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PERTURBATION RESULTS FOR ABSTRACT DEGENERATE VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate additive perturbation theorems for abstract Volterra integro-differential equations in sequentially complete locally convex spaces. We also provide a few instructive examples emphasizing that certain perturbation properties of abstract degenerate Volterra integro-differential equations can be analyzed by using the results from the perturbation theory for non-degenerate equations. Basically, we follow the multivalued linear operator approach to abstract degenerate differential equations.

1. Introduction and preliminaries

The notion of an (a, k)-regularized C-resolvent family generated by a multivalued linear operator has been recently introduced in [19]. The main aim of this paper is to reconsider, in a brief and concise manner, perturbation results for abstract non-degenerate Volterra integro-differential equations ([14, Section 2.6], [17]) from the point of view of the theory of multivalued linear operators. We provide several illustrative applications, primarily to abstract degenerate fractional differential equations with Caputo derivatives.

Chronologically, G. A. Sviridyuk and N. A. Manakova were the first to investigate perturbations of a class of abstract degenerate differential equations of first order ([30], 2003). Using the perturbation theory for strongly continuous semigroups and the theory developed by G. A. Sviridyuk in his fundamental paper [29], V. E. Fedorov and O. A. Ruzakova have analyzed in [11] the unique solvability for the Cauchy problem and Showalter problem for a class of perturbations of abstract degenerate differential equations of first order (cf. [4], [7]-[11], [15], [18]-[20], [23] and [27]-[30]) for the basic source of information on abstract degenerate differential equations with integer order derivatives). The paper [11] contains a great number of applications to initial boundary value problems and we can freely say that this is the first systematic study of perturbations of abstract degenerate differential equations. Recently, A. Favini [10] has considered inverse problems of degenerate differential equations by using perturbation results for linear relations (cf. also M.

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A. Horani, A. Favini [7] and A. Favaron, A. Favini, H. Tanabe [9]). For some other recent results on perturbations of various classes of abstract (non-scalar) Volterra integro-differential equations [15], we refer the reader to [14, Proposition 2.1.12, Theorem 2.6.7] and [18, Theorem 3.12].

The paper is essentially organized as follows. In the second section, we provide the basic information concerning multivalued linear operators in locally convex spaces (cf. [5], [8] and [19] for more details) and remind us of definition of an (a, k)-regularized C-resolvent family generated by a multivalued linear operator. Formulation and proof of our main results, as well as some examples and applications, are given in the third section of paper.

We use the standard notation throughout the paper. Unless specifed otherwise, we shall always assume henceforth that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short; the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. By $L(E)$ we denote the space consisting of all continuous linear mappings from E into E. Denote by $\mathcal{B}$ the family which consists of all bounded subsets of E and set $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(E)$. Then $p_B(\cdot)$ is a seminorm on $L(E)$ and the system $(p_B)_{(p,B)\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E)$. Let us recall that the space $L(E)$ is sequentially complete provided that E is barreled ([22]). If E is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in E$. If A is a closed linear operator acting on E, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Set $p_A(x) := p(x)+p(Ax)$, $x \in D(A)$, $p \in \circledast$. Then the calibration $(p_A)_{p\in\circledast}$ induces the Hausdorff sequentially complete locally convex topology on $D(A)$; we denote this space simply by $[D(A)]$. If F is a linear submanifold of E, then the part of A in F, denoted by A_F, is a linear operator defined by $D(A_F) := \{x \in D(A)\cap F : Ax \in F\}$ and $A_F x := Ax$, $x \in D(A_F)$.

Concerning the integration of functions with values in sequentially complete locally convex spaces, we will follow the approach of C. Martinez and M. Sanz (cf. [21, pp. 99-102]).

If V is a general topological vector space, then a function $f : \Omega \to V$, where Ω is an open non-empty subset of $\mathbb{C}$, is said to be analytic if it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in V. We refer the reader to [1], [14, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our approach the space X is sequentially complete, so that the analyticity of a mapping $f : \Omega \to X$ is equivalent with its weak analyticity.

Given $s \in \mathbb{R}$ in advance, set $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. Define $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$ $(\alpha \in (0, \pi])$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ $(\zeta > 0,\, t > 0)$, $0^0 := 1$ and $g_0(t) :=$ the Dirac δ-distribution.

Fairly complete information about fractional calculus and fractional differential equations can be obtained by consulting [3], [6], [12], [24] and [26]. In this paper, we will use the Caputo fractional derivatives. Let $\zeta > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ ([3], [14]) is defined for those functions $u \in C^{\lceil\zeta\rceil-1}([0,\infty) : E)$ for which $g_{\lceil\zeta\rceil-\zeta} * (u - \sum_{j=0}^{\lceil\zeta\rceil-1} u^{(j)}(0)g_{j+1}) \in C^{\lceil\zeta\rceil}([0,\infty) : E)$, by

$$\mathbf{D}_t^{\zeta} u(t) := \frac{d^{\lceil\zeta\rceil}}{dt^{\lceil\zeta\rceil}} \Biggl[g_{\lceil\zeta\rceil-\zeta} * \Biggl(u - \sum_{j=0}^{\lceil\zeta\rceil-1} u^{(j)}(0) g_{j+1} \Biggr) \Biggr].$$

Mittag-Leffler functions naturally occur as the solutions of fractional order differential equations. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}.$$

In this place, we assume that $1/\Gamma(\alpha n + \beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

Throughout the paper, we assume that $k(t)$ is a scalar-valued continuous function defined on $[0,\tau)$, where $0 < \tau \leq \infty$. The following condition on function $k(t)$ will be used occasionally:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} k(t)\,dt := \int_0^\infty e^{-\lambda t} k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $abs(k) := \inf\{\Re\lambda : \tilde{k}(\lambda) \text{ exists}\}$.

The reader may consult [1], [19], [31] and [14] for further information concerning the Laplace transform of functions with values in Banach and sequentially complete locally convex spaces. In this paper, we will follow our recent approach from [19].

Let $0 < \tau \leq \infty$ and $\mathcal{F} : [0,\tau) \to P(E)$. A single-valued function $f : [0,\tau) \to E$ is called a section of $\mathcal{F}$ iff $f(t) \in \mathcal{F}(t)$ for all $t \in [0,\tau)$. We denote the set consisting of all continuous sections of $\mathcal{F}$ by $\sec_c(\mathcal{F})$.

2. Multivalued linear operators in locally convex spaces

A multivalued map (multimap) $\mathcal{A} : E \to P(E)$ is said to be a multivalued linear operator (MLO) in E, or simply MLO, iff the following holds:

(i) $D(\mathcal{A}) := \{x \in E : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of E;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

An almost immediate consequence of definition is that $\mathcal{A}x + \mathcal{A}y = \mathcal{A}(x+y)$ for all $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x = \mathcal{A}(\lambda x)$ for all $x \in D(\mathcal{A})$, $\lambda \neq 0$. Furthermore, for any $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda|+|\eta| \neq 0$, we have $\lambda\mathcal{A}x+\eta\mathcal{A}y = \mathcal{A}(\lambda x+\eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in E and $\mathcal{A}x = f+\mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted henceforth by $N(\mathcal{A})$ or $\mathrm{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. It is worth noting that $\mathcal{A}x = \mathcal{A}y$ for some two elements x and $y \in D(\mathcal{A})$, iff $\mathcal{A}x \cap \mathcal{A}y \neq \emptyset$; moreover, if $\mathcal{A}$ is injective, then the equality $\mathcal{A}x = \mathcal{A}y$ holds iff $x = y$.

If $\mathcal{A},\ \mathcal{B} : E \to P(E)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A})\cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x+\mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It can be simply verified that $\mathcal{A}+\mathcal{B}$ is likewise an MLO in E. We write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two MLOs in E. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A}$ is an MLO in E and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A}$ in E with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A}$ is an MLO in E and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A}$ are defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \big\{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\big\},$$

and

$$\mathcal{A}^n x := \big(\mathcal{A}\mathcal{A}^{n-1}\big)x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

We can prove inductively that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$. Moreover, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$.

We say that an MLO operator $\mathcal{A}$ in E is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in E such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. Suppose that $\mathcal{A}$ is a closed MLO in E, Ω is a locally compact, separable metric space, and μ is a locally finite Borel measure defined on Ω. If $f : \Omega \to E$ and $g : \Omega \to E$ are μ-integrable, and $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$, then we know that $\int_\Omega f\, d\mu \in D(\mathcal{A})$ and $\int_\Omega g\, d\mu \in \mathcal{A}\int_\Omega f\, d\mu$.

Unless stated otherwise, it will be always assumed that $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Suppose that $\mathcal{A}$ is an MLO in E. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued continuous operator on E.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of C-resolvent sets of single-valued linear operators ([13]-[14]) continue to hold in our framework (observe, however, that there exist certain differences that we will not discuss here). For example, if $\rho(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is closed; it is well known that this statement does not hold if $\rho_C(\mathcal{A}) \neq \emptyset$ for some $C \neq I$.

We need the following important lemma from [19].

Lemma 1 We have

$$\big(\lambda - \mathcal{A}\big)^{-1}C\mathcal{A} \subseteq \lambda\big(\lambda - \mathcal{A}\big)^{-1}C - C \subseteq \mathcal{A}\big(\lambda - \mathcal{A}\big)^{-1}C, \quad \lambda \in \rho_C(\mathcal{A}).$$

The operator $(\lambda - \mathcal{A})^{-1}C\mathcal{A}$ is single-valued on $D(\mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C\mathcal{A}x = (\lambda - \mathcal{A})^{-1}Cy$, whenever $y \in \mathcal{A}x$ and $\lambda \in \rho_C(\mathcal{A})$.

The notion of an (a,k)-regularized C-resolvent family plays a crucial role in the analysis of abstract Volterra equations in locally convex spaces. We will use the following definition of an (a,k)-regularized C-resolvent family subgenerated by an MLO ([19]).

Definition 1 Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : E \to P(E)$ is an MLO, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(E)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ iff $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ ($t \in [0,\tau)$), as well as

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \text{ whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}.$$

We will occasionally use the following condition:

$$\left(\int_0^t a(t-s)R(s)x\,ds, R(t)x - k(t)Cx\right) \in \mathcal{A}, \; t \in [0,\tau), \; x \in E. \tag{1}$$

An (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous iff, for every $t \in (0,\tau)$, the family $\{R(s) : s \in [0,t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family.

3. Perturbations of abstract degenerate Volterra integro-differential equations

We start this section by observing that the following simple lemma holds for multivalued linear operators in locally convex spaces.

Lemma 2 Let $\mathcal{A}$ be an MLO in E, and let $B \in L(E)$. If $\lambda \in \rho(\mathcal{A})$ and $1 \in \rho(B(\lambda-\mathcal{A})^{-1})$, then $\lambda \in \rho(\mathcal{A}+B)$ and

$$\big(\lambda - (\mathcal{A}+B)\big)^{-1} = \big(\lambda-\mathcal{A}\big)^{-1}\Big(1 - B(\lambda-\mathcal{A})^{-1}\Big)^{-1}. \tag{2}$$

Proof. Clearly,

$$\begin{aligned}
&\big(\lambda-\mathcal{A}\big)^{-1}\Big(1 - B(\lambda-\mathcal{A})^{-1}\Big)^{-1} \\
&= \left(\Big(1 - B(\lambda-\mathcal{A})^{-1}\Big)(\lambda-\mathcal{A})\right)^{-1} \\
&= \left(\lambda-\mathcal{A} - B(\lambda-\mathcal{A})^{-1}(\lambda-\mathcal{A})\right)^{-1} \supseteq \big(\lambda-\mathcal{A}-B\big)^{-1}.
\end{aligned}$$

Therefore, it suffices to show that

$$x \in (\lambda-\mathcal{A}-B)\big(\lambda-\mathcal{A}\big)^{-1}\Big(1 - B(\lambda-\mathcal{A})^{-1}\Big)^{-1}x, \quad x \in E.$$

But, this is an immediate consequence of the fact that $x = (1-B(\lambda-\mathcal{A})^{-1})^{-1}x - B(\lambda-\mathcal{A})^{-1}(1-B(\lambda-\mathcal{A})^{-1})^{-1}x$, $x \in E$.

Keeping in mind Lemma 2, [19, Proposition 2.6], the identity [8, (1.2)] and the argumentation already seen in non-degenerate case, the assertions of [14, Theorem

2.6.18-Theorem 2.6.19] can be reformulated for (a,k)-regularized resolvent families in Banach spaces, more or less, without some substantial difficulties. The situation is much more simpler with the assertions of [16, Theorem 4.1, Corollary 4.5], which can be almost straightforwardly reformulated for certain classes of K-convoluted semigroups generated by mutivalued linear operators. Details are left to the interested reader.

The main problem in transferring [14, Theorem 2.6.3] to (a,k)-regularized resolvent families subgenerated by mutivalued linear operators lies in the fact that it is not clear how one can prove that the operator $I-(\mathcal{A}+B)/\lambda^{\alpha}$, appearing in the final part of the proof of this theorem, is injective for $\Re\lambda>0$ suff. large and $\tilde{k}(\lambda)\tilde{a}(\lambda)\neq 0$ (cf. also [14, Theorem 2.6.5, Corollary 2.6.6-Corollary 2.6.9] for further information on this type of bounded commuting perturbations). Nevertheless, the following illustrative example shows that there exist some situations when we can directly apply [14, Theorem 2.6.3] (here, concretely, one of its most important consequences, [14, Corollary 2.6.6]) in the study of perturbation properties of some well-known degenerate equations of mathematical physics and their fractional analogues:

Example 1 Assume that $n\in\mathbb{N}$ and $iA_j,\ 1\leq j\leq n$ are commuting generators of bounded C_0-groups on $E=L^p(\mathbb{R}^n)$, for some $1\leq p<\infty$ (possible applications can be given in $L^p(\mathbb{R}^n)_l$-type spaces, as well; cf. [31]). Set $\mathbf{A}:=(A_1,\cdot\cdot\cdot,A_n)$ and $\mathbf{A}^{\eta}:=A_1^{\eta_1}\cdots A_n^{\eta_n}$ for any $\eta=(\eta_1,\cdots,\eta_n)\in\mathbb{N}_0^n$. If $N\in\mathbb{N}$, and $P(x)=\sum_{|\eta|\leq N}a_{\eta}x^{\eta}$, $x\in\mathbb{R}^n$ is a complex polynomial, define $P(\mathbf{A}):=\sum_{|\eta|\leq N}a_{\eta}\mathbf{A}^{\eta}$. Then we know that the operator $P(\mathbf{A})$ is closable; for more details about functional calculus for commuting generators of bounded C_0-groups, cf. [14]. Suppose now that $P_1(x)$ and $P_2(x)$ are two non-zero complex polynomials in n variables and $0<\alpha<2$; put $N_1:=dg(P_1(x))$ and $N_2:=dg(P_2(x))$. Let $\omega\geq 0$, $N\in\mathbb{N}$, $r\in(0,N]$, let $Q(x)$ be an r-coercive complex polynomial of degree N (cf. [1] for the notion), $a\in\mathbb{C}\setminus Q(\mathbb{R}^n)$ and $\gamma=\frac{n}{r}|\frac{1}{p}-\frac{1}{2}|\max(N,\frac{N_1+N_2}{\min(1,\alpha)})$. Suppose that $P_2(x)\neq 0$, $x\in\mathbb{R}^n$, $P_2(x)$ is an elliptic polynomial, and

$$\sup_{x\in\mathbb{R}^n}\Re\left(\left(\frac{P_1(x)}{P_2(x)}\right)^{1/\alpha}\right)\leq\omega.$$

Then [1, Corollary 8.3.4] yields that $\overline{P_2(\mathbf{A})}^{-1}\in L(E)$ (the violation of this condition has some obvious unpleasant consequenes on the existence and uniqueness of solutions of perturbed problems); hence, $\overline{P_1(\mathbf{A})}\ \overline{P_2(\mathbf{A})}^{-1}$ is a closed linear operator in E. Set

$$R_{\alpha}(t):=\left(E_{\alpha}\left(t^{\alpha}\frac{P_1(x)}{P_2(x)}\right)\big(a-Q(x)\big)^{-\gamma}\right)(A),\ t\geq 0.$$

Then the analysis contained in the proof of [15, Theorem 2.1.22], combined with [19, Remark 4.2(v)], implies that $(R_{\alpha}(t))_{t\geq 0}\subseteq L(E)$ is a global exponentially bounded $(g_{\alpha},R_{\alpha}(0))$-regularized resolvent family generated by $\overline{P_1(\mathbf{A})}\,\overline{P_2(\mathbf{A})}^{-1}$. Set $Df(x):=\int_{-\infty}^{\infty}\psi(x-y)f(y)\,dy,\ f\in E$, where $\psi\in L^1(\mathbb{R}^n)$. Then $D\in L(E)$ and commutes with $\overline{P_1(\mathbf{A})}\ \overline{P_2(\mathbf{A})}^{-1}$. Applying [14, Corollary 2.6.6], we get that the operator $\overline{P_1(\mathbf{A})}\ \overline{P_2(\mathbf{A})}^{-1}+D$ generates an exponentially bounded $(g_{\alpha},R_{\alpha}(0))$-regularized resolvent family, which can be applied in the study of the following perturbation of the abstract fractional Barenblatt-Zheltov-Kochina equation

$$(\eta\Delta - 1)\mathbf{D}_t^{\alpha} u(t) + \Delta u = \int_{-\infty}^{\infty} \psi(x-y)(\eta\Delta - 1)u(t,y)\,dy, \quad (\eta > 0,\ \cos(\pi/\alpha) \leq 0),$$

equipped with the usual initial conditions. We can similarly consider the following perturbation of abstract Boussinesq equation of second order

$$\big(\sigma^2\Delta - 1\big)u_{tt} + \gamma^2\Delta u = \int_{-\infty}^{\infty} \psi(x-y)\big(\sigma^2\Delta - 1\big)u(t,y)\,dy \quad (\sigma > 0,\ \gamma > 0).$$

We shall present one more example in support of use of perturbation theory for abstract non-degenerate differential equations (a similar approach works in the analysis of analytical solutions of perturbed abstract fractional Barenblatt-Zheltov-Kochina equations in finite domains; cf. [20, Definition 4.1, Example 4.4(ii)] for further information):

Example 2 In [19], we have recently applied some results from the theory of abstract non-degenerate differential equations in the study of the following fractional analogue of Benney-Luke equation:

$$(P)_{\eta,f} : \begin{cases} (\lambda - \Delta)\mathbf{D}_t^{\eta} u(t,x) = \big(\alpha\Delta - \beta\Delta^2\big)u(t,x) + f(t,x), \quad t \geq 0,\ x \in \Omega, \\ \Big(\frac{\partial^k}{\partial t^k} u(t,x)\Big)_{t=0} = u_k(x), \quad x \in \Omega,\ 0 \leq k \leq \lceil \eta \rceil - 1, \\ u(t,x) = \Delta u(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \end{cases}$$

where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary, Δ is the Dirichlet Laplacian in $E = L^2(\Omega)$, acting with domain $H^2(\Omega) \cap H_0^1(\Omega)$, $\lambda \in \sigma(\Delta)$, $0 < \eta < 2$ and $\alpha,\ \beta > 0$. Denote by $\{\lambda_k\}$ $[= \sigma(\Delta)]$ the eigenvalues of Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leq -\lambda_2 \cdots \leq -\lambda_k \leq \cdots \to +\infty$ as $k \to \infty$) numbered in nonascending order with regard to multiplicities; by $\{\phi_k\} \subseteq C^{\infty}(\Omega)$ we denote the corresponding set of mutually orthogonal eigenfunctions. Let E_0 be the closed subspace of E consisting of those functions from E that are orthogonal to the eigenfunctions $\phi_k(\cdot)$ for $\lambda_k = \lambda$. Define the closed single-valued linear operator $\mathbf{A}$ in E_0 by its graph: $\mathbf{A} = \{(f,g) \in E_0 \times E_0 : (\lambda - \lambda_k)\langle g, \phi_k\rangle = (\alpha\lambda_k - \beta\lambda_k^2)\langle f, \phi_k\rangle$ for all $k \in \mathbb{N}$ with $\lambda_k \neq \lambda\}$. Then the operator $\mathbf{A}$ generates an exponentially bounded, analytic (g_η, I)-regularized resolvent family of angle $\theta \equiv \min((\pi/\eta) - (\pi/2), \pi/2)$. Suppose that B is a closed linear operator in E satisfying that there exists a number $a > 0$ such that for all sufficiently small numbers $b > 0$ we have $D(\mathbf{A}) \subseteq D(B)$ and $\|Bf\| \leq a\|f\| + b\|\mathbf{A}f\|$, $f \in D(\mathbf{A})$. Applying [3, Theorem 2.25] and the analysis from [19, Example 5.17], we get that the problem $(P)_{\eta,B,f}$, obtained by replacing the term $f(t,x)$ on the right-hand side of the first equation of problem $(P)_{\eta,f}$ by $(\lambda - \Delta)Bu(t,x) + f(t,x)$, has a unique solution provided that $x_0 \in D(\Delta^2) \cap E_0$, $x_1 \in D(\Delta) \cap E_0$, if $\eta > 1$, $\sum_{k|\lambda_k \neq \lambda} \frac{\langle f(\cdot), \phi_k\rangle}{\lambda - \lambda_k}\phi_k = h \in W_{loc}^{1,1}([0,\infty) : E_0)$ satisfies

$$t \mapsto \sum_{k|\lambda_k \neq \lambda} \big(\alpha\lambda_k - \beta\lambda_k^2\big)\Big\langle \frac{d}{dt}(g_\eta * h)(t), \phi_k\Big\rangle \phi_k \in L_{loc}^1([0,\infty) : E_0),$$

$B\phi_k = 0$ for $\lambda = \lambda_k$, and the condition (Q) holds, where

(Q) : $\mathbf{D}_t^{\eta}\langle f(t), \phi_k\rangle$ exists in $L^2(\Omega)$ for $k|\lambda = \lambda_k$, $\langle x_0, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $\langle x_1, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $1 < \eta < 2$, $\langle x_0, \phi_k\rangle = \frac{\langle f(0), \phi_k\rangle}{\beta\lambda_k^2 - \alpha\lambda_k}$ for $k|\lambda = \lambda_k$, and $\langle x_1, \phi_k\rangle = \frac{\langle f'(0), \phi_k\rangle}{\beta\lambda_k^2 - \alpha\lambda_k}$ for $k|\lambda = \lambda_k$, $1 < \eta < 2$.

Finally, we would like to observe that V. E. Fedorov and O. A. Ruzakova have analyzed in [11, Section 5], by using a completely different method, perturbations of degenerate differential equations of first order involving polynomials of elliptic selfadjoint operators.

The assertion of [14, Theorem 2.6.11] admits an extension in our context. We will give some details of the proof immediately after its formulation, when we will be considering Theorem 2.

Theorem 1 Suppose $M > 0$, $\omega \geq 0$, the functions $|a|(t)$ and $k(t)$ satisfy (P1), as well as $\mathcal{A}$ is a densely defined, closed subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ which satisfies that, for every seminorm $p \in \circledast$, we have $p(R(t)x) \leq Me^{\omega t}p(x)$, $x \in E$, $t \geq 0$. Suppose, further, $C^{-1}B \in L(E)$, $BCx = CBx$, $x \in D(\mathcal{A})$, there exist a locally integrable function $b(t)$ and a number $\omega_0 \geq \omega$ such that $|b|(t)$ satisfies (P1) and $\tilde{b}(\lambda) = \frac{\tilde{a}(\lambda)}{\tilde{k}(\lambda)}$, $\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$. Let $\mu > \omega_0$ and $\gamma \in [0,1)$ be such that

$$\int\limits_0^\infty e^{-\mu t} p\Bigg(C^{-1}B\int_0^t b(t-s)R(s)x\,ds\Bigg)\,dt \leq \gamma p(x),\ x \in D(\mathcal{A}),\ p \in \circledast. \tag{3}$$

Then the operator $\mathcal{A}+B$ is a closed subgenerator of an (a,k)-regularized C-resolvent family $(R_B(t))_{t\geq 0}$ which satisfies $p(R_B(t)x) \leq \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$ and

$$R_B(t)x = R(t)x + \int\limits_0^t R_B(t-r)C^{-1}B\int\limits_0^r b(r-s)R(s)x\,ds\,dr,\ t \geq 0,\ x \in D(\mathcal{A}).$$

Furthermore, the equation (1) holds with $R(t)$ replaced by $R_B(t)$ therein.

As observed in [14, Theorem 2.6.12], in many cases we do not have the existence of a function $b(t)$ and a complex number z such that $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda)+z$, $\Re\lambda > \omega_1$, $\tilde{k}(\lambda) \neq 0$. The above-mentioned theorem admits an extension in our context, as well. Before we formulate this extension, let us only outline a few relevant details needed for its proof. First of all, suppose that $\mathcal{A}$ is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $l \in \mathbb{N}$ and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq l$. Then we can prove inductively that, for every $t \in [0,\tau)$,

$$R(t)x_0 = k(t)Cx_0 + \sum_{j=1}^{l-1}\big(a^{*,j} * k\big)(t)Cx_j + \big(a^{*,l} * R(\cdot)x_l\big)(t).$$

In the case that $\tau = \infty$ and the Laplace transform can be applied, the above equation implies that, for certain values of complex parameter λ, we have:

$$\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}x_0 = \tilde{k}(\lambda)Cx_0 + \sum_{j=1}^{l-1}\tilde{a}(\lambda)^j\tilde{k}(\lambda)Cx_j + \tilde{a}(\lambda)^l\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}x_l.$$

If we define the operator family $(S(t))_{t\geq 0}$ as explained below, then the previous equation implies that the identity [14, (180)] continues to hold with the single-valued operator A replaced by the MLO $\mathcal{A}$, provided in addition that the number λ in this equation satisfies $\tilde{a}(\lambda) \neq 0$. Furthermore, the identities [14, (181), (183)] also hold, and the assumption $y \in (I-\tilde{a}(\lambda)(\mathcal{A}+B))x$ implies on account of Lemma 1 and the validity of identity [14, (180)] that $\widetilde{R_B}(\lambda)y = (I - \tilde{S}(\lambda))^{-1}\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cy = \tilde{k}(\lambda)Cx$ for $\Re\lambda > 0$ suff. large and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Owing to the condition (i) in Theorem 2, we have that the operator $\mathcal{A} + B$ is closed and commutes with C. The representation $(I - \tilde{S}(\lambda))^{-1} = \sum_{n=0}^{\infty}[(\frac{1}{\tilde{a}(\lambda)} - \mathcal{A})^{-1}CC^{-1}B]^n$ implies along with the closedness of $\mathcal{A}$ that

$$\tilde{k}(\lambda)Cx \in \left(\frac{1}{\tilde{a}(\lambda)} - (\mathcal{A} + B)\right)\left(I - \tilde{S}(\lambda)\right)^{-1}\tilde{k}(\lambda)\left(\frac{1}{\tilde{a}(\lambda)} - \mathcal{A}\right)^{-1} Cx,\ x \in E,$$

and $R(C) \subseteq R(I - \tilde{a}(\lambda)(\mathcal{A} + B))$ for $\Re\lambda > 0$ suff. large and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Now it is clear that the Laplace transfrom identity [14, (182)] holds with the operator $A + B$ replaced by $\mathcal{A} + B$, provided in addition that the number λ in this equation satisfies $\tilde{a}(\lambda) \neq 0$. After that, we can apply [19, Theorem 5.5(ii)] to complete the whole analysis:

Theorem 2 Suppose $M,\ M_1 > 0,\ \omega \geq 0,\ l \in \mathbb{N}$ and $\mathcal{A}$ is a closed subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ such that $p(R(t)x) \leq Me^{\omega t}p(x)$, $x \in E,\ t \geq 0,\ p \in \circledast$ and (1) holds. Let $|a|(t)$ and $k(t)$ satisfy (P1), and let the following conditions hold:

(i) $BCx = CBx,\ x \in D(\mathcal{A})$; if $x = x_0 \in \overline{D(\mathcal{A})}$, then $C^{-1}Bx \in D(\mathcal{A}^l)$ and there exists a sequence $(x_j)_{1\leq j\leq l}$ such that $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq l$, as well as that:

$$p\big(Cx_j\big) \leq M_1 p(x),\ x \in \overline{D(\mathcal{A})},\ p \in \circledast,\ 0 \leq j \leq l - 1,\ \text{and}$$
$$p\big(x_l\big) \leq M_1 p(x),\ x \in \overline{D(\mathcal{A})},\ p \in \circledast.$$

(ii) There exist a locally integrable function $b(t)$ and a complex number z such that $|b|(t)$ satisfies (P1) and

$$\tilde{a}(\lambda)^{l+1}/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z,\ \Re\lambda > \max(\omega, abs(|a|), abs(k)),\ \tilde{k}(\lambda) \neq 0.$$

(iii) $\lim_{\lambda\to+\infty}\int_0^{\infty} e^{-\lambda t}|a(t)|\, dt = 0$ and $\lim_{\lambda\to+\infty}\int_0^{\infty} e^{-\lambda t}|b(t)|\, dt = 0$.

Define, for every $x = x_0 \in \overline{D(\mathcal{A})}$ and $t \geq 0$,

$$S(t)x := \sum_{j=0}^{l-1} a^{*,j+1}(t)Cx_j + \int_0^t b(t - s)R(s)x_l\, ds + zR(t)x_l,$$

where $(x_j)_{1\leq j\leq l}$ is an arbitrary sequence satisfying the assumptions prescribed in (i). Then, for every $x \in E$, there exists a unique solution of the integral equation

$$R_B(t)x = R(t)x + (S * R_B)(t)x,\ t \geq 0; \tag{4}$$

furthermore, $(R_B(t))_{t\geq 0}$ is an (a, k)-regularized C-resolvent family with a closed subgenerator $\mathcal{A} + B$, there exist $\mu \geq \max(\omega, abs(|a|), abs(k))$ and $\gamma \in [0, 1)$ such that $p(R_B(t)x) \leq \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$ and (1) holds with $R(t)$ replaced by $R_B(t)$ therein.

Remark 1 It is worth noting that Theorem 1 continues to hold, with appropriate changes, in the case that B is not necessarily bounded operator from $\overline{D(\mathcal{A})}$ into E. More precisely, suppose that E is complete, B is a closed linear operator in E, and the requirements of Theorem 1 hold with the condition $C^{-1}B \in L(E)$ replaced by that $D(\mathcal{A}) \subseteq D(C^{-1}B)$ and the mapping $t \mapsto C^{-1}B(b * R)(t)x$, $t \geq 0$ is well-defined, continuous and Laplace transformable for all $x \in D(\mathcal{A})$. Then the final conclusions established in Theorem 1 continue to hold; here, it is only worth noting that the closedness of the operator $\mathcal{A}+B$ can be proved (cf. [14, Remark 2.6.13] for more details, especially, the condition (♮) therein) by using the inclusion $(I-\tilde{S}(\lambda))^{-1}(I-\tilde{a}(\lambda)(\mathcal{A}+B))^{-1}x \subseteq (I-\tilde{a}(\lambda)\mathcal{A})x$, $x \in D(\mathcal{A})$.

Remark 2 The method proposed in the proofs of [25, Theorem 1.2, Theorem 2.3] and [14, Theorem 2.6.13] enables one to deduce some results on the well-posedness of perturbed abstract Volterra inclusion:

$$u(t) \in f(t) + (a + a * k)(t) * \mathcal{A}u(t) + (b * u)(t), \quad t \in [0,\tau), \tag{5}$$

provided that $\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$, b, $k \in L^1_{loc}([0,\infty))$ and $f \in C([0,\infty))$. The starting point is the observation that the regularized resolvent families for (5) satisfy the integral equations like [25, (1.28)] or (4).

Now it quite easy to formulate the following extension of [14, Corollary 2.6.15].
Corollary 1 Suppose M, $M_1 > 0$, $\omega \geq 0$, $\alpha > 0$, $\beta \geq 0$, $\mathcal{A}$ is a closed subgenerator of a $(g_\alpha, g_{\alpha\beta+1})$-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying $p(R(t)x) \leq Me^{\omega t}p(x)$, $x \in E$, $t \geq 0$, $p \in ⊛$ and (1) holds with $a(t) = g_\alpha(t)$ and $k(t) = g_{\alpha\beta+1}(t)$. Assume exactly one of the following conditions:

- (i) $\alpha - 1 - \alpha\beta \geq 0$, $BCx = CBx$, $x \in D(\mathcal{A})$, and (a) $\vee$ (b), where:
  - (a) $p(C^{-1}Bx) \leq M_1 p(x)$, $x \in \overline{D(\mathcal{A})}$, $p \in ⊛$.
  - (b) E is complete, (3) holds, $D(\mathcal{A}) \subseteq D(C^{-1}B)$, as well as the mapping $t \mapsto C^{-1}B(b * R)(t)x$, $t \geq 0$ is well-defined, continuous and Laplace transformable for all $x \in D(\mathcal{A})$.
- (ii) $\alpha - 1 - \alpha\beta < 0$, $BCx = CBx$, $x \in D(\mathcal{A})$, $l = \lceil \frac{\alpha\beta+1-\alpha}{\alpha} \rceil$ and (i) of Theorem 2 holds.

Then there exist $\mu > \omega$ and $\gamma \in [0,1)$ such that $\mathcal{A}+B$ is a closed subgenerator of a $(g_\alpha, g_{\alpha\beta+1})$-regularized C-resolvent family $(R_B(t))_{t\geq 0}$ satisfying $p(R_B(t)x) \leq \frac{M}{1-\gamma}e^{\mu t}p(x)$, $x \in E$, $t \geq 0$, $p \in ⊛$, and (1) holds with $R(t)$ replaced by $R_B(t)$ therein, with $a(t) = g_\alpha(t)$ and $k(t) = g_{\alpha\beta+1}(t)$.

Observe that the local Hölder continuity is an example of the property that is stable under perturbations described in the previous three assertions (cf. [14, Remark 2.6.14] for more details, and [14, Remark 2.6.16] for inheritance of analytical properties under perturbations described in Corollary 1).

Taking into account the assertions of [19, Theorem 2.4(i), Theorem 5.18-Theorem 5.19], and the fact that the identity (2) can be reconsidered for C-resolvents of the operator $\mathcal{A}+B$ (cf. the formula following the equation [14, (204)] for more details), we can repeat almost literally the arguments contained in the proof of [14, Theorem 2.6.22] so as to conclude that the following perturbation result holds true.
Theorem 3 Let $k(t)$ and $|a|(t)$ satisfy (P1). Suppose $\delta \in (0,\pi/2]$, $\omega \geq \max(0, abs(|a|), abs(k))$, there exist analytic functions $\hat{k} : \omega + \Sigma_{\frac{\pi}{2}+\delta} \to \mathbb{C}$ and $\hat{a} : \omega + \Sigma_{\frac{\pi}{2}+\delta} \to \mathbb{C}$ such that $\hat{k}(\lambda) = \tilde{k}(\lambda)$, $\Re\lambda > \omega$, $\hat{a}(\lambda) = \tilde{a}(\lambda)$, $\Re\lambda > \omega$ and $\hat{k}(\lambda)\hat{a}(\lambda) \neq 0$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\delta}$. Let $\mathcal{A}$ be a closed subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle δ, and let (1) hold. Suppose that,

for every $\eta \in (0,\delta)$, there exists $c_\eta > 0$ such that

$$p\big(e^{-\omega \Re z}R(z)x\big) \leq c_\eta p(x),\ x \in E,\ p \in ⊛,\ z \in \Sigma_\eta,$$

as well as $b,\ c \geq 0$, B is a linear operator satisfying $D(C^{-1}\mathcal{A}C) \subseteq D(B)$, $BCx = CBx$, $x \in D(C^{-1}\mathcal{A}C)$ and

$$p\big(C^{-1}Bx\big) \leq bp(y) + cp(x),\ \text{whenever } (x,y) \in C^{-1}\mathcal{A}C,\ p \in ⊛.$$

Assume that at least one of the following conditions holds:

(i) $\mathcal{A}$ is densely defined, the numbers b and c are sufficiently small, there exists $|C|_⊛ > 0$ such that $p(Cx) \leq |C|_⊛ p(x)$, $x \in E$, $p \in ⊛$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \geq \omega$ such that $|\hat{k}(\lambda)^{-1}| = O(|\lambda|)$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$ and $|\hat{a}(\lambda)/\hat{k}(\lambda)| = O(|\lambda|)$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

(ii) $\mathcal{A}$ is densely defined, the number b is sufficiently small, there exists $|C|_⊛ > 0$ such that $p(Cx) \leq |C|_⊛ p(x)$, $x \in E$, $p \in ⊛$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \geq \omega$ such that $|\hat{k}(\lambda)^{-1}| = O(|\lambda|)$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$ and $\hat{a}(\lambda)/(\lambda\hat{k}(\lambda)) \to 0$, $|\lambda| \to \infty$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

(iii) $\mathcal{A}$ is densely defined, the number c is sufficiently small, $b = 0$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \geq \omega$ such that $|\hat{a}(\lambda)/\hat{k}(\lambda)| = O(|\lambda|)$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

(iv) $b = 0$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \geq \omega$ such that $\hat{a}(\lambda)/(\lambda\hat{k}(\lambda)) \to 0$, $|\lambda| \to \infty$, $\lambda \in \omega_\eta + \Sigma_{\frac{\pi}{2}+\eta}$.

Then $C^{-1}(C^{-1}\mathcal{A}C + B)C$ is the integral generator of an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family $(R_B(t))_{t\geq 0}$ of angle δ, which satisfies $R_B(z)[C^{-1}(C^{-1}\mathcal{A}C+B)C] \subseteq [C^{-1}(C^{-1}\mathcal{A}C+B)C]R_B(z)$, $z \in \Sigma_\delta$ and the following condition:

$$\forall \eta \in (0,\delta)\ \exists \omega_\eta' > 0\ \exists M_\eta > 0\ \forall p \in ⊛ :$$
$$p\big(R_B(z)x\big) \leq M_\eta e^{\omega_\eta' \Re z}p(x),\ x \in E,\ z \in \Sigma_\eta.$$

Furthermore, in cases (iii) and (iv), the above remains true with the operator $C^{-1}(C^{-1}\mathcal{A}C + B)C$ replaced by $C^{-1}\mathcal{A}C + B$.

In this paper, we will not discuss possibilities to generalize results on rank 1-perturbations [2] and time-dependent perturbations [32] to (a,k)-regularized C-resolvent families subgenerated by mutivalued linear operators; for more details about non-degenerate case, we refer the reader to [14, Lemma 2.6.26-Theorem 2.6.33] and [14, Theorem 2.6.34, Corollary 2.6.35-Corollary 2.6.38, Theorem 2.6.40, Corollary 2.6.42-Corollary 2.6.45]. In the following theorem, we will extend the assertions of time-dependent perturbations [3, Theorem 2.26] and [14, Theorem 2.6.46(i)] to multivalued linear operators. The proof is very similar to that of [14, Theorem 2.6.46(i)] and therefore omitted.

Theorem 4 Suppose $\alpha \geq 1$, $M \geq 1$, $\omega \geq 0$ and $\mathcal{A}$ is a closed subgenerator of a (local) (g_α, C)-regularized resolvent family $(S_\alpha(t))_{t\in[0,\tau)}$ satisfying $p(S_\alpha(t)x) \leq Me^{\omega t}p(x)$, $t \in [0,\tau)$, $x \in E$, $p \in ⊛$ and (1) with $R(t)$ and $a(t)$ replaced by $S_\alpha(t)$ and $g_\alpha(t)$, respectively. Let $(B(t))_{t\in[0,\tau)} \subseteq L(E)$, $\mathrm{R}(B(t)) \subseteq \mathrm{R}(C)$, $t \in [0,\tau)$ and $C^{-1}B(\cdot) \in C([0,\tau) : L(E))$. Assume that $t \mapsto C^{-1}f(t)$, $t \in [0,\tau)$ is a locally integrable E-valued mapping such that the mapping $t \mapsto (d/dt)C^{-1}f(t)$ is defined for a.e. $t \in [0, \tau)$ and locally integrable on $[0, \tau)$ (in the sense of [21, Definition 4.4.3]). Then there exists a unique solution of the integral Volterra inclusion:

$$u(t, f) \in f(t) + \mathcal{A} \int_0^t g_\alpha(t-s)u(s, f)\, ds + \int_0^t g_\alpha(t-s)B(s)u(s, f)\, ds; \qquad (6)$$

here, by a solution of (6) we mean any continuous function $u \in C([0, \tau) : E)$ such that there exists a continuous section $u_{\mathcal{A},\alpha,f}(t) \in sec_c(\mathcal{A} \int_0^t g_\alpha(t-s)u(s, f)\, ds)$ for $t \in [0, \tau)$, with the property that $u(t, f) = f(t) + u_{\mathcal{A},\alpha,f}(t) + \int_0^t g_\alpha(t-s)B(s)u(s, f)\, ds$, $t \in [0, \tau)$. The solution $u(t, f)$ is given by $u(t, f) := \sum_{n=0}^{\infty} S_{\alpha,n}(t)$, $t \in [0, \tau)$, where we define $S_{\alpha,n}(t)$ recursively by

$$S_{\alpha,0}(t) := S_\alpha(t)C^{-1}f(0) + \int_0^t S_\alpha(t-s)\big(C^{-1}f\big)'(s)\, ds, \quad t \in [0, \tau) \qquad (7)$$

and

$$S_{\alpha,n}(t) := \int_0^t \int_0^{t-\sigma} g_{\alpha-1}(t-\sigma-s)S_\alpha(s)C^{-1}B(\sigma)S_{\alpha,n-1}(\sigma)\, ds\, d\sigma, \quad t \in [0, \tau). \qquad (8)$$

Denote, for every $T \in (0, \tau)$ and $p \in \circledast$, $K_{T,p} := \max_{t\in[0,T]} p(C^{-1}B(t))$ and $F_{T,p} := p(C^{-1}f(0)) + \int_0^T e^{-\omega s} p((C^{-1}f)'(s))\, ds$. Then, for every $p \in \circledast$, we have:

$$p\big(u(t, f)\big) \leq Me^{\omega t}E_\alpha\big(MK_{T,p}t^\alpha\big)F_{T,p},\ t \in [0, T]$$

and

$$p\big(u(t, f) - S_{\alpha,0}(t)\big) \leq Me^{\omega t}\Big(E_\alpha\big(MK_{T,p}t^\alpha\big) - 1\Big)F_{T,p},\ t \in [0, T].$$

In [8, Chapter III], A. Favini and A. Yagi have analyzed a class of infinitely differentiable semigroups generated by multivalued linear operators. Motivated by their research, we introduce the following definition (for the sake of convenience, we shall work only in Banach spaces).

Definition 2 Suppose that $(E, \|\cdot\|)$ is a Banach space, $\alpha > 0$, $\zeta \in (0, 1)$, $0 < \tau \leq \infty$, $\mathcal{A}$ is an MLO in E, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in(0,\tau)} \subseteq L(E)$ is a (g_α, C)-regularized resolvent family of growth order ζ, with a subgenerator $\mathcal{A}$, iff the family $\{t^\zeta R(t) : t \in (0, \tau)\} \subseteq L(E)$ is bounded, as well as that $R(t)C = CR(t)$, $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in (0, \tau))$ and

$$\int_0^t g_\alpha(t-s)R(s)y\, ds = R(t)x - Cx, \text{ whenever } t \in (0, \tau) \text{ and } (x, y) \in \mathcal{A}.$$

It directly follows from definition that, for every $\nu > \zeta$, the operator family $((g_\nu * R)(t))_{t\in[0,\tau)}$ is a g_ν-times integrated (g_α, C)-regularized resolvent family with a subgenerator $\mathcal{A}$.

Consider now the following abstract integral Volterra inclusion:

$$u(t,f) \in f(t)+\mathcal{A}\int_0^t g_\alpha(t-s)u(s,f)\,ds+\int_0^t g_\alpha(t-s)B(s)u(s,f)\,ds, \quad t\in(0,\tau), \tag{9}$$

where $B(\cdot)\in C([0,\tau):L(E))$ and $f\in C((0,\tau):E)$. By a solution of (9) we mean any continuous function $u\in C((0,\tau):E)$ such that the mapping $t\mapsto u(t,f)$, $t\in(0,\tau)$ is locally integrable at the point $t=0$ and there exists a continuous section $u_{\mathcal{A},\alpha,f}(t)\in sec_c(\mathcal{A}\int_0^t g_\alpha(t-s)u(s,f)\,ds)$ for $t\in(0,\tau)$, with the property that $u(t,f)=f(t)+u_{\mathcal{A},\alpha,f}(t)+\int_0^t g_\alpha(t-s)B(s)u(s,f)\,ds$, $t\in(0,\tau)$.

The subsequent theorem is very similar to [3, Theorem 2.26] and [14, Theorem 2.6.46(i)]. For the sake of clarity, we will include the proof.

Theorem 5 Suppose $\alpha\geq 1$, $M\geq 1$, $\omega\geq 0$ and $\mathcal{A}$ is a closed subgenerator of a (local) (g_α,C)-regularized resolvent family $(S_\alpha(t))_{t\in(0,\tau)}$ of growth order $\zeta\in(0,1)$, satisfying that $\|g_{\zeta+1}(t)S_\alpha(t)\|\leq Me^{\omega t}$, $t\in(0,\tau)$ and that (1) holds for $t\in(0,\tau)$ with $R(t)$ and $a(t)$ replaced by $S_\alpha(t)$ and $g_\alpha(t)$, respectively. Let $(B(t))_{t\in[0,\tau)}\subseteq L(E)$, $\mathrm{R}(B(t))\subseteq \mathrm{R}(C)$, $t\in[0,\tau)$ and $C^{-1}B(\cdot)\in C([0,\tau):L(E))$. Assume that $t\mapsto C^{-1}f(t)$, $t\in[0,\tau)$ is a continuous E-valued mapping such that the mapping $t\mapsto S_{\alpha,0}(t)$, defined by (7), is a solution of problem

$$v(t,f)\in f(t)+\mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)\,ds, \quad t\in(0,\tau)$$

and satisfies $\|g_{\zeta+1}(t)S_{\alpha,0}(t)\|\leq Me^{\omega t}$, $t\in(0,\tau)$ (cf. [8, Theorem 3.7-Theorem 3.13] for more details). Then there exists a unique solution $u(t,f)$ of the abstract integral Volterra inclusion (6) on the interval $(0,\tau)$. Moreover, the solution $u(t,f)$ is given by $u(t,f):=\sum_{n=0}^{\infty}S_{\alpha,n}(t)$, $t\in(0,\tau)$, where we define $S_{\alpha,n}(t)$ for $t\in(0,\tau)$ recursively by (8). Denote, for every $T\in(0,\tau)$, $K_T:=\max_{t\in[0,T]}\|C^{-1}B(t)\|$. Then there exists a constant $c_{\alpha,\gamma}>0$ such that:

$$\|u(t,f)\|\leq c_{\alpha,\gamma}e^{\omega t}t^{-\eta}E_{\alpha-\zeta,1-\zeta}\big(MK_Tt^{\alpha-\zeta}\big),\ t\in(0,T] \tag{10}$$

and

$$\big\|u(t,f)-S_{\alpha,0}(t)\big\|\leq c_{\alpha,\gamma}e^{\omega t}t^{-\eta}\Big(E_{\alpha-\zeta,1-\zeta}\big(MK_Tt^{\alpha-\zeta}\big)-1\Big),\ t\in(0,T]. \tag{11}$$

Proof. It is very simple to prove that there exists a constant $c_{\alpha,\gamma}>0$ such that:

$$\|S_{\alpha,n}(t)\|\leq M^{n+1}K_T^n e^{\omega t}\big(g_{\eta+1}(t)\big)^{-1}\frac{t^{(\alpha-\eta)n}}{\Gamma\big((\alpha-\eta)n+1-\eta\big)}, \quad t\in(0,T],\ n\in\mathbb{N}_0,$$

which implies that the series $\sum_{n=0}^{\infty}S_{\alpha,n}(t)$ converges uniformly on compact subsets of $[\epsilon,T]$ and (10)-(11) hold $(0<\epsilon<T)$. Clearly, $u(t,f)=S_{\alpha,0}(t)+\int_0^t(g_{\alpha-1}*$

$S_\alpha)(t-s)C^{-1}B(s)u(s,f)\,ds$, $t\in[0,T]$. With the help of Lemma 1, this implies

$$u(t,f)\in f(t)+\mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)\,ds+\Big[g_{\alpha-1}*S_\alpha*C^{-1}B(\cdot)u(\cdot,f)\Big](t)$$

$$\in f(t)+\mathcal{A}\int_0^t g_\alpha(t-s)v(s,f)\,ds+\mathcal{A}\int_0^t g_\alpha(t-s)\big(g_{\alpha-1}*S_\alpha*C^{-1}B(\cdot)u(\cdot,f)\big)(s)\,ds$$
$$+\big[g_{\alpha-1}*C*C^{-1}B(\cdot)u(\cdot,f)\big](t)$$
$$=f(t)+\mathcal{A}\int_0^t g_\alpha(t-s)u(s,f)\,ds+\int_0^t g_\alpha(t-s)B(s)u(s,f)\,ds,\ \ t\in(0,\tau).$$

Therefore, $u(t,x)$ is a solution of (6). Since the variation of parameters formula holds in our framework, the uniqueness of solutions follows similarly as in the proof of [3, Theorem 2.26].

We close the paper with the following illustrative example.

Example 3

(i) It is clear that Theorem 5 can be applied in the analysis of a great number of the abstract degenerate Cauchy problems of first order appearing in [8, Chapter III] (applications can be also made to some time-oscillation degenerate equations for which the range of possible values of corresponding Caputo fractional derivative depends directly on the value of constant $c>0$ in condition [8, (P), p. 47], provided that $\alpha=1$ in (P)). For example, we can consider the following time-dependent perturbation of the Poisson heat equation in the space $E=L^p(\Omega)$:

$$(P):\left\{\begin{array}{l}\frac{\partial}{\partial t}[m(x)v(t,x)]=(\Delta+b)v(t,x)+m(x)B(t)v(t,x),\ \ t\geq 0,\ \ x\in\Omega;\\ v(t,x)=0,\ \ (t,x)\in[0,\infty)\times\partial\Omega,\\ m(x)v(0,x)=u_0(x),\ \ x\in\Omega,\end{array}\right.$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $b>0$, $m(x)\geq 0$ a.e. $x\in\Omega$, $m\in L^\infty(\Omega)$, $1<p<\infty$ and $B\in C([0,\infty):L(E))$.

(ii) Suppose that A, B and C are three closed linear operators in E, $D(B)\subseteq D(A)\cap D(C)$, $B^{-1}\in L(E)$ and the conditions [8, (6.4)-(6.5)] hold with certain numbers $c>0$ and $0<\beta\leq\alpha\leq 1$. In [8, Chapter VI], the second order differential equation

$$\frac{d}{dt}\big(Cu'(t)\big)+Bu'(t)+Au(t)=f(t),\ \ t>0,$$

has been considered by the usual converting into the first order matricial system

$$\frac{d}{dt}Mz(t)=Lz(t)+F(t),\ \ t>0,$$

where

$$M=\begin{bmatrix} I & O\\ O & C\end{bmatrix},\ L=\begin{bmatrix} O & I\\ -A & -B\end{bmatrix}\text{ and }F(t)=\begin{bmatrix} 0\\ f(t)\end{bmatrix}\ (t>0).$$

The argumentation contained in the proof of [8, Theorem 6.1] shows that the multivalued linear operator $L_{[D(B)]\times E}(M_{[D(B)]\times E})^{-1}$ generates a (g_1, I)-regularized resolvent family $(S_1(t))_{t>0}$ of growth order $\zeta = ((1-\beta)/\alpha)$ in the pivot space $[D(B)] \times E$, satisfying additionally that there exists $\omega \geq 0$ with the property that $\|g_{\zeta+1}(t)S_1(t)\| \leq Me^{\omega t}$, $t > 0$. Assuming that the mappings $t \mapsto B_{1,3}(t) \in L([D(B)])$, $t \geq 0$ and $t \mapsto B_{2,4}(t) \in L(E)$, $t \geq 0$ are continuous, Theorem 5 is susceptible to applications so that we are in a position to consider the-wellposedness of the following system of equations:

$$u_1'(t) = u_2(t) + B_1(t)u_1(t) + B_2(t)u_2(t) + f_1(t),\ t > 0;$$

$$\frac{d}{dt}\big(Cu_2(t)\big) = -Au_1(t) - Bu_2(t) + B_3(t)u_1(t) + B_4(t)u_2(t) + f_2(t),\ t > 0.$$

Many concrete examples of applications can be found in [8, Section 6.2].

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