex counterpart. In their procedure, the idempotent representation of bicomplex numbers plays a vital role.

Hjalmar Mellin (1854-1933, see, e.g. [2]) gave his name to the Mellin transform that associates to a complex-valued function $f(t)$ defined over the interval $(0, \infty)$, the function of complex variable s, as

$$\bar{f}(s) = \int_0^\infty t^{s-1} f(t) dt.$$

The change of variables $t = e^{-x}$ shows that the Mellin transform is closely related to the Laplace transform. General properties of the Mellin transform are usually treated in detail in books on integral transforms, like those of Poularikas A.D. [2] and Davies B. [5]. In 1959, Francis R.G. [7] discussed the application of complex Mellin transform to networks with time-varying parameters. In 1995, Flajolet P. et al. [18] used Mellin transform for the asymptotic analysis of harmonic sums. In 2016, Kilicamn and Omran [3] established some results on Mellin transform of fractional integral and differential operators and discussed their properties.

Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

(where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$).

This representation of a bicomplex number is known as *Idempotent Representation* of ξ. The coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the *Idempotent Components* of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called *Idempotent Basis*.

Cartesian Set: The Auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{w_1 = z_1 - i_1 z_2, \ \forall\, z_1, z_2 \in C_1\}, \ A_2 = \{w_2 = z_1 + i_1 z_2, \ \forall\, z_1, z_2 \in C_1\}.$$

A cartesian set $X_1 \times_e X_2$ determined by $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ and is defined as:

$$X_1 \times_e X_2 = \{z_1 + i_2 z_2 \in C_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2, \ w_1 \in X_1, w_2 \in X_2\}.$$

With the help of idempotent representation, we define projection mappings $P_1 : C_2 \to A_1 \subseteq C_1$, $P_2 : C_2 \to A_2 \subseteq C_1$ as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1, \ \forall\, z_1 + i_2 z_2 \in C_2,$$

$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2, \ \forall\, z_1 + i_2 z_2 \in C_2.$$

In the following theorem, Price G.B. discuss the convergence of bicomplex function with respect to it's idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1.1 (Price G.B. [9]). *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ under functions $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$ are convergent in domains D_1 and D_2, respectively.*

1.1. **Bicomplex Mellin Transform.** Agarwal R. et al. [22] defined Mellin transform in bicomplex variable and discussed its properties. Also, discussed its application in solving the transmission line equation using bicomplex form.

Definition 1.2. [22] *Let $f(t)$ be a bicomplex-valued continuous function on the interval $(0,\infty)$ with $f(t)=O\left(t^{-\alpha}\right)$ as $t\to 0^{+}$ and $f(t)=O\left(t^{-\beta}\right)$ as $t\to\infty$, where $\alpha<\beta$. Then bicomplex Mellin transform of $f(t)$ defined as*

$$\mathfrak{M}[f(t);\xi]=\int_0^{\infty} t^{\xi-1}f(t)dt=\bar{f}(\xi),\quad \xi\in\Omega$$

where $\bar{f}(\xi)$ is analytic and convergent in Ω defined in

$$\Omega=\{\xi\in C_2:\alpha+|Im_j(\xi)|<Re(\xi)<\beta-|Im_j(\xi)|\} \tag{1.1}$$

where $Im_j(\xi)$ denotes the imaginary part w.r.t. j unit of a bicomplex number.

1.2. **Inverse of Bicomplex Mellin Transform.** Agarwal R. et al. [22] defined inverse Mellin transform in bicomplex variable as

Definition 1.3. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued continuous function $f(t)$. Then inverse formula for bicomplex Mellin transform as*

$$f(t)=\frac{1}{2\pi i_1}\int_{\Omega} t^{-\xi}\bar{f}(\xi)d\xi, \tag{1.2}$$

where $\Omega=(\Omega_1,\Omega_2)$ and $\Omega_1,\ \Omega_2$ defined as

$$\Omega_1=\{s_1\in C_1:\alpha<Re(s_1)<\beta\} \tag{1.3}$$

and

$$\Omega_2=\{s_2\in C_1:\alpha<Re(s_2)<\beta\}. \tag{1.4}$$

In [28], Goyal S.P. et al. defined bicomplex gamma and beta function and discussed its various properties.

Definition 1.4 (Bicomplex Gamma function)**.** *Let $\xi\in C_2$, $p=p_1e_1+p_2e_2\in C_2$, $p_1,\ p_2\in(0,\infty)$, then*

$$\Gamma(\xi)=\int_H e^{-p}p^{\xi-1}dp \tag{1.5}$$

where

$$H=(\gamma_1,\gamma_2),\ \gamma_1\equiv\gamma_1(p_1),\ \gamma_2\equiv\gamma_2(p_2). \tag{1.6}$$

$\Gamma(\xi)$ exist provided the integral exist.

Definition 1.5 (Bicomplex Beta function)**.** *Let $\xi=u_1+i_2u_2,\ \eta=v_1+i_2v_2\in C_2$, $p=p_1e_1+p_2e_2\in C_2$, $p_1,\ p_2\in[0,1]$ with $Re(u_1)>|Im(u_2)|$ and $Re(v_1)>|Im(v_2)|$ then*

$$B(\xi,\eta)=\int_H p^{\xi-1}(1-p)^{\eta-1}dp \tag{1.7}$$

where $H=(\gamma_1,\gamma_2)$, $\gamma_1\equiv\gamma_1(p_1)$ and $\gamma_2\equiv\gamma_2(p_2)$.

We shall be requiring following result in the sequel.

Bicomplex Mellin transform of $g(t) = \begin{cases} (1-t)^{\alpha-1}, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$ as follows:

$$\begin{aligned} \mathfrak{M}[g(t);\xi] &= \int_0^\infty t^{\xi-1} g(t) dt \\ &= \int_0^1 t^{\xi-1}(1-t)^{\alpha-1} dt \end{aligned}$$

where $\xi = \xi_1 + i_2\xi_2$ and $\alpha = \alpha_1 + i_2\alpha_2$ with $\mathrm{Re}(\xi_1) > |\mathrm{Im}(\xi_2)|$ and $\mathrm{Re}(\alpha_1) > |\mathrm{Im}(\alpha_2)|$, then

$$\begin{aligned} \mathfrak{M}[g(t);\xi] &= B(\xi,\alpha) \\ &= \frac{\Gamma(\alpha)\Gamma(\xi)}{\Gamma(\xi+\alpha)}. \end{aligned} \tag{1.8}$$

Mellin convolution of two bicomplex-valued functions can be defined as:

$$\begin{aligned} f(t) * g(t) &= \int_0^\infty \frac{1}{x} f(x) g\left(\frac{t}{x}\right) dx \\ f(t) \circ g(t) &= \int_0^\infty f(xt) g(x) dx. \end{aligned}$$

2. Basics of Fractional Calculus

Fractional calculus is a generalization of the classical calculus and it has been used in various fields of science and engineering. The fractional calculus is a powerful mathematical tool for the physical description systems that have long-term memory and long term spatial interactions (see, for details, Podlubny [12], Miller and Ross [13], Hilfer [26], Kilbas et al. [1] and Samko et al. [27]).

In [17], Klimek M. and Dziembowski D. applied Mellin transform to find the solution of fractional differential equations of complex-valued function. In [8], Francisco G.A.J. et al. proposed a fractional differential equation for the electrical RC and LC circuit in terms of the fractional time derivative of the Caputo type. In [10], Liang G. and Liu X. deduced a fractional-order model based on skin effect for frequency dependent transmission line model. In this paper voltage and currents at any location in transmission line can be calculated by the proposed fractional partial differential equations.

In this section, we give the definitions of Riemann-Liouville and Caputo fractional operators along the main properties.

Definition 2.1. *(see, e.g. Miller and Ross* [13, p. 45]*). Let $\alpha > 0$ and f be piecewise continuous on* $(0,\infty)$ *and integrable on any finite subinterval of* $[0,\infty)$*. Then for $t > 0$*

$${}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx$$

the Riemann-Liouville fractional integral of f of order α.

Definition 2.2. *(see, e.g. Miller and Ross* [13, p. 82]*). Let f be a function of class* $\boldsymbol{C}$ *and let $\alpha > 0$. Let n be the smallest integer that exceeds α. Then the fractional* *derivative of f of order α is defined as*

$$ {}_0D_t^{\alpha} f(t) = {}_0D_t^{n}\left[{}_0D_t^{-\beta} f(t)\right], \quad \alpha > 0,\ t > 0 $$

where $\beta = n - \alpha > 0$.

Some properties of Riemann-Liouville fractional operator are as follows:

Theorem 2.3. *(see, e.g. Miller and Ross* [13, Eq. 5.25, 6.1]*). Let α, β are two positive real number, then*

(a) ${}_0D_t^{\alpha}\left({}_0D_t^{-\beta} f(t)\right) = {}_0D_t^{\alpha-\beta} f(t)$,
(b) ${}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t) = {}_0D_t^{-\alpha-\beta} f(t)$,
(c) ${}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t) = {}_0D_t^{-\beta}\,{}_0D_t^{-\alpha} f(t)$.

For Riemann-Liouville operator ${}_0D_t^{\alpha}$ and $\alpha, n > 0$ the fractional derivative of the power function t^n (see, e.g. Miller and Ross [13, p. 36]) is given by

$$ {}_0D_t^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}. \tag{2.1} $$

Definition 2.4. *(Caputo M.* [14] *and see, e.g. Podlubny I.* [12, Eq. (2.138)]*). The Caputo fractional derivative of f for $\alpha > 0$ is defined as*

$$ {}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, \quad n-1 < \alpha \le n \tag{2.2} $$

$$ {}_0^CD_t^{\alpha} f(t) = {}_0D_t^{-(n-\alpha)} g(t), \quad g(t) = f^{(n)}(t),\ n-1 < \alpha \le n \tag{2.3} $$

provided the integral exist.

Some properties of Caputo fractional derivative are as follows:

Theorem 2.5. *(See, e.g. Kilbas A.A. et al.* [1, p. 95, 96]*). If $m-1 < \alpha \le m$, $m \in \mathbb{N}$ and function f s.t. integral (2.2) exist, then*

(a) ${}_0^CD_t^{\alpha}\left({}_0D_t^{-\alpha} f(t)\right) = f(t)$,
(b) ${}_0D_t^{-\alpha}\left({}_0^CD_t^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\left(\frac{x^k}{k!}\right)$.

The organization of this paper is as follows:
In Section 3, we present some useful properties of bicomplex Mellin transform in fractional calculus. In section 4, we discuss application of bicomplex Mellin transform in finding the solution of bicomplex partial differential equation generated by network model and last Section 5, contains the conclusion.

3. Properties of Bicomplex Mellin Transform

In this section, we discuss bicomplex Mellin transform of convolution of functions, Riemann-Liouville fractional integral and Caputo derivative of order $\alpha \ge 0$ of certain functions and some of their properties.

Theorem 3.1. *Let $\bar{f}(\xi)$ and $\bar{g}(\xi)$ are bicomplex Mellin transforms of bicomplex-valued functions $f(t)$ and $g(t)$ respectively. Then*

$$ \mathfrak{M}[f(t) * g(t); \xi] = \mathfrak{M}\left[\int_0^{\infty} \frac{1}{x} f(x) g\left(\frac{t}{x}\right) dx; \xi\right] = \bar{f}(\xi)\bar{g}(\xi), \quad \xi \in \Omega \tag{3.1} $$

and

$$\mathfrak{M}[f(t)\circ g(t);\xi]=\mathfrak{M}\left[\int_0^\infty f(xt)g(x)dx;\xi\right]=\bar{f}(\xi)\bar{g}(1-\xi),\quad \xi\in\Omega \tag{3.2}$$

where Ω *is defined in (1.1).*

Proof. We have, by definition,

$$\begin{aligned}\mathfrak{M}[f(t)*g(t);\xi]&=\mathfrak{M}\left[\int_0^\infty \frac{1}{x}f(x)g\left(\frac{t}{x}\right)dx;\xi\right]\\&=\int_0^\infty t^{\xi-1}dt\int_0^\infty f(x)g\left(\frac{t}{x}\right)\frac{dx}{x}\end{aligned}$$

By changing the order of integration

$$\begin{aligned}&=\int_0^\infty f(x)\frac{dx}{x}\int_0^\infty t^{\xi-1}g\left(\frac{t}{x}\right)dt\\&=\int_0^\infty f(x)dx\int_0^\infty (xy)^{\xi-1}g(y)dy,\quad \left[y=\frac{t}{x}\right]\\&=\int_0^\infty x^{\xi-1}f(x)dx\int_0^\infty y^{\xi-1}g(y)dy\\&=\bar{f}(\xi)\bar{g}(\xi).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathfrak{M}[f(t)\circ g(t);\xi]&=\mathfrak{M}\left[\int_0^\infty f(xt)g(x)dx;\xi\right]\\&=\int_0^\infty t^{\xi-1}dt\int_0^\infty f(xt)g(x)dx\end{aligned}$$

By changing the order of integration

$$\begin{aligned}&=\int_0^\infty g(x)dx\int_0^\infty t^{\xi-1}f(xt)dt\\&=\int_0^\infty g(x)dx\int_0^\infty y^{\xi-1}x^{1-\xi}f(y)\frac{dy}{x},\quad [y=xt]\\&=\int_0^\infty x^{1-\xi-1}g(x)dx\int_0^\infty y^{\xi-1}f(y)dy\\&=\bar{f}(\xi)\bar{g}(1-\xi).\end{aligned}$$

□

In the following theorem, we make efforts to find the bicomplex Mellin transform of the Riemann-Liouville fractional integrals.

Theorem 3.2. *Let* $\bar{f}(\xi)$ *be the bicomplex Mellin transform of bicomplex-valued function* $f(t)$*. Then for* $\alpha>0$

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}f(t);\xi\right]=\frac{\Gamma(1-\xi-\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi+\alpha),\quad \xi+\alpha\in\Omega \tag{3.3}$$

where Ω *is defined in (1.1).*

Proof. Since we know that

$$\begin{aligned}
{}_0D_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} f(tx)dx, \quad \left[x = \frac{\tau}{t}\right] \\
&= \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 f(tx)g(x)dx \qquad (3.4)
\end{aligned}$$

where

$$g(t) = \begin{cases} (1-t)^{\alpha-1}, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases} \qquad (3.5)$$

Then using equations (1.8), (3.2), (3.4) and (3.5), we get

$$\begin{aligned}
\mathfrak{M}\left[{}_0D_t^{-\alpha} f(t); \xi\right] &= \frac{1}{\Gamma(\alpha)} \bar{f}(\xi+\alpha) B(\alpha, 1-\xi-\alpha) \\
&= \frac{\Gamma(1-\xi-\alpha)}{\Gamma(1-\xi)} \bar{f}(\xi+\alpha).
\end{aligned}$$

□

In the following theorem, we make efforts to find the bicomplex Mellin transform of the Riemann-Liouville fractional derivative.

Theorem 3.3. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of the bicomplex-valued function $f(t)$. Then for $0 \le n-1 < \alpha < n$*

$$\mathfrak{M}\left[{}_0D_t^{\alpha} f(t); \xi\right] = \sum_{k=0}^{n-1} \frac{\Gamma(1-\xi+k)}{\Gamma(1-\xi)} \left[{}_0D_t^{\alpha-k-1} f(t) t^{\xi-k-1}\right]_0^\infty + \frac{\Gamma(1-\xi+\alpha)}{1-\xi} \bar{f}(\xi-\alpha), \qquad (3.6)$$

$$\xi - \alpha \in \Omega$$

where Ω defined in (1.1).

Proof. By taking the bicomplex Mellin transform, we get

$$\begin{aligned}
&\mathfrak{M}\left[{}_0D_t^{\alpha} f(t); \xi\right] = \int_0^\infty t^{\xi-1}\, {}_0D_t^{\alpha} f(t)dt \\
&= \left(\int_0^\infty t^{s_1-1}\, {}_0D_t^{\alpha} f_1(t)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1}\, {}_0D_t^{\alpha} f_2(t)dt\right) e_2 \\
&\qquad [\text{where } \xi = s_1e_1 + s_2e_2 \text{ and } f(t) = f_1(t)e_1 + f_2(t)e_2] \\
&= \left(\sum_{k=0}^{n-1} \frac{\Gamma(1-s_1+k)}{\Gamma(1-s_1)} \left[{}_0D_t^{\alpha} f_1(t) t^{s_1-k-1}\right]_0^\infty + \frac{\Gamma(1-s_1+\alpha)}{\Gamma(1-s_1)} \bar{f}_1(s_1-\alpha)\right) e_1 \\
&\quad + \left(\sum_{k=0}^{n-1} \frac{\Gamma(1-s_2+k)}{\Gamma(1-s_1)} \left[{}_0D_t^{\alpha} f_2(t) t^{s_2-k-1}\right]_0^\infty + \frac{\Gamma(1-s_2+\alpha)}{\Gamma(1-s_2)} \bar{f}_2(s_2-\alpha)\right) e_2
\end{aligned}$$

[using [12, Eq. (2.287)]]

$$= \sum_{k=0}^{n-1} \frac{\Gamma(1 - s_1e_1 - s_2e_2 + k)}{\Gamma(1 - s_1e_1 - s_2e_2)} \left[{}_0D_t^\alpha (f_1(t)e_1 + f_2(t)e_2)t^{s_1e_1+s_2e_2-k-1}\right]_0^\infty$$
$$+ \frac{\Gamma(1 - s_1e_1 - s_2e_2 + \alpha)}{\Gamma(1 - s_1e_1 - s_2e_2)} (\bar{f}_1(s_1 - \alpha)e_1 + \bar{f}_2(s_2 - \alpha)e_2)$$
$$= \sum_{k=0}^{n-1} \frac{\Gamma(1 - \xi + k)}{\Gamma(1 - \xi)} \left[{}_0D_t^\alpha f(t)t^{\xi-k-1}\right]_0^\infty + \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha). \tag{3.7}$$

□

Remark 3.4. In its particular case, if $0 < \alpha < 1$, then (3.7) becomes

$$\mathfrak{M}\left[{}_0D_t^\alpha f(t); \xi\right] = \left[{}_0D_t^\alpha f(t)t^{\xi-k-1}\right]_0^\infty + \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha). \tag{3.8}$$

If the function $f(t)$, $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$, where $\xi = s_1e_1 + s_2e_2$ are such that the substitutions of the limit $t = 0$ and $t = \infty$ make the first term of (3.8) zero, then (3.8) reduces to the

$$\mathfrak{M}\left[{}_0D_t^\alpha f(t); \xi\right] = \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha). \tag{3.9}$$

In the following theorem, we have found the bicomplex Mellin transform of the Caputo fractional derivative.

Theorem 3.5. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \le n - 1 \le \alpha < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0^CD_t^\alpha f(t); \xi\right] = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha + k - \xi)}{\Gamma(1 - \xi)} \left[f^{(k)}(t)t^{\xi-\alpha+k}\right]_0^\infty + \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha),$$
$$\xi - \alpha \in \Omega \tag{3.10}$$

where Ω defined in (1.1).

Proof. By taking the bicomplex Mellin transform, we get

$$\mathfrak{M}\left[{}_0^CD_t^\alpha f(t); \xi\right] = \int_0^\infty t^{\xi-1}\, {}_0^CD_t^\alpha f(t)dt$$
$$= \left(\int_0^\infty t^{s_1-1}\, {}_0^CD_t^\alpha f_1(t)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1}\, {}_0^CD_t^\alpha f_2(t)dt\right) e_2$$

[where $\xi = s_1e_1 + s_2e_2$ and $f(t) = f_1(t)e_1 + f_2(t)e_2$

$$= \left(\sum_{k=0}^{n-1} \frac{\Gamma(\alpha + k - s_1)}{\Gamma(1 - s_1)} \left[f_1^{(k)}(t)t^{s_1-\alpha+k}\right]_0^\infty + \frac{\Gamma(1 - s_1 + \alpha)}{\Gamma(1 - s_1)} \bar{f}_1(s_1 - \alpha)\right) e_1$$
$$+ \left(\sum_{k=0}^{n-1} \frac{\Gamma(\alpha + k - s_2)}{\Gamma(1 - s_2)} \left[f_2^{(k)}(t)t^{s_2-\alpha+k}\right]_0^\infty + \frac{\Gamma(1 - s_2 + \alpha)}{\Gamma(1 - s_2)} \bar{f}_2(s_2 - \alpha)\right) e_2$$

[using [12, Eq. (2.291)]]

$$= \sum_{k=0}^{n-1} \frac{\Gamma(\alpha + k - s_1e_1 - s_2e_2)}{\Gamma(1 - s_1e_1 - s_2e_2)} \left[\left(f_1^{(k)}(t)e_1 + f_2^{(k)}(t)e_2 \right) t^{s_1e_1+s_2e_2-\alpha+k} \right]_0^\infty$$
$$+ \frac{\Gamma(1 - s_1e_1 - s_2e_2 + \alpha)}{\Gamma(1 - s_1e_1 - s_2e_2)} \left(\bar{f}_1(s_1 - \alpha)e_1 + \bar{f}_2(s_2 - \alpha)e_2 \right)$$
$$= \sum_{k=0}^{n-1} \frac{\Gamma(\alpha + k - \xi)}{\Gamma(1 - \xi)} \left[f^{(k)}(t)t^{\xi-\alpha+k} \right]_0^\infty + \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha). \tag{3.11}$$

□

Remark 3.6. In its particular case, if $0 < \alpha < 1$, then (3.11) becomes

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t); \xi\right] = \frac{\Gamma(\alpha - \xi)}{\Gamma(1 - \xi)} \left[f(t)t^{\xi-\alpha} \right]_0^\infty + \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha) \tag{3.12}$$

If the function $f(t)$, $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$, where $\xi = s_1e_1 + s_2e_2$ are such that the substitutions of the limit $t = 0$ and $t = \infty$ make the first term of (3.11) zero, then (3.11) reduces to the

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t); \xi\right] = \frac{\Gamma(1 - \xi + \alpha)}{\Gamma(1 - \xi)} \bar{f}(\xi - \alpha). \tag{3.13}$$

Theorem 3.7. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \leq n - 1 < \alpha < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t); \xi\right] = \bar{f}(\xi), \quad \xi \in \Omega \tag{3.14}$$

where Ω defined in (1.1).

Proof. Since we know that

$${}_0^C D_t^\alpha \left[{}_0D_t^{-\alpha} f(t) \right] = f(t).$$

By taking the bicomplex Mellin transform on both side, we have

$$\mathfrak{M}\left[{}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t); \xi\right] = \bar{f}(\xi).$$

□

Deduction 3.8. If we take $f(t) = t^n U(t - t_0)$, then

$${}_0D_t^{-\alpha} t^n U(t - t_0) = \frac{(t - t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \sum_{r=0}^{n} (-1)^r \, {}^nC_r t^{n-r} u^r (t - t_0)^r du. \tag{3.15}$$

where $U(t - t_0)$ is unit step function and hence

$$\mathfrak{M}\left[\frac{1}{\Gamma(\alpha)} \, {}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t); \xi\right] = -\frac{t_0^{\xi+n}}{\xi + n}, \quad \mathrm{Re}(\xi + n) < -\left|\mathrm{Im}_j(\xi + n)\right|. \tag{3.16}$$

Proof. By applying the definition of Riemann-Liouville integral operator on $t^n U(t-t_0)$

$$
\begin{aligned}
{}_0D_t^{-\alpha} t^n U(t-t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} x^n U(x-t_0) dx \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-x)^{\alpha-1} x^n dx \\
&= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \left[t-u(t-t_0)\right]^n du, \ \left[\text{put } u = \frac{t-x}{t-t_0}\right] \\
&= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \sum_{r=0}^{n} (-1)^r \, {}^nC_r t^{n-r} u^r (t-t_0)^r du.
\end{aligned}
$$

Changing the order of integration and summation which is valid under the conditions of convergence, we get

$$
\begin{aligned}
{}_0D_t^{-\alpha} t^n U(t-t_0) &= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \sum_{r=0}^{n} (-1)^r \, {}^nC_r t^{n-r} (t-t_0)^r \int_0^1 u^{\alpha+r-1} du \\
&= \frac{1}{\Gamma(\alpha)} \sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{\alpha+r} t^{n-r} (t-t_0)^{\alpha+r}. \qquad (3.17)
\end{aligned}
$$

Therefore, making use of (3.14) and (3.17), we get the desired result (3.16). □

Theorem 3.9. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \leq n-1 < \alpha < n$, $n \in \mathbb{N}$, then*

$$
\mathfrak{M}\left[{}_0D_t^{-\alpha} \, {}_0^C D_t^\alpha f(t); \xi\right] = \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!(k+\xi)}, \quad |Im_j(k+\xi)| < Re(k+\xi) \quad (3.18)
$$

where Ω defined in (1.1).

Proof. Since we know that

$$
{}_0D_t^{-\alpha} \left[{}_0^C D_t^\alpha f(t)\right] = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \left(\frac{t^k}{k!}\right).
$$

By taking the bicomplex Mellin transform on both side, we have

$$
\begin{aligned}
\mathfrak{M}\left[{}_0D_t^{-\alpha} \, {}_0^C D_t^\alpha f(t); \xi\right] &= \mathfrak{M}\left[f(t); \xi\right] - \mathfrak{M}\left[\sum_{k=0}^{m-1} f^{(k)}(0) \left(\frac{t^k}{k!}\right); \xi\right] \\
&= \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} \int_0^\infty t^{\xi+k-1} dt \\
&= \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!(k+\xi)}.
\end{aligned}
$$

□

Theorem 3.10. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \leq n-1 < \alpha,\ \beta < n,\ n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t);\xi\right] = \frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\bar{f}(\alpha+\beta+\xi), \tag{3.19}$$

$$Re(\alpha+\beta+\xi) < 1 - |Im_j(\alpha+\beta+\xi)|\,,\ \alpha+\beta+\xi \in \Omega$$

where Ω defined in (1.1).

Proof. Since we know that

$${}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t) = {}_0D_t^{-\alpha-\beta} f(t).$$

By taking the bicomplex Mellin transform on both side, we have

$$\begin{aligned}\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t);\xi\right] &= \mathfrak{M}\left[{}_0D_t^{-\alpha-\beta} f(t);\xi\right]\\ &= \int_0^\infty t^{\xi-1}\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-x)^{\alpha+\beta-1} f(x)dxdt\end{aligned}$$

By changing the order of integration

$$= \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty f(x)dx\int_x^\infty t^{\xi-1}(t-x)^{\alpha+\beta-1}dt$$

Put $t = \frac{x}{u}$, then

$$\int_x^\infty t^{\xi-1}(t-x)^{\alpha+\beta-1}dt = x^{\alpha+\beta+\xi-1}\int_0^1 u^{-\alpha-\beta-\xi}(1-u)^{\alpha+\beta-1}du$$

Therefore,

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t);\xi\right] = \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty x^{\alpha+\beta+\xi-1} f(x)dx\int_0^1 u^{-\alpha-\beta-\xi}(1-u)^{\alpha+\beta-1}du$$

where $\alpha+\beta > 0,$ and $\mathrm{Re}(\alpha+\beta+\xi) < 1 - |\mathrm{Im}_j(\alpha+\beta+\xi)|$

After using beta function (1.7), we have

$$\begin{aligned}\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta} f(t);\xi\right] &= \frac{B(1-\alpha-\beta-\xi,\alpha+\beta)}{\Gamma(\alpha+\beta)}\int_0^\infty x^{\alpha+\beta+\xi-1} f(x)dx\\ &= \frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\bar{f}(\alpha+\beta+\xi).\end{aligned}$$

□

Deduction 3.11. For $0 \leq n-1 < \alpha,\ \beta < n,\ n \in \mathbb{N}$

$$\mathfrak{M}\left[\frac{1}{\Gamma(\beta)}\,{}_0D_t^{-\alpha}\left\{\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{\beta+r}t^{n-r}(t-t_0)^{\beta+r}\right\};\xi\right] = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}, \tag{3.20}$$

$$\mathrm{Re}(\xi+\alpha+\beta) < 1 - |\mathrm{Im}_j(\xi+\alpha+\beta)|\,,\ \mathrm{Re}(\xi+\alpha+\beta+n) < -|\mathrm{Im}_j(\xi+\alpha+\beta+n)|\,.$$

Proof. In the similar manner of equation (3.17) and using result (3.19), we get

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta}t^n U(t-t_0)\right] = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}$$

$$\mathfrak{M}\left[\frac{1}{\Gamma(\beta)}\,{}_0D_t^{-\alpha}\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{\beta+r}t^{n-r}(t-t_0)^{\beta+r};\xi\right] = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}.$$

□

Theorem 3.12. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$(a)\quad \mathfrak{M}\left[{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{1}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{1}{2}\right], \qquad \xi-\frac{1}{2}\in\Omega \tag{3.21}$$

$$(b)\quad \mathfrak{M}\left[{}_t^C D_\infty^{\frac{3}{2}} f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{3}{2}\right], \qquad \xi-\frac{3}{2}\in\Omega \tag{3.22}$$

where Ω defined in (1.1) and ${}_0D_t^{\alpha+j-n} t^{s_i-1}\,{}_0D_t^{n-1-j} f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi = s_1e_1+s_2e_2$, $f(t) = f_1(t)e_1+f_2(t)e_2$ and $\alpha=\frac{1}{2},\frac{3}{2}$.

Proof. (a) By applying the definition of bicomplex Mellin transform, we get

$$\begin{aligned}\mathfrak{M}\left[{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] &= \int_0^\infty t^{\xi-1}\,{}_t^C D_\infty^{\frac{1}{2}} f(t)dt \\ &= \left(\int_0^\infty t^{s_1-1}\,{}_t^C D_\infty^{\frac{1}{2}} f_1(t)dt\right)e_1 + \left(\int_0^\infty t^{s_2-1}\,{}_t^C D_\infty^{\frac{1}{2}} f_2(t)dt\right)e_2 \\ &\qquad\qquad [\text{where } \xi = s_1e_1+s_2e_2 \text{ and } f(t)=f_1(t)e_1+f_2(t)e_2].\end{aligned}$$

We know the result of fractional integration by parts (see, e.g. Almeida R. and Torres D.F.M. [24]) as

$$\int_a^b g(t)\,{}_t^C D_b^\alpha f(t)dt = \int_a^b f(t)\,{}_aD_t^\alpha g(t)dt + \sum_{j=0}^{n-1}\left[(-1)^{n+j}\,{}_aD_t^{\alpha+j-n} g(t)\,{}_aD_t^{n-1-j} f(t)\right]_a^b. \tag{3.23}$$

By using (3.23) and using given conditions, we obtain

$$\begin{aligned}\mathfrak{M}\left[{}_t D_\infty^{\frac{1}{2}} f(t);\xi\right] &= \left(\int_0^\infty f_1(t)\,{}_0D_t^{\frac{1}{2}} t^{s_1-1}dt\right)e_1 + \left(\int_0^\infty f_2(t)\,{}_0D_t^{\frac{1}{2}} t^{s_2-1}dt\right)e_2 \\ &= \frac{\Gamma(s_1)}{\Gamma\left(s_1-\frac{1}{2}\right)}\mathfrak{M}\left[f_1(t);s_1-\frac{1}{2}\right]e_1 + \frac{\Gamma(s_2)}{\Gamma\left(s_2-\frac{1}{2}\right)}\mathfrak{M}\left[f_2(t);s_2-\frac{1}{2}\right]e_2 \\ &\qquad\qquad [\text{using } (2.1)] \\ &= \frac{\Gamma(s_1e_1+s_2e_2)}{\Gamma\left(s_1e_1+s_2e_2-\frac{1}{2}\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2-\frac{1}{2}\right] \\ &= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{1}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{1}{2}\right].\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\mathfrak{M}\left[{}^C_tD^{\frac{3}{2}}_{\infty}f(t);\xi\right] &= \left(\int_0^{\infty} f_1(t)\,{}_0D_t^{\frac{3}{2}}t^{s_1-1}dt\right)e_1 + \left(\int_0^{\infty} f_2(t)\,{}_0D_t^{\frac{3}{2}}t^{s_2-1}dt\right)e_2\\ &= \frac{\Gamma(s_1)}{\Gamma\left(s_1-\frac{3}{2}\right)}\mathfrak{M}\left[f_1(t);s_1-\frac{3}{2}\right]e_1 + \frac{\Gamma(s_2)}{\Gamma\left(s_2-\frac{3}{2}\right)}\mathfrak{M}\left[f_2(t);s_2-\frac{3}{2}\right]e_2\\ &= \frac{\Gamma(s_1e_1+s_2e_2)}{\Gamma\left(s_1e_1+s_2e_2-\frac{3}{2}\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2-\frac{3}{2}\right]\\ &= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{3}{2}\right].\end{aligned}$$

□

Continuing by the induction, the results in Theorem 3.12 can be extended further to fractional derivatives as in the following theorem:

Theorem 3.13. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$ for all $n-1<\alpha<n$, $n\in\mathbb{N}$, then*

$$\mathfrak{M}\left[{}^C_tD^{\alpha}_{\infty}f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma(\xi-\alpha)}\mathfrak{M}\left[f(t);\xi-\alpha\right], \qquad \xi-\alpha\in\Omega \tag{3.24}$$

where Ω defined in (1.1) and ${}_0D_t^{\alpha+j-n}t^{s_i-1}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$.

Theorem 3.14. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$(a)\quad \mathfrak{M}\left[t^{\frac{1}{2}}\,{}^C_tD^{\frac{1}{2}}_{\infty}f(t);\xi\right] = \frac{\Gamma\left(\xi+\frac{1}{2}\right)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\frac{1}{2}\in\Omega \tag{3.25}$$

$$(b)\quad \mathfrak{M}\left[t^{\frac{3}{2}}\,{}^C_tD^{\frac{3}{2}}_{\infty}f(t);\xi\right] = \frac{\Gamma\left(\xi+\frac{3}{2}\right)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\frac{3}{2}\in\Omega \tag{3.26}$$

where Ω defined in (1.1) and ${}_0D_t^{\alpha+j-n}t^{s_i-\alpha}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$, $f(t)=f_1(t)e_1+f_2(t)e_2$ and $\alpha=\frac{1}{2},\frac{3}{2}$.

Proof. (a) By applying the definition of bicomplex Mellin transform, we get

$$\begin{aligned}\mathfrak{M}\left[t^{\frac{1}{2}}\,{}^C_tD^{\frac{1}{2}}_{\infty}f(t);\xi\right] &= \int_0^{\infty} t^{\xi-\frac{1}{2}}\,{}^C_tD^{\frac{1}{2}}_{\infty}f(t)dt\\ &= \left(\int_0^{\infty} t^{s_1-\frac{1}{2}}\,{}^C_tD^{\frac{1}{2}}_{\infty}f_1(t)dt\right)e_1 + \left(\int_0^{\infty} t^{s_2-\frac{1}{2}}\,{}^C_tD^{\frac{1}{2}}_{\infty}f_2(t)dt\right)e_2\end{aligned}$$

[where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$].

By using (3.23) and using given conditions, we obtain

$$\mathfrak{M}\left[t^{\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] = \left(\int_0^\infty f_1(t)\,{}_0D_t^{\frac{1}{2}} t^{s_1-\frac{1}{2}}dt\right)e_1 + \left(\int_0^\infty f_2(t)\,{}_0D_t^{\frac{1}{2}} t^{s_2-\frac{1}{2}}dt\right)e_2$$

$$= \frac{\Gamma(s_1+\frac{1}{2})}{\Gamma(s_1)}\left(\int_0^\infty t^{s_1-1}f_1(t)dt\right)e_1 + \frac{\Gamma(s_2+\frac{1}{2})}{\Gamma(s_2)}\left(\int_0^\infty t^{s_2-1}f_2(t)dt\right)e_2$$

[using (2.1)]

$$= \frac{\Gamma\left(s_1e_1+s_2e_2+\frac{1}{2}\right)}{\Gamma\left(s_1e_1+s_2e_2\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2\right]$$

$$= \frac{\Gamma\left(\xi+\frac{1}{2}\right)}{\Gamma\left(\xi\right)}\mathfrak{M}\left[f(t);\xi\right].$$

(b) Similarly,

$$\begin{aligned}\mathfrak{M}\left[t^{\frac{3}{2}}\,{}_t^C D_\infty^{\frac{3}{2}} f(t);\xi\right] &= \left(\int_0^\infty f_1(t)\,{}_0D_t^{\frac{3}{2}} t^{s_1+\frac{1}{2}}dt\right)e_1 + \left(\int_0^\infty f_2(t)\,{}_0D_t^{\frac{3}{2}} t^{s_2+\frac{1}{2}}dt\right)e_2\\ &= \frac{\Gamma\left(s_1+\frac{3}{2}\right)}{\Gamma(s_1)}\mathfrak{M}\left[f_1(t);s_1\right]e_1 + \frac{\Gamma\left(s_2+\frac{3}{2}\right)}{\Gamma(s_2)}\mathfrak{M}\left[f_2(t);s_2\right]e_2\\ &= \frac{\Gamma\left(s_1e_1+s_2e_2+\frac{1}{2}\right)}{\Gamma(s_1e_1+s_2e_2)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2\right]\\ &= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi\right].\end{aligned}$$

□

Following the similar technique as in the above theorem, follows Theorem 3.15.

Theorem 3.15. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$ for all $n-1<\alpha<n$, $n\in\mathbb{N}$, then*

$$\mathfrak{M}\left[t^\alpha\,{}_t^C D_\infty^\alpha f(t);\xi\right] = \frac{\Gamma(\xi+\alpha)}{\Gamma\left(\xi\right)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\alpha\in\Omega \tag{3.27}$$

where Ω defined in (1.1) and ${}_0D_t^{\alpha+j-n}t^{s_i-\alpha}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$.

4. Application

In this section, we discuss the application of bicomplex Mellin transform in solving Caputo fractional equation of bicomplex-valued function.

In [22], Agarwal et al. defined bicomplex scalar field as

$$F\equiv V+i_2I \tag{4.1}$$

where voltage V and current I are complex scalars field. In this paper, authors discussed an equivalent circuit of a transmission line of small length Δx containing resistance $R\Delta x$, capacitance $C\Delta x$ and inductance $L\Delta x$. Then differential equation of bicomplex-valued function as

$$\frac{\partial^2}{\partial x^2}F(x,t) = t^2\frac{\partial^2}{\partial t^2}F(x,t)+t\frac{\partial}{\partial t}F(x,t).$$

Agarwal et al. find the solution of the above differential equation and separate the voltage and current.

In similar manner, we can write a Caputo fractional differential equation of bicomplex-valued function of a circuit of transmission line as follows:

$$t^{\alpha}\,{}_{t}^{C}D_{\infty}^{\alpha}F(x,t)+t^{\beta}\,{}_{t}^{C}D_{\infty}^{\beta}F(x,t)=A\delta(t-a)\delta(x-a),\quad A\in C_2 \tag{4.2}$$

Taking the bicomplex Mellin transform of (4.2) w.r.t. t, we get

$$\frac{\Gamma(\xi+\alpha)}{\Gamma(\xi)}\bar{F}(x,\xi)+\frac{\Gamma(\xi+\beta)}{\Gamma(\xi)}\bar{F}(x,\xi)=A\delta(x-a)a^{\xi-1}$$

$$\therefore\ \bar{F}(x,\xi)=A\delta(x-a)\frac{\Gamma(\xi)a^{\xi-1}}{\Gamma(\xi+\alpha)+\Gamma(\xi+\beta)}.$$

Taking the inverse bicomplex Mellin transform of $\bar{F}(x,\xi)$, we get

$$F(x,t)=V+i_2I=\frac{A}{2\pi i_1}\delta(x-a)\int_{\Omega}t^{-\xi}\frac{\Gamma(\xi)a^{\xi-1}}{\Gamma(\xi+\alpha)+\Gamma(\xi+\beta)}d\xi \tag{4.3}$$

where $\Omega=(\Omega_1,\Omega_2)$ as in Definition 1.3. By separating the bi-real and bi-imaginary part of (4.3), we obtain the voltage and current of the given circuit of transmission line.

5. Conclusion

The concept of bicomplex functions has been applied for finding the solution of transmission line equations in fractional form. In this paper, we find bicomplex Mellin transform of some useful properties of fractional operators, which are useful for finding the solution of fractional differential equation of bicomplex-valued function.

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