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SOME PROPERTIES AND APPLICATIONS OF CONFORMABLE FRACTIONAL LAPLACE TRANSFORM (CFLT)

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Abstract. This paper aims to elucidate properties of fractional Laplace transformation in conjunction with conformable fractional integral. Some significant descriptions for conformable fractional Laplace transform (CFLT) for derivative and integral functions are given. The transforms of particular functions, such as error function, Bessel function and periodic functions are discussed in detail along with the convolution theorem and various examples. Moreover, CFLT of n-dimensional space functions are also elaborated, which is further exemplified with example.

1. Introduction

Modern definitions of fractional derivative and fractional integral real valued functions play a vital role in the theory of fractional calculus theory. Each of these innovative definitions has made this study unproblematic and easygoing. For instance, Riemann-Liouville fractional definitions, Caputo fractional definitions, Grnwald-Letnikov fractional derivative, Atangana-Baleanu fractional definitions, Hadamard fractional integral, Caputo-Fabrizio fractional derivative and conformable fractional definitions (CFD) [1]-[6]. In conjunction with these definitions, many numerical and analytical methods have been reconstructed to solve numerous fractional integrals and differential problems [7]-[11]. Recently, conformable fractional definitions have gained significant attention due to its natural definition and simplicity. Its applications are rapidly increasing in remodelling different dynamical models and emerging variety of methods with this definition [12]-[16].
Adding to this, integral transforms are also ground-breaking inventions in calculus. The capability of integral transforms to manipulate several problems by altering the domain of the equation, have made it persistently important. Although, there are plenty of integral transforms in the literature, but Fourier series, Laplace transforms and Sumudu transforms are most commonly exercised [17]-[19]. These transformations are widely studied in integer as well as fractional form in scrutinizing several differential problems [20]-[23].

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In this endeavour, elaboration of the fractional Laplace transform in the conformable fractional integral sense is represented. Many articles exist in literature, which are related to fractional Laplace transforms with its properties and applications [24]-[26]. Nevertheless, Abdeljawad [15] described the conformable fractional Laplace transform with a few straightforward basic properties. Here, further exploration of this study is made meticulously with many other important properties. For example, we discuss the transformations of Bessel and error functions in the form of series, theorems for CFLT of trigonometric and exponential integrals are provided. Moreover, periodic functions and convolution of functions are also added with experiments.The noteworthy investigation of this attempt is the manifestation of CFLT of functions in $\Re^n$ space.

2. Preliminaries

2.1. **Conformable Fractional Derivative.** For any function $\phi : [0, \infty) \to \Re$ the conformable fractional derivative of order ϖ is given as

$$\mathfrak{T}_\varpi \phi(\tau) = \lim_{\hbar \to 0} \frac{\phi\left(\tau + \hbar\tau^{1-\varpi}\right) - \phi(\tau)}{\hbar} \tag{1}$$

for all $\tau > 0$, $\varpi \in (0, 1)$. In addition, if ϕ is $(\lceil\varpi\rceil + 1)$-differentiable and continuous at $\tau > 0$, then for $\varpi \in (\lceil\varpi\rceil, \lceil\varpi\rceil + 1]$,

$$\mathfrak{T}_\varpi \phi(\tau) = \lim_{\hbar \to 0} \frac{\phi^{(\lceil\varpi\rceil-1)}\left(\tau + \hbar\tau^{(\lceil\varpi\rceil-1)}\right) - \phi^{(\lceil\varpi\rceil-1)}(\tau)}{\hbar} \tag{2}$$

Theorem 2.1.1 Let $\varpi \in (0, 1]$ and ϕ, ψ be ϖ-differentiable at a point $\tau > 0$, then

(1) $\mathfrak{T}_\varpi (a\phi + b\psi) = a\mathfrak{T}_\varpi(\phi) + b\mathfrak{T}_\varpi(\psi)$, $\forall a, b \in \Re$.
(2) $\mathfrak{T}_\varpi(\tau^p) = p\tau^{p-\varpi}$,$\forall p \in \Re$.
(3) $\mathfrak{T}_\varpi(\gamma) = 0$, for all constant function $\phi(\tau) = \gamma$.
(4) $\mathfrak{T}_\varpi(\phi\psi) = \phi\mathfrak{T}_\varpi(\psi) + \psi\mathfrak{T}_\varpi(\phi)$.
(5) $\mathfrak{T}_\varpi\left(\frac{\phi}{\psi}\right) = \frac{\psi\mathfrak{T}_\varpi(\phi) - \phi\mathfrak{T}_\varpi(\psi)}{\psi^2}$.
(6) If ϕ is differentiable, then $\mathfrak{T}_\varpi\phi(\tau) = \tau^{1-\varpi}\frac{d\phi}{d\tau}$. Analogously, if ϕ is $(\lceil\varpi\rceil + 1)$-differentiable, then $\mathfrak{T}_\varpi\phi(\tau) = \tau^{\lceil\varpi\rceil-\varpi}\phi^{(\lceil\varpi\rceil)}(\tau)$.

Moreover, conformable fractional derivative of some fundamental functions, which will be informative for the remaining paper, are found as,

(1) $\mathfrak{T}_\varpi\left(\frac{\tau^\varpi}{\varpi}\right) = 1$.
(2) $\mathfrak{T}_\varpi \sin\left(\frac{\tau^\varpi}{\varpi}\right) = \cos\left(\frac{\tau^\varpi}{\varpi}\right)$.
(3) $\mathfrak{T}_\varpi \cos\left(\frac{\tau^\varpi}{\varpi}\right) = -\sin\left(\frac{\tau^\varpi}{\varpi}\right)$.
(4) $\mathfrak{T}_\varpi \exp\left(\frac{\tau^\varpi}{\varpi}\right) = \exp\left(\frac{\tau^\varpi}{\varpi}\right)$.

2.2. **Conformable Fractional Integral.** Let ϕ be a continuous function in $[a,b]$, then conformable fractional integral of order $\varpi \in (0,1)$ is given as

$$\mathfrak{I}_{\varpi}^{a}\phi(\tau)=\mathfrak{I}_{1}^{a}\left(\tau^{\varpi-1}\phi(\tau)\right)=\int_{a}^{\tau}x^{\varpi-1}\phi(x)\,dx \tag{3}$$

Theorem 2.2.1 For any continuous function ϕ in the domain of $\mathfrak{I}_{\varpi}$, for $\tau \geq a$.

$$\mathfrak{T}_{\varpi}\mathfrak{I}_{\varpi}^{a}\phi(\tau)=\phi(\tau) \tag{4}$$

For further details and proofs, one would see refs. [6],[15].

3. Properties of Conformable Fractional Laplace Transform

Definition 3.1 Let $\phi:[\tau_0,\infty)\to\Re$ be a real valued function with $\tau_0\in\Re$, then the conformable fractional Laplace transform (CFLT) of order $0<\varpi\leq 1$ is defined as

$$\begin{aligned}\mathfrak{L}_{\varpi}^{\tau_0}\{\phi(\tau)\}(\varepsilon)=\Phi_{\varpi}^{\tau_0}(\varepsilon)&=\int_{\tau_0}^{\infty}\exp\left(-\varepsilon\frac{(\tau-\tau_0)^{\varpi}}{\varpi}\right)\phi(\tau)\,d\varpi(\tau,\tau_0) \\ &=\int_{\tau_0}^{\infty}\exp\left(-\varepsilon\frac{(\tau-\tau_0)^{\varpi}}{\varpi}\right)\phi(\tau)(\tau-\tau_0)^{\varpi-1}\,d\tau\end{aligned} \tag{5}$$

Theorem 3.1 *Conformable Fractional Derivatives* Let $\phi:[p_0,\infty)\to\Re$ be a continuous and differentiable real valued function with $p_0\in\Re$, then the conformable fractional Laplace transform (CFLT) $\mathfrak{L}_{\varpi}^{p_0}$ of $\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)$ is defined as

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon)=\varepsilon\Phi_{\varpi}^{\tau_0}(\varepsilon)-\phi(p_0) \tag{6}$$

Proof From the Definition 3.1 we have

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon)=\int_{p_0}^{\infty}\exp\left(-\varepsilon\frac{(\tau-p_0)^{\varpi}}{\varpi}\right)\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\,d\varpi(\tau,\tau_0) \tag{7}$$

Following sixth property of Theorem 2.1.1 and expanding the usual integration by part we reach at

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon)=-\phi(p_0)+\varepsilon\int_{p_0}^{\infty}\exp\left(-\varepsilon\frac{(\tau-p_0)^{\varpi}}{\varpi}\right)\phi(\tau)(\tau-p_0)^{\varpi-1}\,d\tau \tag{8}$$

Hence

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon)=\varepsilon\Phi_{\varpi}^{\tau_0}(\varepsilon)-\phi(p_0) \tag{9}$$

Proposition 3.1 Let $\phi:[p_0,\infty)\to\Re$ be such that $\phi(\tau)$ and $\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)$ are continuous, then for $\varpi\in(\lceil\varpi\rceil-1,\lceil\varpi\rceil]$, the conformable fractional Laplace transform (CFLT) $\mathfrak{L}_{\varpi}^{p_0}$ of ${}^{(\lceil\varpi\rceil)}\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)$ is given as

$$\begin{aligned}\mathfrak{L}_{\varpi}^{p_0}\left\{{}^{(\lceil\varpi\rceil)}\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\right\}(\varepsilon)&=\varepsilon^{\lceil\varpi\rceil}\Phi_{\varpi}^{\tau_0}(\varepsilon)-\varepsilon^{\lceil\varpi\rceil-1}\phi(p_0)-\varepsilon^{\lceil\varpi\rceil-2}\mathfrak{T}_{\varpi}^{p_0}\phi(p_0) \\ &\quad-\cdots-\varepsilon^{\ (\lceil\varpi\rceil-2)}\mathfrak{T}_{\varpi}^{p_0}\phi(p_0)-\varepsilon^{\ (\lceil\varpi\rceil-1)}\mathfrak{T}_{\varpi}^{p_0}\phi(p_0)\end{aligned} \tag{10}$$

The proof follows easily by resuming aforementioned theorem inductively.

Theorem 3.2 *Conformable Fractional Integrals* Let $\phi:[p_0,\infty)\to\Re$ be a continuous and differentiable real valued function such that $\mathfrak{L}_{\varpi}^{p_0}\{\phi(\tau)\}(\varepsilon)=\Phi_{\varpi}^{p_0}(\varepsilon)$, then CFLT $\mathfrak{L}_{\varpi}^{p_0}$ of $\mathfrak{I}_{\varpi}^{p_0}\phi(\tau)$ is defined as

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{I}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon)=\frac{1}{\varepsilon}\left(\Phi_{\varpi}^{p_0}(\varepsilon)+\lambda\right) \tag{11}$$

where $\lambda = \lim_{\tau \to p_0} \phi(\tau)$.
Proof Taking $\mathfrak{I}_{\varpi}^{p_0}\phi(\tau) = \psi(\tau)$, then $\phi(\tau) = \mathfrak{T}_{\varpi}^{p_0}\psi(\tau)$ and on applying CFLT on both sides, we have

$$\Phi_{\varpi}^{p_0}(\varepsilon) = \varepsilon \mathfrak{L}_{\varpi}^{p_0}\{\psi(\tau)\} - \psi(p_0) \tag{12}$$

Therefore, we come up with

$$\mathfrak{L}_{\varpi}^{p_0}\{\psi(\tau)\} = \frac{\Phi_{\varpi}^{p_0}(\varepsilon) + \lambda}{\varepsilon} \tag{13}$$

that completes the proof.
Theorem 3.3 *Multiplication by* $(\tau - \tau_0)^{m\varpi}$ Let $\phi : [\tau_0, \infty) \to \Re$ be a continuous and differentiable real valued function such that $\mathfrak{L}_{\varpi}^{\tau_0}\{\phi(\tau)\}(\varepsilon) = \Phi_{\varpi}^{\tau_0}(\varepsilon)$, then CFLT $\mathfrak{L}_{\varpi}^{\tau_0}$ of $(\tau - \tau_0)^{m\varpi}\phi(\tau)$ is defined as

$$\mathfrak{L}_{\varpi}^{\tau_0}\{(\tau - \tau_0)^{m\varpi}\phi(\tau)\}(\varepsilon) = (-1)^m \varpi^m \left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(m)} \tag{14}$$

Proof Using Eq.5, applying Leibnitz's rule and taking the conformable fractional derivative with respect to ε, we get

$$\mathfrak{L}_{\varpi}^{\tau_0}\{(\tau - \tau_0)^{\varpi}\phi(\tau)\}(\varepsilon) = -\varpi \left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(1)} \tag{15}$$

which proves to be true for $m = 1$. Inductively, assume that it is true for $m = q$ i.e.

$$\mathfrak{L}_{\varpi}^{\tau_0}\{(\tau - \tau_0)^{q\varpi}\phi(\tau)\}(\varepsilon) = (-1)^q \varpi^q \left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(q)} \tag{16}$$

Then, fractionally differentiating Eq.14 with respect to and by Leibnitz's rule, we obtain

$$\mathfrak{L}_{\varpi}^{\tau_0}\left\{(\tau - \tau_0)^{(q+1)\varpi}\phi(\tau)\right\}(\varepsilon) = (-1)^{q+1} \varpi^{q+1} \left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(q+1)} \tag{17}$$

which follows that if the theorem holds for $m = q$, then it also holds for $m = q + 1$. Hence, from Eqs.15-17 theorem is determined to be true for positive integer values of m.
Proposition 3.2 Let $\phi : [\tau_0, \infty) \to \Re$ be a continuous and differentiable real valued function such that $\mathfrak{L}_{\varpi}^{\tau_0}\{\phi(\tau)\}(\varepsilon) = \Phi_{\varpi}^{\tau_0}(\varepsilon)$, then CFLT $\mathfrak{L}_{\varpi}^{\tau_0}$ of $(\tau - \tau_0)^{m\varpi}\phi(\tau)$, for negative integer values of m, is shown as

$$\mathfrak{L}_{\varpi}^{\tau_0}\{(\tau - \tau_0)^{r\varpi}\phi(\tau)\}(\varepsilon) = (-1)^r \varpi^r \left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(r)} \tag{18}$$

where $r = -m$ and $\left(\Phi_{\varpi}^{\tau_0}(\varepsilon)\right)^{(r)} = \underbrace{\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty}\cdots\int_{\varepsilon}^{\infty}}_{m\text{-}times} \Phi_{\varpi}^{\tau_0}(\eta)(d\eta)^m$. It can be easily proved by using the mathematical induction, same as Theorem 3.3.
Theorem 3.4 Sine, Cosine and Exponential Integrals Let the sine, cosine and exponential integrals, with the definition conformable fractional integral, be defined as

$$Si\left(\frac{\tau^{\varpi}}{\varpi}\right) = \varpi \int_0^{\tau} \frac{\sin\left(\frac{\upsilon^{\varpi}}{\varpi}\right)}{\upsilon^{\varpi}} \upsilon^{\varpi - 1} d\upsilon \tag{19}$$

$$Ci\left(\frac{\tau^{\varpi}}{\varpi}\right) = \varpi \int_{\tau}^{\infty} \frac{\cos\left(\frac{\upsilon^{\varpi}}{\varpi}\right)}{\upsilon^{\varpi}} \upsilon^{\varpi - 1} d\upsilon \tag{20}$$

and

$$Ei\left(\frac{\tau^{\varpi}}{\varpi}\right)=\varpi\int_{\tau}^{\infty}\frac{\exp\left(\frac{\upsilon^{\varpi}}{\varpi}\right)}{\upsilon^{\varpi}}\upsilon^{\varpi-1}d\upsilon \tag{21}$$

Then the CFLT of Eqs.19-21 can be expressed as

$$\mathfrak{L}_{\varpi}^{0}\left\{Si\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{\cot^{-1}\varepsilon}{\varepsilon} \tag{22}$$

$$\mathfrak{L}_{\varpi}^{0}\left\{Ci\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{ln\left(\varepsilon^{2}+1\right)}{2\varepsilon} \tag{23}$$

and

$$\mathfrak{L}_{\varpi}^{0}\left\{Ei\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{ln\left(\varepsilon+1\right)}{\varepsilon} \tag{24}$$

respectively.

Proof Consider Eq.19 and take $\upsilon=\tau\xi$, then using Eq.5 we have

$$\mathfrak{L}_{\varpi}^{0}\left\{Si\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\int_{0}^{\infty}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}\left(\varpi\int_{0}^{1}\frac{\sin\left(\frac{(\tau\xi)^{\varpi}}{\varpi}\right)}{\xi^{\varpi}}d\xi\right)d\tau \tag{25}$$

On some manipulation we come up with

$$\mathfrak{L}_{\varpi}^{0}\left\{Si\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\varpi\int_{0}^{1}\frac{\xi^{\varpi-1}}{\varepsilon^{2}+\xi^{2\varpi}}d\xi \tag{26}$$

hence

$$\mathfrak{L}_{\varpi}^{0}\left\{Si\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{\cot^{-1}\varepsilon}{\varepsilon} \tag{27}$$

Analogously, proofs of Eqs. 20 and 21 are straightforward.

Theorem 3.5 *Error Function*

The CFLT of error function, defined as $erf\left(\sqrt{\tau}\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{\sqrt{\tau}}\exp\left(-\upsilon^{2}\right)d\upsilon$, is expressed by

$$\mathfrak{L}_{\varpi}^{0}\left\{erf\left(\sqrt{\tau}\right)\right\}(\varepsilon)=\frac{2}{\sqrt{\pi}}\sum_{j=0}^{\infty}a_{j}N_{j}^{\varpi}(\varepsilon) \tag{28}$$

where $a_{j}=\frac{(-1)^{j}}{j!\,(2j+1)}$ and $N_{j}^{\varpi}(\varepsilon)=\frac{\varpi^{(2j+1)/2\varpi}}{\varepsilon^{1+(2j+1)/2\varpi}}\Gamma\left(1+\frac{(2j+1)}{2\varpi}\right)$.

Proof Using Eq.5 we have

$$\mathfrak{L}_{\varpi}^{0}\left\{erf\left(\sqrt{\tau}\right)\right\}(\varepsilon)=\int_{0}^{\infty}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}\left(\frac{2}{\sqrt{\pi}}\int_{0}^{\sqrt{\tau}}\exp\left(-\upsilon^{2}\right)d\upsilon\right)d\tau \tag{29}$$

On expanding the integral and using the equality [15]

$$\mathfrak{L}_{\varpi}^{0}\left\{\tau^{\eta}\right\}(\varepsilon)=\frac{\varpi^{\eta/\varpi}}{\varepsilon^{1+\eta/\varpi}}\Gamma\left(1+\frac{\eta}{\varpi}\right),\ \eta\in\Re \tag{30}$$

we come up with

$$\mathfrak{L}_{\varpi}^{0}\left\{erf\left(\sqrt{\tau}\right)\right\}(\varepsilon)=\frac{2}{\sqrt{\pi}}\left(\frac{\varpi^{1/2\varpi}}{\varepsilon^{1+1/2\varpi}}\Gamma\left(1+\frac{1}{2\varpi}\right)-\frac{1}{3}\frac{\varpi^{3/2\varpi}}{\varepsilon^{1+3/2\varpi}}\Gamma\left(1+\frac{3}{2\varpi}\right)\right. \quad (31)$$

$$\left.+\frac{1}{5.2!}\frac{\varpi^{5/2\varpi}}{\varepsilon^{1+5/2\varpi}}\Gamma\left(1+\frac{5}{2\varpi}\right)-\frac{1}{5.3!}\frac{\varpi^{7/2\varpi}}{\varepsilon^{1+7/2\varpi}}\Gamma\left(1+\frac{7}{2\varpi}\right)+\cdots\right. \quad Raw$$

which on closing in summation form completes the proof i.e.

$$\mathfrak{L}_{\varpi}^{0}\left\{erf\left(\sqrt{\tau}\right)\right\}(\varepsilon)=\frac{2}{\sqrt{\pi}}\sum_{j=0}^{\infty}a_{j}N_{j}^{\varpi}(\varepsilon) \quad (32)$$

Theorem 3.6 *Bessel Function* The Bessel function of the first kind can be defined by the expression

$$J_{\sigma}(\tau)=\left(\frac{\tau}{2}\right)^{\sigma}\sum_{k=0}^{\infty}\frac{\left(-\tau^{2}\right)^{k}}{4^{k}k!\Gamma(\sigma+1+k)} \quad (33)$$

which are finite solutions of the Bessel equation at $\tau=0$ and for integer and positive real values of σ. The CFLT of Eq.33 can be constructed as

$$\mathfrak{L}_{\varpi}^{0}\left\{J_{\sigma}(\tau)\right\}(\varepsilon)=\sum_{k=0}^{\infty}c_{k}^{\sigma}g_{k}^{\varpi,\sigma}(\varepsilon) \quad (34)$$

where $c_{k}^{\sigma}=\dfrac{(-1)^{k}}{2^{2k+\sigma}k!\Gamma(\sigma+1+k)}$ and $g_{k}^{\varpi,\sigma}(\varepsilon)=\dfrac{\varpi^{(2k+\sigma)/\varpi}}{\varepsilon^{1+(2k+\sigma)/\varpi}}\Gamma(1+(2k+\sigma)/\varpi)$. It can be proved straightforwardly by using the Eq.30. As a consequence, the CFLT of the Bessel function of any order can be attained easily. For instance, transformation of $J_0(\tau)$ and $J_1(\tau)$ can be attained as $\mathfrak{L}_{\varpi}^{0}\{J_0(\tau)\}(\varepsilon)=\sum_{k=0}^{\infty}c_k^0 g_k^{\varpi,0}(\varepsilon)$ and $\mathfrak{L}_{\varpi}^{0}\{J_1(\tau)\}(\varepsilon)=\sum_{k=0}^{\infty}c_k^1 g_k^{\varpi,1}(\varepsilon)$, whereas CFLT of $J_{3/2}(\tau)$ is $\mathfrak{L}_{\varpi}^{0}\left\{J_{3/2}(\tau)\right\}(\varepsilon)=\sum_{k=0}^{\infty}c_k^{3/2}g_k^{\varpi,3/2}(\varepsilon)$.

Theorem 3.7 *Periodic Function* Assume that $\Psi:[0,\infty)\rightarrow\Re$ be a continuous and differentiable periodic function of period $\theta>0$ such that $\Psi\left(\dfrac{\tau^{\varpi}}{\varpi}\right)=\Psi\left(\dfrac{\tau^{\varpi}}{\varpi}+i\dfrac{\theta^{\varpi}}{\varpi}\right)$, for all $\tau\geq 0$ and $i\in\aleph$, if $\mathfrak{L}_{\varpi}^{0}\left\{\Psi\left(\dfrac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)$ exists, then

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\frac{\mu(\varepsilon)}{1-\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)} \quad (35)$$

where $\mu(\varepsilon)=\int_{0}^{\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\dfrac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau$.

Proof From Eq.5 we have

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\int_{0}^{\infty}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau$$

$$=\int_{0}^{\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau+\int_{\theta^{\varpi}/\varpi}^{2\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau$$

$$+\int_{2\theta^{\varpi}/\varpi}^{3\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau\cdots \quad (36)$$

that can be summed up as

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\sum_{i=0}^{\infty}\int_{i\theta^{\varpi}/\varpi}^{(i+1)\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau \tag{37}$$

Let, $\dfrac{\nu^{\varpi}}{\varpi}=\left(\dfrac{\tau^{\varpi}}{\varpi}-i\dfrac{\theta^{\varpi}}{\varpi}\right)$, thus

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\sum_{i=0}^{\infty}\exp\left(-\varepsilon i\theta^{\varpi}/\varpi\right)\left(\int_{0}^{\theta^{\varpi}/\varpi}\exp\left(-\varepsilon\nu^{\varpi}/\varpi\right)\Psi\left(\frac{\nu^{\varpi}}{\varpi}\right)\nu^{\varpi-1}d\tau\right) \tag{38}$$

where, $\sum_{i=0}^{\infty}\exp\left(-\varepsilon i\theta^{\varpi}/\varpi\right)$ is a geometric series with common ratio $\exp\left(-\varepsilon i\theta^{\varpi}/\varpi\right)<1$ for $\varepsilon>0$. Therefore

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\frac{\mu(\varepsilon)}{1-\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)}$$

Theorem 3.8 *Convolution Theorem* Consider two continuous real valued functions i.e. $\phi,\psi:[0,\infty)\rightarrow\Re$. If, by the conformable fractional derivative, the convolution of ϕ and ψ of order ϖ is expressed as

$$(\phi*\psi)=\mathfrak{I}_{\varpi}^{0}\left(\phi\left(\frac{\tau^{\varpi}}{\varpi}-\frac{\upsilon^{\varpi}}{\varpi}\right)\psi\left(\frac{\upsilon^{\varpi}}{\varpi}\right)\right) \tag{39}$$

Then, one can obtain the CFLT as

$$\mathfrak{L}_{\varpi}^{0}\left\{(\phi*\psi)\right\}(\varepsilon)=\Phi_{\varpi}^{0}(\varepsilon)\,\Psi_{\varpi}^{0}(\varepsilon) \tag{40}$$

where $\Phi_{\varpi}^{0}(\varepsilon)$ and $\Psi_{\varpi}^{0}(\varepsilon)$ are the CFLT of ϕ and ψ, respectively.
Proof Applying CFLT on Eq.39 and using Eq.3 we get

$$\mathfrak{L}_{\varpi}^{0}\left\{(\phi*\psi)\right\}(\varepsilon)=\int_{0}^{\infty}\exp\left(-\varepsilon\tau^{\varpi}/\varpi\right)\left(\int_{0}^{\tau}\phi\left(\frac{\tau^{\varpi}}{\varpi}-\frac{\upsilon^{\varpi}}{\varpi}\right)\psi\left(\frac{\upsilon^{\varpi}}{\varpi}\right)\upsilon^{\varpi-1}d\upsilon\right)\tau^{\varpi-1}d\tau \tag{41}$$

Let $\left(\dfrac{\tau^{\varpi}}{\varpi}-\dfrac{\upsilon^{\varpi}}{\varpi}\right)=\dfrac{\xi^{\varpi}}{\varpi}$, then we get

$$\mathfrak{L}_{\varpi}^{0}\left\{(\phi*\psi)\right\}(\varepsilon)=\int_{0}^{\infty}\left(\int_{0}^{\infty}\exp\left(-\varepsilon\left(\frac{\xi^{\varpi}}{\varpi}+\frac{\upsilon^{\varpi}}{\varpi}\right)\right)\phi\left(\frac{\xi^{\varpi}}{\varpi}\right)\xi^{\varpi-1}d\xi\right)\psi\left(\frac{\upsilon^{\varpi}}{\varpi}\right)\upsilon^{\varpi-1}d\upsilon \tag{42}$$

which can be written as

$$\mathfrak{L}_{\varpi}^{0}\left\{(\phi*\psi)\right\}(\varepsilon)=\Phi_{\varpi}^{0}(\varepsilon)\,\Psi_{\varpi}^{0}(\varepsilon)$$

Definition 3.2 Let $\phi(\mathbf{x},\tau)$ be a real valued function, such that $\mathbf{x}\in\Re^{n}$. Then the conformable fractional Laplace transform of function in $\Re^{n}$ space, of order $0<\varpi\leq1$, is defined as

$$\mathfrak{L}_{\varpi}^{0}\left\{\phi(\mathbf{x},\tau)\right\}(\wp)=\mathbf{\Phi}_{\varpi}^{\mathbf{x}_0}(\wp,\tau)=\int_{\mathbf{x}_0}^{\infty}\exp\left(-\wp\frac{(\mathbf{x}-\mathbf{x}_0)^{\varpi}}{\varpi}\right)\phi(\mathbf{x},\tau)(\mathbf{x}-\mathbf{x}_0)^{\varpi-1}d\mathbf{x} \tag{43}$$

where $\mathbf{x}_0$ is the lower limit for the respective variables $\mathbf{x}$.
Proposition 3.3 All the aforementioned properties are straightforward for the function $\phi(\mathbf{x},\tau)$.

- $\mathfrak{L}_{\varpi}^{\mathbf{x}_0}\left\{^{(\lceil\varpi\rceil)}\mathfrak{T}_{\varpi}^{\mathbf{x}_0}\phi(\mathbf{x},\tau)\right\}(\wp)=\wp^{\lceil\varpi\rceil}\mathbf{\Phi}_{\varpi}^{\mathbf{x}_0}(\wp,\tau)-\sum_{j=0}^{\lceil\varpi\rceil-1}\wp^{\lceil\varpi\rceil-j-1(j)}\mathfrak{T}_{\varpi}^{\mathbf{x}_0}\phi(\mathbf{x}_0,\tau)$
- $\mathfrak{L}_{\varpi}^{\mathbf{x}_0}\left\{\mathfrak{I}_{\varpi}^{\mathbf{x}_0}\phi(\mathbf{x},\tau)\right\}(\wp)=\dfrac{1}{\wp}\left(\mathbf{\Phi}_{\varpi}^{\mathbf{x}_0}(\wp,\tau)+\kappa\right)$, $\kappa=\lim_{\mathbf{x}\rightarrow\mathbf{x}_0}\phi(\mathbf{x},\tau)$

- $\mathfrak{L}_{\varpi}^{\mathbf{x}_0}\left\{(\mathbf{x}-\mathbf{x}_0)^{m\varpi}\phi(\mathbf{x},\tau)\right\}(\wp)=(-1)^m\varpi^m\left(\Phi_{\varpi}^{\mathbf{x}_0}(\wp,\tau)\right)^{(m)}$

4. Explanatory Examples

Example 4.1 Consider a periodic function

$$\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)=\begin{cases}3\dfrac{\tau^{\varpi}}{\varpi}, & 0<\tau<\dfrac{2^{\varpi}}{\varpi}\\ 6, & \dfrac{2^{\varpi}}{\varpi}<\tau<\dfrac{4^{\varpi}}{\varpi}\end{cases}\tag{44}$$

with the period 4. Then from Eq.5 we have the CFLT of Eq.44 as

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\frac{\mu(\varepsilon)}{1-\exp\left(-\varepsilon 4^{\varpi}/\varpi\right)}\tag{45}$$

where $\mu(\varepsilon)$ can be attained as

$$\mu(\varepsilon)=\int_0^{4^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau$$

$$=\int_0^{2^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\left(\frac{3\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau+\int_{2^{\varpi}/\varpi}^{4^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)(6)\,\tau^{\varpi-1}d\tau$$

$$=3\varpi^{1+\varpi}\left(1-2\exp\left(-\varepsilon 4^{\varpi^2}/\varpi^{1+\varpi}\right)\right)+\exp\left(-\varepsilon 2^{\varpi^2}/\varpi^{1+\varpi}\right)\left(-3\varepsilon 2^{\varpi}+\varpi^{1+\varpi}(-3+6\varepsilon)\right)\tag{46}$$

Example 4.2 Consider two functions $\psi\left(\frac{\tau^{\varpi}}{\varpi}\right)=\frac{\tau^{\varpi}}{\varpi}\exp\left(-2\tau^{\varpi}/\varpi\right)$ and $\phi\left(\frac{\tau^{\varpi}}{\varpi}\right)=\exp\left(2\tau^{\varpi}/\varpi\right)$, then the convolution can be defined as

$$(\phi*\psi)=\mathfrak{I}_{\varpi}^{0}\exp\left(2\left(\tau^{\varpi}/\varpi-\upsilon^{\varpi}/\varpi\right)\right)\frac{\upsilon^{\varpi}}{\varpi}\exp\left(-2\left(\upsilon^{\varpi}/\varpi\right)\right)\tag{47}$$

Since, $\mathfrak{L}_{\varpi}^{0}\left\{\psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{1}{(\varepsilon+2)^2}$ and $\mathfrak{L}_{\varpi}^{0}\left\{\phi\left(\frac{\tau^{\varpi}}{\varpi}\right)\right\}(\varepsilon)=\frac{1}{(\varepsilon-2)}$, then from Eq.5, CFLT of Eq.47 is easily achieved as

$$\mathfrak{L}_{\varpi}^{0}\left\{(\phi*\psi)\right\}(\varepsilon)=\frac{1}{(\varepsilon+2)^2(\varepsilon-2)}\tag{48}$$

Example 4.3 Consider the following partial differential equation

$$\frac{\partial^{\varpi}U(x,\tau)}{\partial\tau^{\varpi}}=\frac{\partial^2U(x,\tau)}{\partial x^2},\quad 0<\varpi\leq 1\tag{49}$$

which represents the conformable fractional heat equation with coefficient 1 and subjected to the conditions

$$U(x,0)=f(x),\quad U(0,\tau)=U(1,\tau)=0\tag{50}$$

Taking CFLT on both sides of Eq.49 and using Eq.6 we attain

$$\varepsilon\mathbf{U}_{\varpi}^{0}(x,\varepsilon)-f(x)=\frac{\partial^2\mathbf{U}_{\varpi}^{0}(x,\varepsilon)}{\partial x^2}\tag{51}$$

Assume $\mathbf{U}_{\varpi}^{0}(x,\varepsilon)=V(x)$, then

$$\frac{d^2V(x)}{dx^2}-\varepsilon V(x)=-f(x)\tag{52}$$

which is a second order ordinary differential equation with the conditions $V(0) = V(1) = 0$. Let, $f(x) = 3\sin(2\pi x)$ we have

$$V(x) = \frac{3}{\varepsilon + 4\pi^2}\sin(2\pi x) \tag{53}$$

Thus, the solution can be easily achieved as

$$U(x,\tau) = 3\exp\left(-4\pi^2\tau\right)\sin(2\pi x) \tag{54}$$

5. Conclusion

This work constitutes the descriptions of fractional Laplace transform with the conformable fractional definition. Transformations of the functions with algebraic multiples and divisions were discussed. In addition, delineation of conformable fractional Laplace of trigonometric and exponential conformable fractional integrals, Bessel functions, error functions, periodic functions, convolution of functions and functions in were illustrated. Moreover, results procured from theorems are further supported with illustrative examples. Thus, the following outcomes are distinguished

- Conformable fractional Laplace transforms of algebraic multiplication and division of functions, conformable fractional derivative and integral of functions, trigonometric and exponential integrals are related to the usual Laplace transform if the function is taken as $\phi\left(\tau_0 + (\varpi\tau)^{1/\varpi}\right)$.
- On the other hand, the CFLTs of Bessel and error functions are in the form of fractional infinite series, which corresponds to the simple Laplace transform only in the case when $\varpi = 1$.
- Conformable fractional Laplace transforms of convolution of fractional functions, represented by the conformable fractional integral, resulted in the product of the CFLTs of corresponding fractional functions, which might relate to the usual Laplace transform.

References

[1] K.S. Miller, An introduction to fractional calculus and fractional differential equations, J. Wiley Sons, New York, 1993.
[2] D.A. Murio, Stable numerical evaluation of GrnwaldLetnikov fractional derivatives applied to a fractional IHCP, Inverse Prob. Sci. Eng. 17, 229-243, 2009.
[3] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. 20, 763-769, 2016.
[4] A.A Kilbas, Hadamard-type fractional calculus. J. Korean Math. Soc. 38, 6, 1191-1204, 2011.
[5] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Frac. Differ. Appl. 1, 73-85, 2015.
[6] R. Khalil, M. Al-Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264, 65-70, 2014.
[7] Y.Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives. Adv. Differ. Eqs. 2014, 10, 2014.
[8] A. Kilbas, H. Srivastava and J. Trujillo, Theory and applications of fractional differential equations, In: Math. Studies. North-Holland, New York, 2006.
[9] E.F.D. Goufo, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Bergers equation, Math. Model. Anal. 21, 188-198, 2016.
[10] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, Chaos Solit. Fract. 89, 447-454, 2016.

[11] N.A. Khan, O.A. Razzaq, A. Ara and F. Riaz, Numerical solution of system of fractional differential equations in imprecise environment, Numerical Simulation-From Brain Imaging to Turbulent Flow, InTech, (2016) doi: 10.5772/64150.
[12] K. Hosseini, A. Bekir and R. Ansari, New exact solutions of the conformable time-fractional Cahn-Allen and Cahn-Hilliard equations using the modified Kudryashov method, Optik. 132, 203-209, 2017.
[13] M. Eslami and H. Rezazadeh, The first integral method for WuZhang system with conformable time-fractional derivative, Calcolo. 53, 475-485, 2016.
[14] N. Benkhettou, S. Hassani and D.F.M. Torres, A conformable fractional calculus on arbitrary time Scales, J. King Saud Uni. Sci. 28, 93-98, 2016.
[15] T. Abdeljawad. On conformable fractional calculus. J. Comput. Appl. Math. 279, 57-66, 2015.
[16] H. Batarfi, J. Losada, J.J. Nieto and W. Shammakh, Three-point boundary value problems for conformable fractional differential equations. J. Func. Spaces. 2015, Article ID 706383.
[17] J.L. Schiff, The Laplace transform: Theory and applications, Springer, New York, 1999.
[18] G.K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Int. J. Math. Edu. Sci. Tech. 24, 35-43, 1993.
[19] B. Malgrange, Fourier transform and differential equations, Recent Developments in Quantum Mechanics: Proceedings of the Brasov Conference, Poiana Brasov 1989, Romania, Springer Netherlands, 33-48, 1991.
[20] D. Wei, Novel convolution and correlation theorems for the fractional Fourier transform, Optik-Int. J. Light Elect. Optics. 127, 3669-3675, 2016.
[21] Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy and A. Armand, A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, Fuz. Sets Sys. 309, 81-97, 2017.
[22] R. Darzi, Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation, J. Math. Comp. Sci. 6, 79-84, 2013.
[23] N.A. Khan, O.A. Razzaq and M. Ayyaz, On the solution of fuzzy differential equations by fuzzy Sumudu transform, Nonl. Eng. 4, 1, 49-60, 2015.
[24] S. Liang, R. Wu and L. Chen, Laplace transform of fractional order differential equations, Electr. J. Diff. Eqs. 2015, 139, 1-15, 2015.
[25] G. Jumarie, Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative, Appl. Math. Lett. 22, 1659-1664, 2009.
[26] S.D. Lin and C.H. Lu, Laplace transform for solving some families of fractional differential equations and its applications, Adv. Differ. Eqs. 2013, 137, 2013.

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