

## SOME UNIFIED INTEGRALS ASSOCIATED WITH WHITTAKER FUNCTION

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**ABSTRACT.** The main object of this note is to present two unified integral formulas involving the Whittaker function of the first kind  $M_{\rho,\sigma}(z)$ , which are expressed in terms of Kampé de Fériet functions. Some (potentially useful) integrals involving exponential functions, sine hyperbolic functions, modified Bessel functions and Laguerre polynomials are also obtained as special cases of our main results.

### 1. INTRODUCTION

In recent years, a large number of integral formulas involving a variety of special functions of mathematical physics have been developed by several authors (see, [1], [2], [7], [8] [9], [10], [11], [12], [13], [14] [15], [16] etc.). Motivated by such type of works, in this paper, we present two (presumably new) unified integral formulas involving the Whittaker function of the first kind  $M_{\rho,\sigma}(z)$ , which are expressed in terms of the Kampé de Fériet functions.

The Whittaker function  $M_{\rho,\sigma}$  was introduced by Whittaker [3] (see [4, p.337], see also [6, p.39]) in terms of confluent hypergeometric function  ${}_1F_1$  (or Kummer's functions):

$$M_{\rho,\sigma}(z) = z^{\sigma+\frac{1}{2}} e^{-z/2} {}_1F_1\left(\frac{1}{2} + \sigma - \rho; 2\sigma + 1; z\right). \quad (1)$$

In 1921, the four Appell functions were unified and generalized by Kampé de Fériet, who defined a general hypergeometric function of two variables. The notation introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchinal and Chaundy. We recall here the definition of a more general double hypergeometric function in a slightly

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modified notation (see [6, p.63]):

$$F_{l; m; n}^{p; q; k} \left[ \begin{matrix} (a_p) : (b_q); (c_k); \\ (\alpha_l) : (\beta_m); (\gamma_n); \end{matrix} x, y \right] = \sum_{r, s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^l (\alpha_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, \quad (2)$$

where, for convergence,

(i)  $p + q < l + m + 1$ ,  $p + k < l + n + 1$ ,  $|x| < \infty$ ,  $|y| < \infty$ , or

(ii)  $p + q = l + m + 1$ ,  $p + k = l + n + 1$ , and

$$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, & \text{if } p > l, \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq l. \end{cases}$$

In our present investigation, we also need to recall here the following Oberhettinger's integral formula (see [5], see also [7, p.2])

$$\int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\lambda} dx = 2\lambda a^{-\lambda} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda - \mu)}{\Gamma(1 + \lambda + \mu)}, \quad (3)$$

provided  $0 < \Re(\mu) < \Re(\lambda)$ .

## 2. MAIN RESULTS

In this section, we establish two generalized integral formulas involving the Whittaker function of first kind, which are expressed in terms of Kampé de Fériet functions.

**First Integral:** The following integral formula holds true: For  $\Re(\sigma) > -\frac{1}{2}$ ,  $\Re(\sigma - \rho) > -\frac{1}{2}$  and  $0 < \Re(\mu) < \frac{1}{2} + \Re(\lambda + \sigma)$ ,

$$\begin{aligned} & \int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\lambda} M_{\rho, \sigma} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \\ &= y^{\sigma + \frac{1}{2}} a^{\mu - \lambda - \sigma - \frac{1}{2}} \left( \lambda + \sigma + \frac{1}{2} \right) 2^{1-\mu} \frac{\Gamma(2\mu)\Gamma(\lambda + \sigma - \mu + \frac{1}{2})}{\Gamma(\lambda + \sigma + \mu + \frac{3}{2})} \\ & \times F_{2; 1; 0}^{2; 1; 0} \left[ \begin{matrix} \lambda + \sigma + \frac{3}{2}, & \lambda + \sigma - \mu + \frac{1}{2} : & \sigma - \rho + \frac{1}{2}; & -; & y, & -y \\ \lambda + \sigma + \frac{1}{2}, & \lambda + \sigma + \mu + \frac{3}{2} : & 2\sigma + 1; & -; & a, & -2a \end{matrix} \right]. \end{aligned} \quad (4)$$

**Second Integral:** The following integral formula holds true: For  $\Re(\sigma) > -\frac{1}{2}$ ,  $\Re(\sigma - \rho) > -\frac{1}{2}$  and  $-\frac{1}{2} - \Re(\sigma) < \Re(\mu) < \Re(\lambda)$ ,

$$\begin{aligned} & \int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\lambda} M_{\rho, \sigma} \left( \frac{xy}{x + a + \sqrt{x^2 + 2ax}} \right) dx \\ &= y^{\sigma + \frac{1}{2}} a^{\mu - \lambda} \left( \lambda + \sigma + \frac{1}{2} \right) 2^{\frac{1}{2} - \mu - \sigma} \frac{\Gamma(\lambda - \mu)\Gamma(2\mu + 2\sigma + 1)}{\Gamma(2 + \lambda + 2\sigma + \mu)} \end{aligned}$$

$$\times F_{3: 1; 0}^{3: 1; 0} \left[ \begin{matrix} \lambda + \sigma + \frac{3}{2}, \mu + \sigma + \frac{1}{2}, \mu + \sigma + 1 : \sigma - \rho + \frac{1}{2}; -; \frac{y}{2}, -\frac{y}{4} \\ \lambda + \sigma + \frac{1}{2}, \frac{2+\lambda+2\sigma+\mu}{2}, \frac{3+\lambda+2\sigma+\mu}{2} : 2\sigma + 1; -; \end{matrix} \right]. \quad (5)$$

**Proof of (4).** In order to derive our first integral (4), we denote the left-hand side of (4) by  $I$ , expressing  $M_{\rho,\sigma}$  as a series with the help of (1), changing the order of integration and summation (which is verified by uniform convergence of the involved series under the given conditions), we get

$$I = y^{\sigma+\frac{1}{2}} \sum_{r,k=0}^{\infty} \frac{(\sigma - \rho + \frac{1}{2})_r}{(2\sigma + 1)_r} \frac{y^r}{r!} \frac{(-\frac{y}{2})^k}{k!} \int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-(\lambda+\sigma+\frac{1}{2}+r+k)} dx.$$

Evaluating the above integral with the help of (3) and using the result  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ , we get

$$I = y^{\sigma+\frac{1}{2}} a^{\mu-\lambda-\sigma-\frac{1}{2}} 2^{1-\mu} \left( \lambda + \sigma + \frac{1}{2} \right) \frac{\Gamma(2\mu)\Gamma(\lambda + \sigma - \mu + \frac{1}{2})}{\Gamma(\lambda + \sigma + \mu + \frac{3}{2})} \\ \times \sum_{r,k=0}^{\infty} \frac{(\lambda + \sigma + \frac{3}{2})_{r+k} (\lambda + \sigma - \mu + \frac{1}{2})_{r+k} (\sigma - \rho + \frac{1}{2})_r}{(\lambda + \sigma + \frac{1}{2})_{r+k} (\lambda + \sigma + \mu + \frac{3}{2})_{r+k} (2\sigma + 1)_r} \frac{(\frac{y}{a})^r}{r!} \frac{(-\frac{y}{2a})^k}{k!}.$$

Finally, summing up the above series with the help of the definition (2), we arrive at the right-hand side of (4). This completes the proof of our first result.

**Proof of (5).** Similarly, to derive our second integral (5), we denote the left-hand side of (5) by  $I'$ . On expressing  $M_{\rho,\sigma}$  as a series with the help of (1), changing the order of integration and summation (which is verified by uniform convergence of the involved series under the given conditions), we get

$$I' = y^{\sigma+\frac{1}{2}} \sum_{r,k=0}^{\infty} \frac{(\sigma - \rho + \frac{1}{2})_r}{(2\sigma + 1)_r} \frac{y^r}{r!} \frac{(-\frac{y}{2})^k}{k!} \\ \times \int_0^{\infty} x^{\mu+\sigma+\frac{1}{2}+r+k-1} (x + a + \sqrt{x^2 + 2ax})^{-(\lambda+\sigma+\frac{1}{2}+r+k)} dx.$$

Evaluating the above integral with the help of (3) and using the following results;

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ and } (\lambda)_{2n} = 2^{2n} \left( \frac{\lambda}{2} \right)_n \left( \frac{\lambda+1}{2} \right)_n,$$

and after a little simplification, we have

$$I' = y^{\sigma+\frac{1}{2}} a^{\mu-\lambda} 2^{\frac{1}{2}-\mu-\sigma} \left( \lambda + \sigma + \frac{1}{2} \right) \frac{\Gamma(\lambda - \mu) \Gamma(2\mu + 2\sigma + 1)}{\Gamma(2 + \lambda + 2\sigma + \mu)} \\ \times \sum_{r,k=0}^{\infty} \frac{(\lambda + \sigma + \frac{3}{2})_{r+k} (\mu + \sigma + \frac{1}{2})_{r+k} (\mu + \sigma + 1)_{r+k} (\sigma - \rho + \frac{1}{2})_r}{(\lambda + \sigma + \frac{1}{2})_{r+k} (\frac{2+\lambda+2\sigma+\mu}{2})_{r+k} (\frac{3+\lambda+2\sigma+\mu}{2})_{r+k} (2\sigma + 1)_r} \frac{(\frac{y}{2})^r}{r!} \frac{(-\frac{y}{4})^k}{k!},$$

which, upon using the definition (2), yields (5). This completes the proof of our second result.

## 3. SPECIAL CASES

In this section, we derive certain new integral formulas for the exponential functions, modified Bessel functions, sine hyperbolic functions and Laguerre polynomials. To do this, we recall here the following known relations:

$$M_{\rho, -\rho - \frac{1}{2}}(z) = e^{\frac{z}{2}} z^{-\rho}; \quad (6)$$

$$M_{0, \sigma}(2z) = 2^{2\sigma + \frac{1}{2}} \Gamma(1 + \sigma) \sqrt{z} I_{\sigma}(z), \quad (7)$$

where  $I_{\sigma}(z)$  is modified Bessel function [6];

$$M_{0, \frac{1}{2}}(2z) = 2 \sinh z; \quad (8)$$

$$M_{\frac{\alpha}{2} + \frac{1}{2} + n, \frac{\alpha}{2}}(z) = \frac{n!}{(\alpha + 1)_n} e^{-\frac{z}{2}} z^{\frac{\alpha}{2} + \frac{1}{2}} L_n^{\alpha}(z), \quad (9)$$

where  $L_n^{\alpha}(z)$  is the generalized Laguerre polynomial [6].

(i). On setting  $\sigma = -\rho - \frac{1}{2}$  in (4) and then using (6), we get

$$\begin{aligned} & \int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-(\lambda-\rho)} \exp\left\{\frac{y}{2(x+a+\sqrt{x^2+2ax})}\right\} dx \\ &= a^{\mu+\rho-\lambda} 2^{1-\mu} (\lambda-\rho) \frac{\Gamma(2\mu) \Gamma(\lambda-\rho-\mu)}{\Gamma(1+\lambda-\rho+\mu)} \\ & \times {}_2F_2\left[\begin{matrix} \lambda-\rho+1, & \lambda-\rho-\mu; & -\frac{y}{2a} \\ \lambda-\rho, & 1+\lambda-\rho+\mu; \end{matrix}\right], \end{aligned} \quad (10)$$

where  $0 < \Re(\mu) < \Re(\lambda-\rho)$  and  ${}_2F_2$  is the generalized hypergeometric function [6].

(ii). Further, on setting  $\sigma = -\rho - \frac{1}{2}$  in (5) and then using (6), we get

$$\begin{aligned} & \int_0^{\infty} x^{\mu-\rho-1} (x+a+\sqrt{x^2+2ax})^{-(\lambda-\rho)} \exp\left\{\frac{xy}{2(x+a+\sqrt{x^2+2ax})}\right\} dx \\ &= a^{\mu-\lambda} 2^{1-\mu+\rho} (\lambda-\rho) \frac{\Gamma(2\mu-2\rho) \Gamma(\lambda-\mu)}{\Gamma(1+\lambda-2\rho+\mu)} \\ & \times {}_3F_3\left[\begin{matrix} \lambda-\rho+1, & \mu-\rho, & \mu-\rho+\frac{1}{2}; & -\frac{y}{4} \\ \lambda-\rho, & \frac{1+\lambda+\mu-2\rho}{2}, & \frac{2+\lambda+\mu-2\rho}{2}; \end{matrix}\right], \end{aligned} \quad (11)$$

where  $\Re(\rho) < \Re(\mu) < \Re(\lambda)$ .

(iii). On setting  $\rho = 0$  and replacing  $y$  by  $2y$  in (4), and then by using (7), we obtain

$$\begin{aligned} & \int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-(\lambda+\frac{1}{2})} I_{\sigma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= y^{\sigma} a^{\mu-\lambda-\sigma-\frac{1}{2}} 2^{1-\sigma-\mu} \left(\lambda+\sigma+\frac{1}{2}\right) \frac{\Gamma(2\mu) \Gamma(\lambda+\sigma-\mu+\frac{1}{2})}{\Gamma(1+\sigma) \Gamma(\lambda+\sigma+\mu+\frac{3}{2})} \\ & \times F_{2: 1; 0}^{2: 1; 0} \left[\begin{matrix} \lambda+\sigma+\frac{3}{2}, & \lambda+\sigma-\mu+\frac{1}{2}; & \sigma+\frac{1}{2}; & -; & \frac{2y}{a}, & -\frac{y}{a} \\ \lambda+\sigma+\frac{1}{2}, & \lambda+\sigma+\mu+\frac{3}{2}; & 2\sigma+1; & -; \end{matrix}\right], \end{aligned} \quad (12)$$

where  $-\frac{1}{2} < -\frac{1}{2} + \Re(\mu) < \Re(\lambda + \sigma)$ .

(iv). Further, on setting  $\rho = 0$  and replacing  $y$  by  $2y$  in our second integral (5), and then by using (7), we get

$$\begin{aligned} & \int_0^\infty x^{\mu-\frac{1}{2}} (x+a+\sqrt{x^2+2ax})^{-(\lambda+\frac{1}{2})} I_\sigma \left( \frac{xy}{x+a+\sqrt{x^2+2ax}} \right) dx \\ &= y^\sigma a^{\mu-\lambda} 2^{\frac{1}{2}-2\sigma-\mu} \left( \lambda + \sigma + \frac{1}{2} \right) \frac{\Gamma(2\mu+2\sigma+1) \Gamma(\lambda-\mu)}{\Gamma(1+\sigma) \Gamma(\lambda+2\sigma+\mu+2)} \\ & \times F_{3: 1; 0}^{3: 1; 0} \left[ \begin{matrix} \lambda + \sigma + \frac{3}{2}, \mu + \sigma + \frac{1}{2}, \mu + \sigma + 1 : \sigma + \frac{1}{2}; & -; \\ \lambda + \sigma + \frac{1}{2}, \frac{2+\lambda+2\sigma+\mu}{2}, \frac{3+\lambda+2\sigma+\mu}{2} : 2\sigma + 1; & -; \end{matrix} \right. \left. y, -\frac{y}{2} \right], \end{aligned} \tag{13}$$

where  $-\frac{1}{2} - \Re(\sigma) < \Re(\mu) < \Re(\lambda)$ .

(v). On setting  $\rho = 0, \sigma = \frac{1}{2}$  and replacing  $y$  by  $2y$  in our first integral (4), and then by using (8), we get

$$\begin{aligned} & \int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\lambda} \sinh \left( \frac{y}{x+a+\sqrt{x^2+2ax}} \right) dx = y a^{\mu-\lambda-1} 2^{1-\mu} (\lambda+1) \\ & \times \frac{\Gamma(2\mu) \Gamma(\lambda+1-\mu)}{\Gamma(2+\lambda+\mu)} F_{2: 1; 0}^{2: 1; 0} \left[ \begin{matrix} \lambda + 2, \lambda + 1 - \mu : 1; & -; \\ \lambda + 1, 2 + \lambda + \mu : 2; & -; \end{matrix} \right. \left. \frac{2y}{a}, -\frac{y}{a} \right], \end{aligned} \tag{14}$$

where  $0 < \Re(\mu) < 1 + \Re(\lambda)$ .

(vi). Further, on setting  $\rho = 0, \sigma = \frac{1}{2}$  and replacing  $y$  by  $2y$  in our second integral (5), and then by using (8), we get

$$\begin{aligned} & \int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\lambda} \sinh \left( \frac{xy}{x+a+\sqrt{x^2+2ax}} \right) dx = y (\lambda+1) a^{\mu-\lambda} \\ & \times \frac{\Gamma(\lambda-\mu) \Gamma(2\mu+2)}{2^\mu \Gamma(3+\lambda+\mu)} F_{3: 1; 0}^{3: 1; 0} \left[ \begin{matrix} \lambda + 2, \mu + 1, \mu + \frac{3}{2} : 1; & -; \\ \lambda + 1, \frac{3+\lambda+\mu}{2}, \frac{4+\lambda+\mu}{2} : 2; & -; \end{matrix} \right. \left. y, -\frac{y}{2} \right], \end{aligned} \tag{15}$$

where  $-1 < \Re(\mu) < \Re(\lambda)$ .

(vii). On setting  $\rho = \frac{\alpha}{2} + \frac{1}{2} + n$  ( $n$  is non negative integer) and  $\sigma = \frac{\alpha}{2}$  in (4), and then by using (9), we obtain

$$\begin{aligned} & \int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-(\lambda+\frac{\alpha}{2}+\frac{1}{2})} \exp \left\{ -\frac{y}{2(x+a+\sqrt{x^2+2ax})} \right\} \\ & \quad \times L_n^\alpha \left( \frac{y}{x+a+\sqrt{x^2+2ax}} \right) dx \\ &= 2^{1-\mu} a^{\mu-\lambda-\frac{\alpha}{2}-\frac{1}{2}} \left( \lambda + \frac{\alpha}{2} + \frac{1}{2} \right) \frac{(\alpha+1)_n \Gamma(2\mu) \Gamma(\lambda+\frac{\alpha}{2}-\mu+\frac{1}{2})}{n! \Gamma(\frac{3}{2}+\lambda+\frac{\alpha}{2}+\mu)} \\ & \times F_{2: 1; 0}^{2: 1; 0} \left[ \begin{matrix} \lambda + \frac{\alpha}{2} + \frac{3}{2}, \lambda + \frac{\alpha}{2} - \mu + \frac{1}{2} : -n; & -; \\ \lambda + \frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2} + \lambda + \frac{\alpha}{2} + \mu : \alpha + 1; & -; \end{matrix} \right. \left. \frac{y}{a}, -\frac{y}{2a} \right], \end{aligned} \tag{16}$$

where  $0 < \Re(\mu) < \frac{1}{2} + \Re\{\lambda + (\frac{\alpha}{2})\}$ .

(viii). Further, on setting  $\rho = \frac{\alpha}{2} + \frac{1}{2} + n$  and  $\sigma = \frac{\alpha}{2}$  in our second integral (5), and then by using (9), we get

$$\begin{aligned} & \int_0^\infty x^{\mu + \frac{\alpha}{2} - \frac{1}{2}} (x + a + \sqrt{x^2 + 2ax})^{-(\lambda + \frac{\alpha}{2} + \frac{1}{2})} \exp\left\{\frac{-xy}{2(x + a + \sqrt{x^2 + 2ax})}\right\} \\ & \quad \times L_n^\alpha\left(\frac{xy}{x + a + \sqrt{x^2 + 2ax}}\right) dx \\ & = 2^{\frac{1}{2} - \mu - \frac{\alpha}{2}} a^{\mu - \lambda} \left(\lambda + \frac{\alpha}{2} + \frac{1}{2}\right) \frac{(\alpha + 1)_n \Gamma(2\mu + \alpha + 1) \Gamma(\lambda - \mu)}{n! \Gamma(2 + \lambda + \mu + \alpha)} \\ & \times F_{3: 1; 0}^{3: 1; 0} \left[ \begin{matrix} \lambda + \frac{\alpha}{2} + \frac{3}{2}, \mu + \frac{\alpha}{2} + \frac{1}{2}, \mu + \frac{\alpha}{2} + 1 : & -n, & -; & \frac{y}{2}, & -\frac{y}{4} \\ \lambda + \frac{\alpha}{2} + \frac{1}{2}, \frac{2 + \lambda + \mu + \alpha}{2}, \frac{3 + \lambda + \mu + \alpha}{2} : & \alpha + 1; & -; & & \end{matrix} \right], \quad (17) \end{aligned}$$

where  $-\frac{1}{2} - \Re(\frac{\alpha}{2}) < \Re(\mu) < \Re(\lambda)$ .

#### 4. CONCLUDING REMARKS

In this paper, we have established some potentially useful integrals involving Whittaker function of first kind. Also, we have derived various interesting integrals (involving exponential functions, sine hyperbolic functions, modified Bessel functions and Laguerre polynomials) as special cases of the main results. It is noticed that, the modified Bessel function and Laguerre polynomial are the special cases of Bessel function and Hermite polynomial, respectively. Therefore, we can obtain some other interesting integrals involving Bessel functions and Hermite polynomials after some suitable parametric replacement. As the integrals involving various kind of special functions play an important role in many diverse field of physics, such as in the field of neutron physics, plasma physics, radio physics etc. So the results presented in this paper may also be useful in the same directions of physics and engineering sciences.

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