A NEW CLASS OF INTEGRALS INVOLVING EXTENDED MITTAG-LEFFLER FUNCTIONS

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ABSTRACT. The main aim of this paper is to establish two generalized integral formulas involving the extended Mittag-Leffler function based on the well known Lavoie and Trottier integral formula and the results are express in term of extended Wright-type function. Also, we establish certain special cases of our main result.

1. INTRODUCTION

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \ z \in \mathbb{C}, \ \Re(\alpha) > 0, \tag{1}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \ z, \beta \in \mathbb{C}, \ \Re(\alpha) > 0, \tag{2}$$

respectively. For the generalizations and applications of Mittag-Leffler function, the readers can refer to the recent work of researchers ([5, 6, 7, 8, 9, 11, 23, 25, 33]), Kilbas et al. ([12], Chapter 1) and Saigo and Kilbas [13]. In recent years, the Mittag-Leffler function and some of its variety generalizations have been numerically established in the complex plane ([10, 27]). A new generalization of the Mittag-Leffler functions $E_{\alpha,\beta}(z)$ of (2) have defined by Prabhakar [24] as follows:

$$E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^\infty \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \ z, \beta \in \mathbb{C}, \ \Re(\alpha) > 0, \tag{3}$$

where $(\gamma)_n$ denote the well known Pochhammer Symbol which is defined by:

$$(\gamma)_n = \begin{cases} 1, & (n = 0, \ \gamma \in \mathbb{C}), \\ \gamma(\gamma+1)(\gamma+2)...(\gamma+n-1) & (n \in N, \ \gamma \in \mathbb{C}). \end{cases}$$

In fact, the following special cases are satisfied:

$$E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E^1_{\alpha,1}(z) = E_{\alpha,\beta}(z). \tag{4}$$

Recently, many researchers have established the significance and great consideration of Mittag-Leffler functions in the theory of special functions for exploring the generalizations and some applications. Various extensions for these functions are found in ([9],[28, 29, 30]). Srivastava and Tomovski [32] have defined further generalization of the Mittag-Leffler function $E^1_{\alpha,\beta}(z)$ of (3), which is defined as:

$$E^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{5}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(k) - 1\}; \Re(k) > 0$. Very recently, Ozarslan and Yilmaz [22] have investigated an extended Mittag-Leffler function $E^{\gamma,c}_{\alpha,\beta}(z : p)$ which is defined as:

$$E^{\gamma;c}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma + n, c - \gamma)(c)_n}{\mathbf{B}(\gamma, c - \gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{6}$$

where $p > 0$, $\Re(c) > \Re(\gamma) > 0$ and $B_p(x, y)$ is extended beta function defined in Mainardi as follows:

$$B_p(x, y) = \int_0^t t^{(x-1)}(1 - t)^{(y-1)}e^{-(p/t(1-t))}dt, \tag{7}$$

where $\Re(p) > 0, \Re(x) > 0$ and $\Re(y) > 0$. If $p = 0$, then $B_p(x, y)$ reduces to the following beta function:

$$B(x, y) = \int_0^t t^{(x-1)}(1 - t)^{(y-1)}dt. \tag{8}$$

The notion of the extended beta functions given by Chaudhry et al. [2] and further analyzed in [16, 18] The gamma function is defined by:

$$\Gamma(z) = \int_0^t t^{(z-1)}e^{(-t)}dt; \quad \Re(z) > 0. \tag{9}$$

By inspection, we conclude the following relation

$$\Gamma(z + 1) = z\Gamma(z). \tag{10}$$

The generalized hypergeometric function $_pF_q(z)$ is defined in [6] as:

$$\begin{aligned}
_pF_q(z) &= \ _pF_q \left[\begin{array}{cc} (a_1), (a_2), ..., (a_p); & \\ & z, p \\ (b_1), (b_2), ..., (b_q); & \end{array} \right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_q)_n z^n}{(b_1)_n(b_2)_n...(b_p)_n n!}
\end{aligned} \tag{11}$$

where $\alpha_i, \beta_i \in \mathbb{C}; i = 1, 2, ..., p; j = 1, 2, ..., q$ and $b_j \neq 0, -1, -2, ...$ and $(z)_n$ is the Pochhammer symbols.

Sharma and Devi [26] defined extended Wright type function in the following series:

$$
\begin{aligned}
{}_{p+1}F_{q+1}(z;p) &= {}_{p+1}F_{q+1}\left[
\begin{array}{cc}
(a_i, A_i)_{1,p}, (\gamma, 1); & \\
 & z \\
(b_j, \beta_j)_{1,p}, (c, 1); &
\end{array}
\right] \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n)...\Gamma(a_p + A_p n)\mathbf{B}_p(\gamma + n, c - \gamma)z^n}{\Gamma(b_1 + \beta_1 n)...\Gamma(b_q + \beta_q n)\Gamma(c - \gamma)n!} \quad (12)
\end{aligned}
$$

where $i = 1, 2, ..., p, j = 1, 2, ..., q, \Re(p) > 0, \Re(c) > \Re(\gamma) > 0$ and $p \geq 0$. Recently, Mittal et al. [17] defined an extended generalized Mittag-Leffer function as:

$$
E_{\alpha,\beta}^{\gamma,q;c}(z;p) = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma + nq, c - \gamma)(c)_{nq}}{\mathbf{B}(\gamma, c - \gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad (13)
$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q > 0$ and $B_p(x, y)$ is an extended beta function defined in (7). In this paper, we derive new integral formulas involving the Mittag-Leffler function (6) and (13). For the present investigation, we need the following result of Lavoie and Trottier [14]

$$
\int_0^1 x^{(a-1)}(1-x)^{(2b-1)} \left(1 - \frac{x}{3}\right)^{(2a-1)} \left(1 - \frac{x}{4}\right)^{(b-1)} dx = \left(\frac{2}{3}\right)^{2a} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (14)
$$

where $\Re(a) > 0, \Re(b) > 0$.

For more details about the recent works in the field of dynamical systems theory, stochastic systems, non-equilibrium statistical mechanics and quantum mechanics, the readers may refer to the recent work of the researchers[1, 3, 4, 15, 19, 20, 21] and the references cited therein.

2. Main Result

In this section, the generalized integral formulas involving the Mittag-Leffler function (6) and (13), are established here by inserting with the suitable argument in the integrand of (14).

Theorem 1. *Let $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; with $\Re(\alpha) > \max\{0, \Re(c) - 1\}; \Re(c) > 0.$ $\Re(\gamma) > 0, \Re(2\rho + j) > 0$ and $x > 0$. Then the following formula holds true:*

$$
\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)} \left(1 - \frac{x}{3}\right)^{(2(\rho+j)-1)} \left(1 - \frac{x}{4}\right)^{(\rho-1)} E_{\alpha,\beta}^{\gamma;c}\left(y\left(1 - \frac{x}{4}\right)(1-x)^2\right) dx
$$

$$
= \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} \, {}_3\psi_3\left[
\begin{array}{cc}
(\rho, 1), (c, 1), (\gamma, 1); & \\
 & |y; p \\
(\beta, \alpha), (2\rho + j, 1), (c, 1); &
\end{array}
\right]. (15)
$$

Proof. Let S_1 be the left-hand side of (15). Now applying the series representation of the Mittag-Leffler function (6) to the integrand of (14) and by interchanging the order of integration and summation under the given condition in Theorem (1), we

have

$$
\begin{aligned}
S_1 &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} \\
&\times E_{\alpha,\beta}^{\gamma;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx \\
&= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} \\
&\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^n}{n!}dx \\
&= \int_0^1 x^{(\rho+j-1)}(1-x)^{2(\rho+n)-1}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho+n-1)} \\
&\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx.
\end{aligned}
$$

Now using (6) and (14), we get

$$
\begin{aligned}
S_1 &= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\
&= \left(\frac{2}{3}\right)^{(2\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)\Gamma(\rho+n)\Gamma(c+n)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{y^n}{n!},
\end{aligned}
$$

which upon using (12), we get

$$
S_1 = \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{cc}(\rho,1),(c,1),(\gamma,1) & \\ & ;|y;p\\ (\beta,\alpha),(2\rho+j,1),(c,1) & \end{array}\right].
$$

$\square$

Theorem 2. *Let* $y,c,\rho,j,\alpha,\beta,\gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0,\Re(c)-1\}; \Re(c) > 0.$ $\Re(\gamma) > 0, \Re(2\rho+j) > 0$ *and* $x > 0$. *Then the following formula holds true:*

$$
\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)}E_{\alpha,\beta}^{\gamma;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx
$$
$$
= \left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{cc}(\rho,1),(c,1),(\gamma,1); & \\ & ;|\tfrac{4}{9}y;p\\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right]. \tag{16}
$$

Proof. Let S_2 be the left-hand side of (16). Now applying the series representation of the Mittag-Leffler function (6) to the integrand of (14), we have

$$
\begin{aligned}
S_2 &= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\quad \times\ E_{\alpha,\beta}^{\gamma;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx \\
&= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\quad \times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{3}\right)^2\right)^n}{n!}dx \\
&= \int_0^1 x^{(\rho+n-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho+n)-1}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\quad \times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx.
\end{aligned}
$$

Now using (6) and (14), we get

$$
\begin{aligned}
S_2 &= \left(\frac{2}{3}\right)^{2(\rho+n)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\
&= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)\Gamma(\rho+n)\Gamma(c+n)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{(\frac{4}{9}y)^n}{n!},
\end{aligned}
$$

which upon using (12), we get

$$
S_2 = \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ _3\psi_3\left[\begin{array}{cc}(\rho,1),(c,1),(\gamma,1); & \\ & ;|\tfrac{4}{9}y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right].
$$

$\square$

Theorem 3. *Let* $y,c,\rho,j,\alpha,\beta,\gamma\in\mathbb{C};$ *with* $\Re(\alpha)>\max\{0,\Re(c)-1\};\Re(c)>0.$ $\Re(\gamma)>0,\Re(2\rho+j)>0$ *and* $x>0.$ *Then the following formula holds true:*

$$
\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)}E_{\alpha,\beta}^{\gamma,q;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2;p\right)dx
$$

$$
= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ _3\psi_3\left[\begin{array}{cc}(\rho,1),(c,q),(\gamma,1); & \\ & ;|y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right]\quad(17)
$$

Proof. Let S_3 be the left-hand side of (17). Now applying the series representation of the Mittag-Leffler function (6) to the integrand of (14) and by interchanging the order of integration and summation under the given condition in Theorem (1), we

have

$$
\begin{aligned}
S_3 &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} \\
&\times\ E_{\alpha,\beta}^{\gamma,q;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2;p\right)dx \\
&= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} \\
&\times\sum_{n=0}^\infty \frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^n}{n!}dx \\
&= \int_0^1 x^{(\rho+j-1)}(1-x)^{2(\rho+n)-1}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho+n-1)} \\
&\times\sum_{n=0}^\infty \frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx.
\end{aligned}
$$

Now using (6) and (14), we get

$$
\begin{aligned}
S_1 &= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^\infty \frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\
&= \left(\frac{2}{3}\right)^{(2\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\mathbf{B}_p(\gamma+nq,c-\gamma)\Gamma(\rho+n)\Gamma(c+nq)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{y^n}{n!},
\end{aligned}
$$

which upon using (12), we get

$$
S_3 = \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{cc} (\rho,1),(c,1),(\gamma,1); & \\ & ;|y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right].
$$

$\square$

Theorem 4. *Let* $\rho,j,\alpha,\beta,\gamma\in\mathbb{C}$; *with* $R(\alpha)>0,\Re(c)>\Re(\gamma)>0,\Re(2\rho+j)>0$ *and* $x>0$. *Then the following formula holds true:*

$$
\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)}E_{\alpha,\beta}^{\gamma,q;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx
$$

$$
= \left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\rho+j)}{\Gamma(\rho)}\ {}_3\psi_3\left[\begin{array}{cc} (\rho,1),(c,q),(\gamma,1); & \\ & ;|\frac{4}{9}y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right]. \tag{18}
$$

Proof. Let S_4 be the left-hand side of (18). Now applying the series representation of the Mittag-Leffler function (6) to the integrand of (14), we have

$$
\begin{aligned}
S_4 &= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\times\ E_{\alpha,\beta}^{\gamma,q;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx \\
&= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{3}\right)^2\right)^n}{n!}dx \\
&= \int_0^1 x^{(\rho+n-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho+n)-1}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\
&\times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx
\end{aligned}
$$

Now using (13) and (14), we get

$$
\begin{aligned}
S_4 &= \left(\frac{2}{3}\right)^{2(\rho+n)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\
&= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)\Gamma(\rho+n)\Gamma(c+nq)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{(\frac{4}{9}y)^n}{n!},
\end{aligned}
$$

which upon using (12), we get

$$
S_4 = \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{cc} (\rho,1),(c,q),(\gamma,1); & \\ & |\tfrac{4}{9}y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right].
$$

$\square$

3. Particular cases

In this section, we obtained some particular cases of Theorems 1-4. Setting $p=0$ in (6) and (13), then we get the following well known Mittag-Leffler functions.

$$
E_{\alpha,\beta}^{\gamma,c}(z) = \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{19}
$$

and

$$
E_{\alpha,\beta}^{\gamma,q;c}(z) = \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{z^n}{n!}. \tag{20}
$$

Corollary 3.1. *Let* $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0, \Re(c) - 1\}; \Re(c) > 0.$ $\Re(\gamma) > 0, \Re(2\rho + j) > 0$ *and* $x > 0.$ *Then the following formula holds true:*

$$\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} E^\gamma_{\alpha,\beta}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\; {}_2\psi_2\left[\begin{array}{cc} (\rho,1),(\gamma,1); & \\ & |y \\ (\beta,\alpha),(2\rho+j,1); & \end{array}\right].\,(21)$$

Corollary 3.2. *Let* $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0, \Re(c) - 1\}; \Re(c) > 0.$ $\Re(\gamma) > 0, \Re(2\rho + j) > 0$ *and* $x > 0.$ *Then the following formula holds true:*

$$\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E^\gamma_{\alpha,\beta}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\; {}_2\psi_2\left[\begin{array}{cc} (\rho,1),(\gamma,1); & \\ & ;|\frac{4}{9}y \\ (\beta,\alpha),(2\rho+j,1); & \end{array}\right].\quad\quad(22)$$

Corollary 3.3. *Let* $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0, \Re(c) - 1\}; \Re(c) > 0.$ $\Re(\gamma) > 0, \Re(2\rho + j) > 0$ *and* $x > 0.$ *Then the following formula holds true:*

$$\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} E^{\gamma,q}_{\alpha,\beta}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\; {}_2\psi_2\left[\begin{array}{cc} (\rho,1),(\gamma,q); & \\ & ;|y \\ (\beta,\alpha),(2\rho+j,1); & \end{array}\right]\quad\quad(23)$$

Corollary 3.4. *Let* $\rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > 0, \Re(c) > \Re(\gamma) > 0, \Re(2\rho+j) > 0$ *and* $x > 0.$ *Then the following formula holds true:*

$$\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E^{\gamma,q}_{\alpha,\beta}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\rho+j)}{\Gamma(\rho)}\; {}_2\psi_2\left[\begin{array}{cc} (\rho,1),(\gamma,q); & \\ & ;|\frac{4}{9}y \\ (\beta,\alpha),(2\rho+j,1); & \end{array}\right].\quad\quad(24)$$

References

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