Journal of Fractional Calculus and Applications
Vol. 9(2) June 2018, pp. 13-21.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

INTEGRAL TRANSFORMS OF GENERALIZED K-MITTAG-LEFFLER FUNCTION

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Abstract. Remarkably, large number of integral formulas involving a variety of special functions have been developed by many authors. Also many integral formulas involving Mittag-Leffler function have been exhibited. In this paper, we establish two new integral formulas involving the generalized k-Mittag-Leffler function, which are expressed in terms of the generalized(Wright) hypergeometric functions.

1. Introduction

Integral transforms play an important role in many diverse fields of physics and engineering. In this paper, we present two new integral transforms involving generalized k-Mittag-Leffler function, which are expressed in terms of Wright hypergeometric functions. Numerous integral transforms involving a variety of special functions have been established by many researchers (see[18]-[21]). The importance of the Mittag-Leffler function is realized during the last one and a half decades due to its direct involvement in the problems of Physics, Biology, Engineering and Applied Sciences. Mittag-Leffler functions naturally occurs as the solution of fractional order differetial equations and fractional order integral equations. The recent work is in the field of non-equilibrium statistical mechanics, Quantum mechanics and dynamical system theory (see [4],[13],[17],[22],[23] and [25])
The Mittag-Leffler function $E_\alpha(z)$ is defined as follows (see [10] and [11]).

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)},\ (\alpha, z \in \mathbb{C}, \Re(\alpha) > 0) \tag{1}$$

by assigning values $\alpha = 2$ *and* $\alpha = 4$ respectively, we get

$$E_2(z) = \cosh(\sqrt{z}),\ z \in \mathbb{C} \tag{2}$$

$$E_4(z) = \frac{1}{2}\left[\cos(z^{\frac{1}{4}}) + \cosh(z^{\frac{1}{4}})\right],\ z \in \mathbb{C} \tag{3}$$

2010 *Mathematics Subject Classification.* 33C45, 26A33, 05C38, 33C20.
Key words and phrases. k-Mittag-Leffler function, Generalized(Wright)hypergeometric function, Lavoie-Trottier integral formula, Gamma function.
Submitted Jan. 11, 2017.

and its generalization was studied by Wiman [2], defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ (\alpha,\beta,z\in\mathbb{C},[\Re(\alpha),\Re(\beta)]>0) \tag{4}$$

with $\mathbb{C}$ being the set of complex numbers are called Mittag-Leffler functions. The former was introduced by Mittag-Leffler (1903), in connection with his method of summation of some divergent series.
By assigning values $\alpha=1,\ \beta=2\ and\ \alpha=2,\ \beta=2$ respectively, we get

$$E_{1,2}(z)=\frac{e^z-1}{z},\ z\in\mathbb{C} \tag{5}$$

$$E_{2,2}(z)=\frac{\sinh(\sqrt{z})}{\sqrt{z}},\ z\in\mathbb{C} \tag{6}$$

In 1971, Prabhakar [24] intoduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the following form

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)n!},\ (\alpha,\beta,\gamma,z\in\mathbb{C},[\Re(\alpha),\Re(\beta),\Re(\gamma)]>0) \tag{7}$$

The generalization of the Mittag-Leffler function called as k-Mittag Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$ introduced [9] and is defined as.

$$E^{\gamma}_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k} z^n}{\Gamma_k(\beta+\alpha n)n!} \tag{8}$$

where $\alpha,\beta,\gamma,z\in\mathbb{C},\ [\Re(\alpha),\Re(\beta),\Re(\gamma)]>0$ and $k\in\mathbb{R}$ and $(\gamma)_{n,k}$ is the k-Pochammer symbol defined as

$$(\gamma)_{n,k}=\gamma(\gamma+k)(\gamma+2k).....(\gamma+(n-1)k),\ (\gamma\in\mathbb{C},k\in\mathbb{R},n\in\mathbb{N}) \tag{9}$$

The k-Pochhammer symbol in terms of k-Gamma function satisfies the following relation

$$(\gamma)_{n,k}=\frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)},\ (\gamma\in\mathbb{C},k\in(0,\infty),n\in\mathbb{N}) \tag{10}$$

The k-Gamma function satisfies the following relation

$$\Gamma_k(\gamma)=(k)^{\frac{\gamma}{k}-1}\Gamma(\frac{\gamma}{k}),\ (k\in(0,\infty),\Re(\gamma)>0) \tag{11}$$

Newly, a generalization of k-Mittag-Leffler function was introduced (see [15]) as

$$GE^{\gamma,q}_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k} z^n}{\Gamma_k(\beta+\alpha n)n!} \tag{12}$$

where $\alpha,\beta,\gamma,z\in\mathbb{C}\ and\ k\in\mathbb{R}$ and $[\Re(\alpha),\Re(\beta),\Re(\gamma)]>0$ and $q\in(0,1)\cup\mathbb{N}$
In this paper we use the more generalized k-Mittag-Leffler function[16], defined as

$$E^{\gamma,q}_{k,\alpha,\beta,\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k} z^n}{\Gamma_k(\alpha n+\beta)(\delta)_n} \tag{13}$$

provided $\alpha,\beta,\gamma,\delta,z\in\mathbb{C},k\in\mathbb{R},[\Re(\alpha),\Re(\beta)]>0,\delta\neq 0,-1,-2,....$ and nq is a positive integer.
The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [3]

and Wright (see [6]-[8]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [1] and [12, p.21]).

$$ {}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ A_1), & ....., & (\alpha_p\ ,\ A_p); & \\ & & & z \\ (\beta_1\ ,\ B_1), & ....., & (\beta_q\ ,\ B_q); & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!} \tag{14} $$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$ (i) 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \ and \ 0 < |z| < \infty; z \neq 0 $$

$$ (ii) 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \ and \ 0 < |z| < A_1^{-A_1} \cdots A_p^{-A_p} B_1^{-B_1} \cdots B_q^{-B_q} $$

A special case of (14) is

$$ {}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); & \\ & & & z \\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); & \end{array} \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \, {}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right] \tag{15} $$

where ${}_pF_q$ is the generalized hypergeometric function defined by (see [5])

$$ \begin{aligned} {}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z) \end{aligned} \tag{16} $$

For our present investigation, the following interesting and useful results due to Lavoie and Trottier [14] will be required.

$$ \int_0^1 x^{\alpha-1}\ (1-x)^{2\beta-1} \left(1 - \frac{x}{3}\right)^{2\alpha-1} \left(1 - \frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{17} $$

where $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. Main Results

Theorem 1 The following integral formula holds true:

$$ \begin{aligned} &\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1 - \frac{x}{3}\right)^{2\rho-1} \left(1 - \frac{x}{4}\right)^{\sigma-1} E_{k,\alpha,\beta,\delta}^{\gamma,q} \left[yx \left(1 - \frac{x}{3}\right)^2 \right] dx \\ &= \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma\sigma\Gamma\delta(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma} {}_3\Psi_3 \left[\begin{array}{cccc} (\rho, 1) & (1,1) & (\frac{\gamma}{k}, q); & \\ & & & \frac{4y}{9} k^{q-\frac{\alpha}{k}} \\ (\rho+\sigma, 1) & (\delta, 1) & (\frac{\beta}{k}, \frac{\alpha}{k}); & \end{array} \right] \end{aligned} \tag{18} $$

where $\rho, \sigma, \alpha, \beta, \gamma, \delta, y \in \mathbb{C}$, $q \in \mathbb{N}$, $k \in (0, \infty)$ and $[\Re(\rho), \Re(\sigma), \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)] > 0$

Proof. In order to derive the result (18), we denote the left hand side of (18) by I, expanding $E_{k,\alpha,\beta,\delta}^{\gamma,q}(z)$ as a series with the help of (13) and then interchanging the order of integral and summation which is verified by uniform convergence of the involved series under the given conditions, we get

$$I=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}y^n}{\Gamma_k(\alpha n+\beta)(\delta)_n}\int_0^1 x^{\rho+n-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho+2n-1}\left(1-\frac{x}{4}\right)^{\sigma-1}dx.$$

Using (17) in above equation, we get

$$I=\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)(\gamma)_{qn,k}(4y/9)^n n!}{\Gamma_k(\alpha n+\beta)\Gamma(\rho+\sigma+n)(\delta)_n n!}$$

Now using (10) and (11) , we get

$$I=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\sigma)\Gamma(\delta)}{\Gamma(\gamma)}k^{1-\frac{\beta}{k}}\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)\Gamma(1+n)\Gamma(\frac{\gamma}{k},nq)}{\Gamma(\rho+\sigma+n)\Gamma(\delta+n)\Gamma(\frac{\beta}{k},\frac{\alpha n}{k})n!}\left(\frac{4yk^{q-\frac{\alpha}{k}}}{9}\right)^n$$

Using (14), yields (18). This completes the proof of Theorem 2.1 .

Theorem 2 The following integral formula holds true:

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E^{\gamma,q}_{k,\alpha,\beta,\delta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho\Gamma\delta(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma}{}_3\Psi_3\left[\begin{matrix}(\sigma,1) & (1,1) & (\frac{\gamma}{k},q); & \\ & & & yk^{q-\frac{\alpha}{k}}\\ (\rho+\sigma,1) & (\delta,1) & (\frac{\beta}{k},\frac{\alpha}{k}); & \end{matrix}\right] \qquad (19)$$

where $\rho,\sigma,\alpha,\beta,\gamma,\delta,y\in\mathbb{C}$, $q\in\mathbb{N}$, $k\in(0,\infty)$ and $[\Re(\rho),\Re(\sigma),\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)]>0$

Proof. The above Theorem can be obtained by similar steps as in proof of Theorem 2.1 .

3. Special cases

All the following cases are true under the same conditions, as their main result.

Corollaty 1 If we put $\delta=1$ in (18) and use (12) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}GE^{\gamma,q}_{k,\alpha,\beta}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma}{}_2\Psi_2\left[\begin{matrix}(\rho,1) & (\frac{\gamma}{k},q); & \\ & & \frac{4y}{9}k^{q-\frac{\alpha}{k}}\\ (\rho+\sigma,1) & (\frac{\beta}{k},\frac{\alpha}{k}); & \end{matrix}\right] \qquad (20)$$

Corollaty 2 If we put $\delta=q=1$ in (18) and use (8) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E^{\gamma}_{k,\alpha,\beta,}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\sigma(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma}{}_2\Psi_2\left[\begin{matrix}(\rho,1) & (\frac{\gamma}{k},1); & \\ & & \frac{4y}{9}k^{1-\frac{\alpha}{k}}\\ (\rho+\sigma,1) & (\frac{\beta}{k},\frac{\alpha}{k}); & \end{matrix}\right] \qquad (21)$$

Corollaty 3 If we put $\delta = k = q = 1$ in (18) and use (7) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} E^{\gamma}_{\alpha,\beta} \left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma\sigma}{\Gamma\gamma} {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (\gamma,1); \\ (\rho+\sigma,1) & (\beta,\alpha); \end{matrix} \frac{4y}{9}\right] \tag{22}$$

Corollaty 4 If we put $\gamma = \delta = k = q = 1$ in (18) and use (4) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} E_{\alpha,\beta} \left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \Gamma\sigma\ {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (1,1); \\ (\rho+\sigma,1) & (\beta,\alpha); \end{matrix} \frac{4y}{9}\right] \tag{23}$$

Corollaty 5 If we put $\gamma = \delta = \alpha = k = q = 1$ *and* $\beta = 2$ in (18) and use (5) then we get the following integral formula

$$\int_0^1 x^{\rho-2}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-3} \left(1-\frac{x}{4}\right)^{\sigma-1} [e^{yx\left(1-\frac{x}{3}\right)^2}-1]dx$$

$$= y\left(\frac{2}{3}\right)^{2\rho} \Gamma\sigma\ {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (1,1); \\ (\rho+\sigma,1) & (2,1); \end{matrix} \frac{4y}{9}\right] \tag{24}$$

Corollaty 6 If we put $\gamma = \delta = k = q = 1$ *and* $\alpha = \beta = 2$ in (18) and use (6) then we get the following integral formula

$$\int_0^1 x^{\rho-3/2}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-2} \left(1-\frac{x}{4}\right)^{\sigma-1} \sinh\left[\sqrt{xy}\left(1-\frac{x}{3}\right)\right] dx$$

$$= \sqrt{y}\left(\frac{2}{3}\right)^{2\rho} \Gamma\sigma\ {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (1,1); \\ (\rho+\sigma,1) & (2,2); \end{matrix} \frac{4y}{9}\right] \tag{25}$$

Corollaty 7 If we put $\gamma = \delta = k = \beta = q = 1$ in (18) and use (1) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} E_{\alpha} \left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \Gamma\sigma\ {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (1,1); \\ (\rho+\sigma,1) & (1,\alpha); \end{matrix} \frac{4y}{9}\right] \tag{26}$$

Corollaty 8 If we put $\gamma = \delta = k = \beta = q = 1$ *and* $\alpha = 2$ in (18) and use (2) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} \cosh\left[\sqrt{xy}\left(1-\frac{x}{3}\right)\right] dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \Gamma\sigma\ {}_2\Psi_2 \left[\begin{matrix} (\rho,1) & (1,1); \\ (\rho+\sigma,1) & (1,2); \end{matrix} \frac{4y}{9}\right] \tag{27}$$

Corollaty 9 If we put $\gamma = \delta = k = \beta = q = 1$ *and* $\alpha = 4$ in (18) and use (3) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left\{\cos\left[xy\left(1-\frac{x}{3}\right)^2\right]^{\frac{1}{4}}+\cosh\left[9xy\left(1-\frac{x}{3}\right)^2\right]^{\frac{1}{4}}\right\}dx$$

$$=2\left(\frac{2}{3}\right)^{2\rho}\Gamma\sigma\ {}_2\Psi_2\left[\begin{matrix}(\rho,1) & (1,1); \\ (\rho+\sigma,1) & (1,4);\end{matrix}\ \frac{4y}{9}\right] \tag{28}$$

Corollaty 10 If we put $\delta = 1$ in (19) and use (12) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_{k,\alpha,\beta}^{\gamma,q}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma}{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (\frac{\gamma}{k},q); \\ (\rho+\sigma,1) & (\frac{\beta}{k},\frac{\alpha}{k});\end{matrix}\ yk^{q-\frac{\alpha}{k}}\right] \tag{29}$$

Corollaty 11 If we put $\delta = q = 1$ in (19) and use (8) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_{k,\alpha,\beta}^{\gamma}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho(k)^{1-\frac{\beta}{k}}}{\Gamma\gamma}{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (\frac{\gamma}{k},1); \\ (\rho+\sigma,1) & (\frac{\beta}{k},\frac{\alpha}{k});\end{matrix}\ yk^{1-\frac{\alpha}{k}}\right] \tag{30}$$

Corollaty 12 If we put $\delta = k = q = 1$ in (19) and use (7) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_{\alpha,\beta}^{\gamma}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma\rho}{\Gamma\gamma}{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (\gamma,1); \\ (\rho+\sigma,1) & (\beta,\alpha);\end{matrix}\ y\right] \tag{31}$$

Corollaty 13 If we put $\gamma = \delta = k = q = 1$ in (19) and use (4) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_{\alpha,\beta}\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\ {}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ (\rho+\sigma,1) & (\beta,\alpha);\end{matrix}\ y\right] \tag{32}$$

Corollaty 14 If we put $\gamma = \delta = k = q = 1$ *and* $\alpha = 1, \beta = 2$ in (19) and use (5) then we get the following integral formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\sigma-3}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-2}[e^{y(1-x)^2\left(1-\frac{x}{4}\right)}-1]dx$$

$$=y\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\ {}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ (\rho+\sigma,1) & (2,1);\end{matrix}\ y\right] \tag{33}$$

Corollaty 15 If we put $\gamma = \delta = k = q = 1$ *and* $\alpha = 2, \beta = 2$ in (19) and use (6) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-2}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-3/2}\sinh\left[\sqrt{y\left(1-\frac{x}{4}\right)}(1-x)\right]dx$$

$$= \sqrt{y}\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\;{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ & & y \\ (\rho+\sigma,1) & (2,2);\end{matrix}\right] \tag{34}$$

Corollaty 16 If we put $\gamma = \delta = k = \beta = q = 1$ in (19) and use (1) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}E_\alpha\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\;{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ & & y \\ (\rho+\sigma,1) & (1,\alpha);\end{matrix}\right] \tag{35}$$

Corollaty 17 If we put $\gamma = \delta = k = \beta = q = 1$ *and* $\alpha = 2$ in (19) and use (2) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\cosh\left[\sqrt{y\left(1-\frac{x}{4}\right)}(1-x)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\;{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ & & y \\ (\rho+\sigma,1) & (1,2);\end{matrix}\right] \tag{36}$$

Corollaty 18 If we put $\gamma = \delta = k = \beta = q = 1$ *and* $\alpha = 4$ in (19) and use (3) then we get the following integral formula

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left\{\cos\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]^{\frac{1}{4}}+\cosh\left[y(1-x)^2\left(1-\frac{x}{4}\right)\right]^{\frac{1}{4}}\right\}dx$$

$$= 2\left(\frac{2}{3}\right)^{2\rho}\Gamma\rho\;{}_2\Psi_2\left[\begin{matrix}(\sigma,1) & (1,1); \\ & & y \\ (\rho+\sigma,1) & (1,4);\end{matrix}\right] \tag{37}$$

4. Concluding Remark

In the present paper, we have investigated two unified integrals involving k-Mittag-Leffler $E^{\gamma,q}_{k,\alpha,\beta,\gamma,\delta}(z)$,which are expressed in terms of generalized (Wright) hypergeometric functions. Also we have investigated many special cases of our main results. Since Mittag-Leffler functions are associated with wide range of problems in diverse fields of mathematical physics, biology, engineering and applied sciences. The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions.

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