

A TREATMENT OF THE HADAMARD INEQUALITY DUE TO m -CONVEXITY VIA GENERALIZED FRACTIONAL INTEGRALS

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ABSTRACT. Fractional calculus is as important as calculus. This paper is presentation of the Hadamard inequality for m -convex functions via fractional calculus. We present the Hadamard inequality for several generalized fractional integral operators.

1. INTRODUCTION

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1)$$

holds, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. The function f is called concave if reverse of inequality (1) holds.

For any convex function $f : I \rightarrow \mathbb{R}$ where I is an interval in \mathbb{R} , following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}, \quad (2)$$

where $a, b \in I$ and $a < b$.

Inequality (2) is well known in literature as the Hadamard inequality.

In [20] Toader define the concept of m -convexity, an intermediate between usual convexity and star shape function.

Definition 2. A function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be m -convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If we take $m = 1$, then we recapture the concept of convex functions defined on $[0, b]$ and if we take $m = 0$, then we get the concept of starshaped functions on $[0, b]$. We recall that $f : [0, b] \rightarrow \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, 1] \text{ and } x \in [0, b].$$

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Denote by $K_m(b)$ the set of the m -convex functions on $[0, b]$ for which $f(0) < 0$, then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \rightarrow \mathbb{R}$ for which $f(0) \leq 0$ (see [4]).

Example 1. [13] *The function $f : [0, \infty) \rightarrow \mathbb{R}$, given by*

$$f(x) = \frac{1}{12} (4x^3 - 15x^2 + 18x - 5)$$

is $\frac{16}{17}$ -convex function but it is not convex function.

For more results and inequalities related to m -convex functions one can consult for example [3, 4, 11, 7, 15] along with references.

Fractional calculus refers to integration or differentiation of fractional order is as old as calculus. For a historical survey the reader may see [10, 12, 14].

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators (see, [1, 2, 5, 9, 16, 21, 8, 18]).

In the following two sided definition of the generalized fractional integral operator containing the generalized Mittag–Leffler function is given.

Definition 3. *Let $\alpha, \beta, k, l, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator $\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k}$ containing the generalized Mittag–Leffler function for a real-valued continuous function f is defined by:*

$$(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} f)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(x-t)^\alpha) f(t) dt, \quad (3)$$

and

$$(\epsilon_{\alpha, \beta, l, \omega, b^-}^{\gamma, \delta, k} f)(x) = \int_x^b (t-x)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(t-x)^\alpha) f(t) dt, \quad (4)$$

where the function $E_{\alpha, \beta, l}^{\gamma, \delta, k}$ is the generalized Mittag–Leffler function defined as

$$E_{\alpha, \beta, l}^{\gamma, \delta, k}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\alpha n + \beta)} \frac{t^n}{(\delta)_{ln}}, \quad (5)$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\dots(a+n-1)$, $(a)_0 = 1$.

If $\delta = l = 1$ in (3), then integral operator $\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k}$ reduces to an integral operator containing the generalized Mittag–Leffler function $E_{\alpha, \beta, 1}^{\gamma, 1, k}$ introduced by Srivastava and Tomovski in [19]. Along $\delta = l = 1$ in addition if $k = 1$, then (3) reduces to an integral operator defined by Prabhakar in [16] containing the Mittag–Leffler function $E_{\alpha, \beta}^\gamma$. For $\omega = 0$ in (3), integral operator $\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k}$ would correspond essentially to the Riemann–Liouville fractional integral operator (see, [17])

$$I_{a^+}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt, \quad x > a$$

and

$$I_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t) dt, \quad x < b.$$

In [18], Sarikaya et al. proved the following, a version of the Hadamard inequality for convex functions involving the Riemann–Liouville fractional integral operator.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for fractional integral holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[I_{a+}^{\beta} f(b) + I_{b-}^{\beta} f(a) \right] \leq \frac{f(a) + f(b)}{2} \quad (6)$$

with $\beta > 0$.

A generalization of above result is the following Hadamard inequality for generalized fractional integrals [6].

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for fractional integral holds*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) (\epsilon_{\alpha, \beta, l, \omega', a^+}^{\gamma, \delta, k})(b) \\ & \leq \frac{(\epsilon_{\alpha, \beta, l, \omega', a^+}^{\gamma, \delta, k} f)(b) + (\epsilon_{\alpha, \beta, l, \omega', b^-}^{\gamma, \delta, k} f)(a)}{2} \\ & \leq \frac{f(a) + f(b)}{2} (\epsilon_{\alpha, \beta, l, \omega', b^-}^{\gamma, \delta, k})(a), \end{aligned} \quad (7)$$

where $\omega' = \frac{w}{(b-a)^{\alpha}}$.

In this paper we give the Hadamard inequality for m -convex functions via generalized fractional integral operators. We also show that this inequality contains the Hadamard inequality for convex functions via generalized fractional integrals [6] and in particular for Riemann–Liouville fractional integrals given in [18]. In general this contains several related inequalities.

2. HADAMARD INEQUALITY FOR m -CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

The following result holds for m -convex functions.

Theorem 3. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a positive m -convex function and $f \in L_1[0, \infty)$. Then for $0 \leq a < mb$ the following inequality holds*

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right) (\epsilon_{\alpha, \beta, l, \omega', a^+}^{\gamma, \delta, k})(mb) \\ & \leq \frac{(\epsilon_{\alpha, \beta, l, \omega', a^+}^{\gamma, \delta, k} f)(mb) + m^{\beta+1} (\epsilon_{\alpha, \beta, l, m^{\alpha} \omega', b^-}^{\gamma, \delta, k} f)\left(\frac{a}{m}\right)}{2} \\ & \leq \frac{m^{\beta+1}}{2} \left[\left(f(b) + m f\left(\frac{a}{m^2}\right) \right) (\epsilon_{\alpha, \beta, l, m^{\alpha} \omega', b^-}^{\gamma, \delta, k})\left(\frac{a}{m}\right) \right. \\ & \quad \left. + \frac{f(a) - m^2 f\left(\frac{a}{m^2}\right)}{mb-a} (\epsilon_{\alpha, \beta+1, l, m^{\alpha} \omega', b^-}^{\gamma, \delta, k})\left(\frac{a}{m}\right) \right], \end{aligned} \quad (8)$$

where $\omega' = \frac{w}{(mb-a)^{\alpha}}$.

Proof. Since f is a m -convex function, therefore one have

$$\begin{aligned} & f\left(\frac{1}{2}(ta + m(1-t)b) + m\left(1 - \frac{1}{2}\right)\left(\frac{1}{m}(1-t)a + tb\right)\right) \\ & \leq \frac{f(ta + m(1-t)b) + mf\left(\frac{1}{m}(1-t)a + tb\right)}{2} \end{aligned}$$

that gives after multiplying with $t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)$

$$\begin{aligned} & 2t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)f\left(\frac{a+mb}{2}\right) \\ & \leq t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)\left(f(ta + m(1-t)b) + mf\left(\frac{1}{m}(1-t)a + tb\right)\right). \end{aligned}$$

Integrating with respect to t on $[0, 1]$ we have

$$\begin{aligned} & 2f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)dt \\ & \leq \int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)f(ta + m(1-t)b)dt \\ & \quad + m\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)f\left(\frac{1}{m}(1-t)a + tb\right)dt. \end{aligned}$$

If $u = at + m(1-t)b$, then $t = \frac{mb-u}{mb-a}$ and if $v = \frac{1}{m}(1-t)a + tb$, then $t = \frac{mv-a}{mb-a}$ and one can get

$$f\left(\frac{a+mb}{2}\right)(\epsilon_{\alpha,\beta,l,\omega',a+}^{\gamma,\delta,k})(mb) \leq \frac{(\epsilon_{\alpha,\beta,l,\omega',a+}^{\gamma,\delta,k}f)(mb) + m^{\beta+1}(\epsilon_{\alpha,\beta,l,m^\alpha\omega',b-}^{\gamma,\delta,k}f)\left(\frac{a}{m}\right)}{2}. \quad (9)$$

On the other hand using m -convexity of f we have

$$\begin{aligned} & f(ta + m(1-t)b) + mf\left(\frac{1}{m}(1-t)a + tb\right) \\ & \leq tf(a) + m(1-t)f(b) + m^2(1-t)f\left(\frac{a}{m^2}\right) + mtf(b). \end{aligned}$$

Now multiplying with $t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)$ and integrating over $[0, 1]$ we get

$$\begin{aligned} & \int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)f(ta + m(1-t)b)dt \\ & \quad + m\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)f\left(\frac{1}{m}(1-t)a + tb\right)dt \\ & \leq \left(mf(b) + m^2f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)dt \\ & \quad + \left(f(b) - m^2f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^\beta E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)dt \end{aligned}$$

from which by using change of variables as used to obtain (9), one can find

$$\begin{aligned} & (\epsilon_{\alpha,\beta,l,\omega',a+}^{\gamma,\delta,k} f)(mb) + m^{\beta+1} (\epsilon_{\alpha,\beta,l,m^\alpha\omega',b-}^{\gamma,\delta,k} f) \left(\frac{a}{m}\right) \\ & \leq m^{\beta+1} \left[\left(f(b) + mf \left(\frac{a}{m^2}\right) \right) \epsilon_{\alpha,\beta,l,m^\alpha\omega',b-}^{\gamma,\delta,k} \left(\frac{a}{m}\right) \right. \\ & \quad \left. + \frac{f(a) - m^2 f \left(\frac{a}{m^2}\right)}{mb - a} \epsilon_{\alpha,\beta+1,l,m^\alpha\omega',b-}^{\gamma,\delta,k} \left(\frac{a}{m}\right) \right]. \end{aligned} \quad (10)$$

Combining inequality in (9) and inequality in (10) we get inequality in (8). \square

Remark 1. If $m = 1$ in (8), then we get inequality in (7).

Remark 2. If $\delta = l = 1$ in (8), then we have fractional Hadamard inequality for m -convex functions via integral operator introduced by Srivastava and Tomovski in [19]. Along $\delta = l = 1$ in addition if $k = 1$ in (8), then we have fractional Hadamard inequality for m -convex functions via integral operator defined by Prabhakar in [16]. All such results can be obtained for convex functions just by taking $m = 1$.

Remark 3. If we take $\omega = 0$, the above theorem gives the Hadamard inequality for m -convex functions via fractional integrals while the inequality in Theorem 1 can be obtained by taking $m = 1$ along with $\omega = 0$. Moreover if along with $\omega = 0, m = 1$ we take $\alpha = 1$, then we get (2).

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