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GENERALIZATIONS OF SOME HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR p-CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper firstly we obtain a version of the Hermite-Hadamard-Fejér inequality for p-convex functions via generalized fractional integral operator. Secondly we construct an integral identity and some Hermite-Hadamard-Fejér type integral inequalities for p-convex functions via generalized fractional integral operator. Being generalizations, we also deduce some known results.

1. Introduction and Preliminary Results

Inequalities are very useful in the diverse field of mathematics. Hermite-Hadamard type inequalities are one of the important chain of this field. On the basis of vast applications of fractional calculus a class of researchers have extended their work to the Hermite-Hadamard type inequalities involving fractional integrals. Many authors are continuously working on it and several Hermite-Hadamard like integral inequalities have been established for many kinds of functions related to convex functions. Recently a lot of integral inequalities of the Hadamard type for harmonically convex functions via fractional integrals have been published (see, [2, 4, 6, 7, 9, 10, 12, 13] and references there in).
The Hadamard inequality for convex functions is stated in the following theorem.

Theorem 1.1. *Let I be an interval of real numbers and $f : I \to \mathbb{R}$ be a convex function on I. Then for all $a, b \in I$ the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Fejér gave a weighted version of the Hadamard inequality stated as follows.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $g : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

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It is well known as the Fejér-Hadamard inequality, here we use Hermite-Hadamard-Fejér inequality. In the following we give the definition of p-convex functions.

Definition 1. [8] Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. Then a function $f : I \to \mathbb{R}$ is said to be p-convex, if

$$f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) \le tf(a) + (1-t)f(b)$$

holds for $a, b \in I$ and $t \in [0, 1]$. It is easily noted that for $p = 1$ and $p = -1$ p-convexity reduces to ordinary convexity and harmonically convexity respectively.

Definition 2. [2] Let $p \in \mathbb{R} \setminus \{0\}$. Then a function $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ if

$$f(t^{\frac{1}{p}}) = f\left([a^p + b^p - t]^{\frac{1}{p}}\right)$$

holds, for $t \in [a, b]$.

In the following we give the definition of harmonically convex functions.

Definition 3. [7] Let I be an interval of non-zero real numbers. Then a function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{ab}{ta + (1-t)b}\right) \le tf(b) + (1-t)f(a) \tag{1}$$

holds for $a, b \in I$ and $t \in [0, 1]$. If inequality in (1) is reversed, then f is said to be harmonically concave.

In the following we give the Hadamard inequality for hormonically convex functions.

Theorem 1.3. [7] *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality holds:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \le \frac{f(a) + f(b)}{2}. \tag{2}$$

A Fejér-Hadamard inequality for harmonically convex functions is stated as follows.

Theorem 1.4. [4] *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and let $u : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a non negative integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for fractional integral operator holds:*

$$f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{u(x)}{x^2} dx \le \int_a^b \frac{f(x)u(x)}{x^2} dx \le \frac{f(a) + f(b)}{2} \int_a^b \frac{u(x)}{x^2}. \tag{3}$$

The following definition of the Riemann-Liouville fractional integral is the asset of fractional calculus.

Definition 4. [18] Let $f \in L[a, b]$. Then the Riemann-Liouville fractional integrals of f of order $\nu > 0$ are defined as follows

$$I^{\nu}_{a^+} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \ x > a$$

and

$$I^{\nu}_{b_-} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)dt,\ x < b.$$

A version of the Fejér-Hadamard inequality for harmonically convex functions via Riemann-Liouville fractional integrals is stated as follows.

Theorem 1.5. [10] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and f is harmonically convex function, then the following inequality for fractional integral operator holds:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\nu+1)}{2}\left(\frac{ab}{b-a}\right)^{\nu}\left[I^{\nu}_{\frac{1}{b}+}(f\circ h)(\frac{1}{a}) + I^{\nu}_{\frac{1}{a}-}(f\circ h)(\frac{1}{b})\right] \le \frac{f(a)+f(b)}{2} \tag{4}$$

where $h(x) = \frac{1}{x}$ and $x \in [a,b]$.

In the following we give the definition of a generalized fractional integral operator which will help us to give a generalized Fejér-Hadamard inequality for harmonically convex functions and related results.

Definition 5. [16] Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing generalized Mittag-Leffler function for a real valued continuous function f are defined as follows:

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+} f\right)(x) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^{\mu}) f(t)dt,$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b_-} f\right)(x) = \int_x^b (t-x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-x)^{\mu}) f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}. \tag{5}$$

For $\delta = l = 1$ in (5), the integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator $\epsilon^{\gamma,1,k}_{\mu,\nu,1,\omega,a^+}$ containing generalized Mittag-Leffler function $E^{\gamma,1,k}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [17]. Along with $\delta = l = 1$ in addition if $k = 1$, then it reduces to an integral operator defined by Prabhaker in [15] containing Mittag-Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (5), integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to the Riemann-Liouville fractional integral operator [16].

In [16, 17] properties of the generalized fractional integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ and the generalized Mittag-Leffler function $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ are studied in brief. In [16] it is proved that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for $k < l+\mu$ and $t \in \mathbb{R}$.
Since

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \le \sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}\right|.$$

If we say that $\sum_{n=0}^{\infty} |\frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}| = S$, then

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \le S.$$

We use this property of generalized Mittag-Leffler function in sequal in our results.

Also we use in sequal the following special functions known as beta function and hypergeometric functions respectively [14]

$$\beta(\mu,\nu)=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}=\int_0^1 t^{\mu-1}(1-t)^{\nu-1}dt,\quad x,y>0$$

$${}_2F_1(a,b;c;w)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-wt)^{-a}dt$$

where $0<b<c, |z|<1$.
In [1, 5] authors have proved the Hermite-Hadamard and the Hermite-Hadamard-Fejér inequalities for generalized fractional integrals.

In this paper, we give a generalized version of the Hermite-Hadamard-Fejér inequality for p-convex functions via generalized fractional integral operator. We also obtain an integral identity and find bounds of the absolute difference of this generalized Hermite-Hadamard-Fejér inequalities for p-convex functions. Being generalization we deduce the results already have been published.

2. Section-I

In this section we obtain Hermite-Hadamard-Fejér type integral inequalities for p-convex function via generalized fractional integral operators, for this first we prove the following lemma.

Lemma 2.1. *Let a function $u:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an integrable, p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$. Then for $p\in\mathbb{R}\setminus\{0\}$*
(i) if $p>0$, then

$$\begin{aligned}\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}u\circ h\right)(b^p)&=\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}u\circ h\right)(a^p)\\&=\frac{\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}u\circ h\right)(b^p)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^a_-}u\circ h\right)(a^p)}{2}\end{aligned}$$

with $h(t)=t^{\frac{1}{p}}$, $t\in[a^p,b^p]$,
(ii) if $p<0$, then

$$\begin{aligned}\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}u\circ h\right)(a^p)&=\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}u\circ h\right)(b^p)\\&=\frac{\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}u\circ h\right)(a^p)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}u\circ h\right)(b^p)}{2}\end{aligned}$$

with $h(t)=t^{\frac{1}{p}}$, $t\in[b^p,a^p]$.

Proof. (i) Since u is a p-symmetric about $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, therefore $u(x^{\frac{1}{p}}) = u\left([a^p + b^p - x]^{\frac{1}{p}}\right)$ for all $x \in [a^p, b^p]$. By definition of generalized fractional integral operator, we have

$$\begin{aligned}\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p) &= \int_{a^p}^{b^p} (b^p - x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(b^p - x)^{\mu}) u \circ h(x) dx \qquad (6)\\ &= \int_{a^p}^{b^p} (b^p - x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(b^p - x)^{\mu}) u(x^{\frac{1}{p}}) dx\\ &= \int_{a^p}^{b^p} (b^p - x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(b^p - x)^{\mu}) u\left([a^p + b^p - x]^{\frac{1}{p}}\right) dx\end{aligned}$$

setting $t = a^p + b^p - x$ in above equation, we have

$$\begin{aligned}\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p) &= \int_{a^p}^{b^p} (t - a^p)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t - a^p)^{\mu}) u(t^{\frac{1}{p}}) dt\\ &= \int_{a^p}^{b^p} (t - a^p)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t - a^p)^{\mu}) u \circ h(t) dt.\end{aligned}$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p) = \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-} u \circ h\right)(a^p). \qquad (7)$$

By adding $\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p)$ in both sides of (7), we have

$$2\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p) = \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+} u \circ h\right)(b^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-} u \circ h\right)(a^p) \qquad (8)$$

(7) and (8) are our required results.
(ii) Proof is on the same lines of (i). □

Theorem 2.2. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus \{0\}$ and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and also let $u : [a, b] \to \mathbb{R}$ be a non-negative, integrable and both f, u are p-symmetric functions about $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$. Then the following inequalities for generalized fractional integrals hold:*
(i) if $p > 0$, then

$$\begin{aligned}f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right)&\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+} u \circ h\right)(b^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_-} u \circ h\right)(a^p)\right]\\ &\leq \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+} fu \circ h\right)(b^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_-} fu \circ h\right)(a^p)\\ &\leq \frac{f(a) + f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+} u \circ h\right)(b^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_-} u \circ h\right)(a^p)\right],\end{aligned} \qquad (9)$$

where $\omega' = \frac{\omega}{(b^p-a^p)^\mu}$ *and* $h(t) = t^{\frac{1}{p}}$ *for all* $t \in [a^p, b^p]$,
(ii) if $p < 0$, *then*

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_+} u \circ h\right)(a^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_-} u \circ h\right)(b^p)\right] \\ &\le \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_+} fu \circ h\right)(a^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_-} fu \circ h\right)(b^p) \\ &\le \frac{f(a)+f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_+} u \circ h\right)(a^p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_-} u \circ h\right)(b^p)\right], \end{aligned} \tag{10}$$

where $\omega' = \frac{\omega}{(a^p-b^p)^\mu}$ *and* $h(t) = t^{\frac{1}{p}}$ *for all* $t \in [b^p, a^p]$.

Proof. (i) Since f is p-convex function over $[a, b]$, therefore for all $x, y \in I$ and $t = \frac{1}{2}$, we have

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{f(x)+f(y)}{2}$$

setting $x = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and $y = [tb^p + (1-t)a^p]^{\frac{1}{p}}$ in above inequality, we have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{f([ta^p+(1-t)b^p]^{\frac{1}{p}}) + f([tb^p+(1-t)a^p]^{\frac{1}{p}})}{2}. \tag{11}$$

Multiplying both sides of (11) by $2t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^\mu)u([ta^p + (1-t)b^p]^{\frac{1}{p}})$ then integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} 2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^\mu)u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt \\ &\le \int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^\mu)f(ta^p+(1-t)b^p)u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt \\ &+ \int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^\mu)f(tb^p+(1-t)a^p)u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt. \end{aligned} \tag{12}$$

By choosing $x = ta^p + (1-t)b^p$ that is $a^p + b^p - x = tb^p + (1-t)a^p$ in (12), we have

$$\begin{aligned} 2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\int_{b^p}^{a^p}\left(\frac{b^p-x}{b^p-a^p}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{b^p-x}{b^p-a^p}\right)^\mu\right)u(x^{\frac{1}{p}})\left(-\frac{dx}{b^p-a^p}\right) \\ \le &\int_{b^p}^{a^p}\left(\frac{b^p-x}{b^p-a^p}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{b^p-x}{b^p-a^p}\right)^\mu\right)f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})\left(-\frac{dx}{b^p-a^p}\right) \\ + &\int_{a^p}^{b^p}\left(\frac{x-a^p}{b^p-a^p}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{x-a^p}{b^p-a^p}\right)^\mu\right)f([a^p+b^p-x]^{\frac{1}{p}})u(x^{\frac{1}{p}})\left(\frac{dx}{b^p-a^p}\right). \end{aligned} \tag{13}$$

Since f is p-symmetric about $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ therefore $f([a^p+b^p-x]^{\frac{1}{p}})=f(x^{\frac{1}{p}})$, (13) becomes

$$\begin{aligned}
&2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\frac{\omega}{(b^p-a^p)^{\mu}}(b^p-x)^{\mu}\right)u(x^{\frac{1}{p}})dx\\
&\leq\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\frac{\omega}{(b^p-a^p)^{\mu}}(b^p-x)^{\mu}\right)f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})dx \qquad (14)\\
&+\int_{a^p}^{b^p}(x-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\frac{\omega}{(b^p-a^p)^{\mu}}(x-a^p)^{\mu}\right)f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})dx.
\end{aligned}$$

Put $\omega'=\frac{\omega}{(b^p-a^p)^{\mu}}$ in (14), we have

$$\begin{aligned}
&2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(b^p-x)^{\mu}\right)u(x^{\frac{1}{p}})dx\\
&\leq\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(b^p-x)^{\mu}\right)f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})dx \qquad (15)\\
&+\int_{a^p}^{b^p}(x-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(x-a^p)^{\mu}\right)f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})dx.
\end{aligned}$$

Since $f(x^{\frac{1}{p}})u(x^{\frac{1}{p}})=fu(x^{\frac{1}{p}})=fu(h(x))=fu\circ h(x)$, inequality (15) becomes

$$\begin{aligned}
&2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(b^p-x)^{\mu}\right)u\circ h(x)dx\\
&\leq\int_{a^p}^{b^p}(b^p-x)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(b^p-x)^{\mu}\right)fu\circ h(x)dx\\
&+\int_{a^p}^{b^p}(x-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(x-a^p)^{\mu}\right)fu\circ h(x)dx.
\end{aligned}$$

This implies

$$\begin{aligned}
&2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left(\epsilon_{\mu,\nu,l,\omega',a_+^p}^{\gamma,\delta,k}u\circ h\right)(b^p)\\
&\leq\left(\epsilon_{\mu,\nu,l,\omega',a_+^p}^{\gamma,\delta,k}fu\circ h\right)(b^p)+\left(\epsilon_{\mu,\nu,l,\omega',b_-^p}^{\gamma,\delta,k}fu\circ h\right)(a^p).
\end{aligned}$$

Using Lemma 2.1 (i) in above inequality, we have

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left[\left(\epsilon_{\mu,\nu,l,\omega',a_+^p}^{\gamma,\delta,k}u\circ h\right)(b^p)+\left(\epsilon_{\mu,\nu,l,\omega',b_-^p}^{\gamma,\delta,k}u\circ h\right)(a^p)\right] \qquad (16)\\
&\leq\left(\epsilon_{\mu,\nu,l,\omega',a_+^p}^{\gamma,\delta,k}fu\circ h\right)(b^p)+\left(\epsilon_{\mu,\nu,l,\omega',b_-^p}^{\gamma,\delta,k}fu\circ h\right)(a^p).
\end{aligned}$$

To prove the second part of inequalities (9), again from p-convexity of f over $[a,b]$ and for $t\in[0,1]$, we have

$$f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)+f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)\leq f(a)+f(b). \qquad (17)$$

Multiplying both sides of (17) by $2t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})u([ta^p+(1-t)b^p]^{\frac{1}{p}})$ then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt\\&+\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt\\&\leq [f(a)+f(b)]\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})u([ta^p+(1-t)b^p]^{\frac{1}{p}})dt.\end{aligned} \tag{18}$$

Setting $x=ta^p+(1-t)b^p$ and using p-symmetry of f with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ in (18) and after simplification, we have

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+}fu\circ h\right)(b^p)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^a_-}fu\circ h\right)(a^p)\\&\leq [f(a)+f(b)]\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+}u\circ h\right)(b^p).\end{aligned} \tag{19}$$

Using Lemma 2.1 (i), inequality (19) becomes

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+}fu\circ h\right)(b^p)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_-}fu\circ h\right)(a^p)\\&\leq \frac{[f(a)+f(b)]}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^p_+}u\circ h\right)(b^p)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^p_-}g\circ h\right)(a^p)\right].\end{aligned} \tag{20}$$

By combining (16) and (20), we get (9).
(ii) Proof is similar as (i) by using Lemma 2.1 (ii). □

Corollary 2.3. *If we put $p=-1$ in Theorem 2.2, we get the following Hermite-Hadamard-Fejér inequalities for harmonically convex function via generalized fractional integral operator,*

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}u\circ h\right)(\frac{1}{a})+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}u\circ h\right)(\frac{1}{b})\right]\\&\leq\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}fg\circ h\right)(\frac{1}{a})+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}fu\circ h\right)(\frac{1}{b})\\&\leq\frac{f(a)+f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}u\circ h\right)(\frac{1}{a})+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}u\circ h\right)(\frac{1}{b})\right].\end{aligned}$$

Remark 2.4. *From Theorem 2.2, we deduce the following results.*
(i) If we take $\omega=0$, then we get [2, Theorem 9].
(ii) If we take $\omega=0$, $p=-1$, $\nu=1$ and $u(x)=1$, then we get Theorem 1.3.
(iii) If we take $\omega=0$, $p=-1$ and $\nu=1$, then we get Theorem 1.4.
(iv) If we take $\omega=0$, $p=-1$ and $u(x)=1$, then we get Theorem 1.5.

3. **Section-II**

In this section first we prove an integral identity in the following lemma, then we establish the Hermite-Hadamard-Fejér type integral inequalities for p-convex functions via generalized fractional integral operator. Also we deduce some known results.

Lemma 3.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I and $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. Also let $u : [a,b] \to \mathbb{R}$ be an integrable and p-symmetric about $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following equality holds for generalized fractional integral with $p \in \mathbb{R}\setminus\{0\}$*
(i) if $p>0$

$$\begin{aligned}
&\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\\
&-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\\
&=\left[\int_{a^p}^{b^p}\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-s)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt\right.\\
&\left.-\int_{a^p}^{b^p}\left(\int_{t}^{b^p}(s-a^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(s-a^p)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt\right]
\end{aligned}$$

where $h(t)=t^{\frac{1}{p}}$ for $t\in[a^p,b^p]$,
(ii) if $p<0$

$$\begin{aligned}
&\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(u\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(u\circ h)(b^p)\right)\\
&-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(fu\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(fu\circ h)(b^p)\right)\\
&=\left[\int_{b^p}^{a^p}\left(\int_{b^p}^{t}(a^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(a^p-s)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt\right.\\
&\left.-\int_{b^p}^{a^p}\left(\int_{t}^{a^p}(s-b^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(s-b^p)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt\right]
\end{aligned}$$

where $h(t)=t^{\frac{1}{p}}$ for $t\in[b^p,a^p]$.

Proof. (i) To prove the result, one can have

$$\begin{aligned}
&\int_{a^p}^{b^p}\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-s)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt\\
&=\left|\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-s)^\mu\right)(u\circ h)(s)ds\right)(f\circ h)(t)\right|_{a^p}^{b^p}\\
&-\int_{a^p}^{b^p}(b^p-t)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-t)^\mu\right)(u\circ h)(t)(f\circ h)(t)dt\\
&=\left(\int_{a^p}^{b^p}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-s)^\mu\right)(u\circ h)(s)ds\right)f\circ h(b^p)\\
&-\int_{a^p}^{b^p}(b^p-t)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(b^p-t)^\mu\right)(fu\circ h)(t)dt.
\end{aligned}$$

Since $f \circ h(b^p) = f(h(b^p)) = f(b)$, above equation becomes

$$\int_{a^p}^{b^p} \left(\int_{a^p}^{t} (b^p - s)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (b^p - s)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)'(t) dt$$
$$= f(b) \left(\int_{a^p}^{b^p} (b^p - s)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (b^p - s)^{\mu}) (u \circ h)(s) ds \right)$$
$$- \int_{a^p}^{b^p} (b^p - t)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (b^p - t)^{\mu}) (fu \circ h)(t) dt.$$

This implies

$$\int_{a^p}^{b^p} \left(\int_{a^p}^{t} (b^p - s)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (b^p - s)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)'(t) dt \tag{21}$$
$$= f(b) \epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k} (u \circ h)(b^p) - \epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k} (fu \circ h)(b^p).$$

By using Lemma 2.1 (i) in (21), we have

$$\int_{a^p}^{b^p} \left(\int_{a^p}^{t} (b^p - s)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (b^p - s)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)'(t) dt \tag{22}$$
$$= \frac{f(b)}{2} [\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k} (u \circ h)(b^p) + \epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k} (u \circ h)(a^p)] - \epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k} (fu \circ h)(b^p).$$

Similarly, one can have

$$\int_{a^p}^{b^p} \left(\int_{t}^{b^p} (s - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (s - a^p)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)'(t) dt$$
$$= \left| \left(\int_{t}^{b^p} (s - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (s - a^p)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)(t) \right|_{a^p}^{b^p}$$
$$+ \int_{a^p}^{b^p} (t - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (t - b^p)^{\mu}) (u \circ h)(t) (f \circ h)(t) dt$$
$$= - \left(\int_{a^p}^{b^p} (s - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (s - b^p)^{\mu}) (u \circ h)(s) ds \right) f \circ h(a^p)$$
$$+ \int_{a^p}^{b^p} (t - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (t - b^p)^{\mu}) (fu \circ h)(t) dt.$$

Since $f \circ h(a^p) = f(h(a^p)) = f(a)$, above equation becomes

$$\int_{a^p}^{b^p} \left(\int_{t}^{b^p} (s - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (s - a^p)^{\mu}) (u \circ h)(s) ds \right) (f \circ h)'(t) dt$$
$$= -f(a) \left(\int_{a^p}^{b^p} (s - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (s - a^p)^{\mu}) (u \circ h)(s) ds \right)$$
$$+ \int_{a^p}^{b^p} (t - a^p)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k} (\omega (t - a^p)^{\mu}) (fu \circ h)(t) dt.$$

This implies

$$\int_{a^p}^{b^p}\left(\int_t^{b^p}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt \tag{23}$$
$$=-f(a)\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(u\circ h)(a^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(fu\circ h)(a^p).$$

By using Lemma 2.1 (i) in (23), we have

$$\int_{a^p}^{b^p}\left(\int_t^{b^p}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)(f\circ h)'(t)dt \tag{24}$$
$$=-\frac{f(a)}{2}[\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(u\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(u\circ h)(a^p)]+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(fu\circ h)(a^p).$$

Subtract (24) from (22), we get the result.
(ii) Proof is on the same lines of (i) by using Lemma 2.1 (ii). □

Remark 3.2. *In Lemma 3.1, we can deduce the following results.*
(i) If we take $\omega=0$, then we get [2, Lemma 2].
(ii) If we take $p=1$ with $\omega=0$, then we get [9, Lemma 2.4].
(iii) If we take $p=1$, $u(x)=1$ with $\omega=0$, then we get [13, Lemma 2].
(iv) If we take $p=1$, $\nu=1$ with $\omega=0$, then we get [12, Lemma 2.6].
(v) If we take $p=1$, $\nu=1$, $u(x)=1$ with $\omega=0$, then we get [3, Lemma 2.1].
(vi) If we take $p=-1$ with $\omega=0$,then we get [10, Lemma 3].
(vii) If we take $p=-1$, $u(x)=1$ with $\omega=0$, then we get [9, Lemma 3].
(viii) If we take $u(x)=1$, $p=1$, $\nu=1$ with $\omega=0$, then we get [7, Lemma 2.5].

Theorem 3.3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on the interior of I and $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|$ is p-convex function on $[a,b]$, $u:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following inequality for generalized fractional integrals holds with $p\in\mathbb{R}\backslash\{0\}$*
(i) if $p>0$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(u\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(u\circ h)(a^p)\right)\right.$$
$$\left.-\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(fu\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(fu\circ h)(a^p)\right)\right|$$
$$\leq\frac{\parallel u\parallel_\infty S\left(b^p-a^p\right)^{\nu+1}}{\nu}\left(K_1(\nu,p)|f'(a)|+K_2(\nu,p)|f'(b)|\right),$$

where

$$K_1(\nu,p)=\int_0^1\frac{|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}wdw,$$

and

$$K_2(\nu,p)=\int_0^1\frac{|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}(1-w)dw,$$

with $h(t)=t^{\frac{1}{p}}$ *for all* $t\in[a^p,b^p]$,
(ii) if $p<0$

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(u\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(u\circ h)(b^p)\right)\right.\\&-\left.\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(fu\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(fu\circ h)(b^p)\right)\right|\\&\le\frac{\parallel u\parallel_\infty S\,(a^p-b^p)^{\nu+1}}{\nu}\left(K_3(\nu,p)|f'(a)|+K_4(\nu,p)|f'(b)|\right),\end{aligned}$$

where

$$K_3(\nu,p)=\int_0^1\frac{-|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}wdw,$$

and

$$K_4(\nu,p)=\int_0^1\frac{-|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}(1-w)dw,$$

with $h(t)=t^{\frac{1}{p}}$ *for all* $t\in[b^p,a^p]$.

Proof. (i) Let $p>0$ by using Lemma 3.1 (i), we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.\\&-\left.\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|\\&\le\int_{a^p}^{b^p}\left|\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(b^p-s)^\mu)(u\circ h)(s)ds\right)\right.\\&-\left.\left(\int_t^{b^p}(s-a^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-a^p)^\mu)(u\circ h)(s)ds\right)\right||(f\circ h)'(t)|dt.\end{aligned}\tag{25}$$

Since u is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ therefore $u(x^{\frac{1}{p}})=u\left([a^p+b^p-x]^{\frac{1}{p}}\right)$ for all $x\in[a^p,b^p]$, setting $s=a^p+b^p-x$ in first part of right hand side of above inequality, we have

$$\begin{aligned}&\left|\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(b^p-s)^\mu)(u\circ h)(s)ds\right)\right.\\&-\left.\left(\int_t^{b^p}(s-a^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-a^p)^\mu)(u\circ h)(s)ds\right)\right|\\&=\left|\left(\int_{a^p+b^p-t}^{b^p}(x-a^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-a^p)^\mu)(u\circ h)(a^p+b^p-x)dx\right)\right.\\&-\left.\left(\int_t^{b^p}(s-a^p)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-a^p)^\mu)(u\circ h)(s)ds\right)\right|.\end{aligned}\tag{26}$$

As $(u\circ h)(a^p+b^p-x)=u([a^p+b^p-x]^{\frac{1}{p}})=u(x^{\frac{1}{p}})=u(h(x))=u\circ h(x)$ and replacing x by s in (26), we have

$$\begin{aligned}
&\left|\left(\int_{a^p}^{t}(b^p-s)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(b^p-s\right)^{\mu}\right)(u\circ h)(s)ds\right)\right. \\
&\left.-\left(\int_{t}^{b^p}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)\right| \\
&=\left|\left(\int_{a^p+b^p-t}^{b^p}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)\right. \\
&\left.+\left(\int_{b^p}^{t}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)\right| \\
&=\left|\left(\int_{a^p+b^p-t}^{t}(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)ds\right)\right|
\end{aligned} \tag{27}$$

$$\leq\begin{cases}\int_{t}^{a^p+b^p-t}|(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega(s-a^p)^{\mu})u\circ h(s)|ds, & t\in[a^p,\frac{a^p+b^p}{2}]\\[2ex] \int_{a^p+b^p-t}^{t}|(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega(s-a^b)^{\mu})u\circ h(s)|ds, & t\in[\frac{a^p+b^p}{2},b^p].\end{cases} \tag{28}$$

Using (28) and (27) in (25) we have

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(u\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(u\circ h)(a^p)\right)\right. \\
&\left.-\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(fu\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(fu\circ h)(a^p)\right)\right| \\
&\leq\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}\left|(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt \\
&+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left|(s-a^p)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-a^p\right)^{\mu}\right)(u\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt.
\end{aligned} \tag{29}$$

Using $\| u \|_{\infty}=\sup_{t\in[a,b]}|u(t)|$ and absolute convergence of Mittag-Leffer function, above inequality becomes

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(u\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(u\circ h)(a^p)\right)\right. \\
&\left.-\left(\epsilon_{\mu,\nu,l,\omega,a_+^p}^{\gamma,\delta,k}(fu\circ h)(b^p)+\epsilon_{\mu,\nu,l,\omega,b_-^p}^{\gamma,\delta,k}(fu\circ h)(a^p)\right)\right| \\
&\leq\| u \|_{\infty} S\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}(s-a^p)^{\nu-1}ds\right)|f'(t^{\frac{1}{p}})\frac{1}{p}t^{\frac{1}{p}-1}|dt\right. \\
&\left.+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}(s-a^p)^{\nu-1}ds\right)|f'(t^{\frac{1}{p}})\frac{1}{p}t^{\frac{1}{p}-1}|dt\right].
\end{aligned} \tag{30}$$

After simplification, inequality (30) becomes

$$\Big|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right) \tag{31}$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\Big|$$
$$\leq\parallel u\parallel_\infty S\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\frac{(b^p-t)^\nu-(t-a^p)^\nu}{\nu p t^{1-\frac{1}{p}}}|f'(t^{\frac{1}{p}})|dt\right.$$
$$\left.+\int_{\frac{a^p+b^p}{2}}^{b^p}\frac{(t-a^p)^\nu-(b^p-t)^\nu}{\nu p t^{1-\frac{1}{p}}}|f'(t^{\frac{1}{p}})|dt\right].$$

Setting $t=wa^p+(1-w)b^p$ in (31), we have

$$\Big|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right) \tag{32}$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\Big|$$
$$\leq\parallel u\parallel_\infty S\left[(b^p-a^p)^{\nu+1}\int_0^{\frac{1}{2}}\frac{(1-w)^\nu-(w)^\nu}{\nu p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}|f'([wa^p+(1-w)b^p]^{\frac{1}{p}})|dw\right.$$
$$\left.+(b^p-a^p)^{\nu+1}\int_{\frac{1}{2}}^1\frac{(w)^\nu-(1-w)^\nu}{\nu p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}|f'([wa^p+(1-w)b^p]^{\frac{1}{p}})|dw\right].$$

Since $|f'|$ is p-convex on $[a,b]$, it can be written as

$$\left|f'\left([wa^p+(1-w)bp]^{\frac{1}{p}}\right)\right|\leq w|f'(a)|+(1-w)|f'(b)|. \tag{33}$$

Using (33) in (32), we have

$$\Big|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right) \tag{34}$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\Big|$$
$$\leq\frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\int_0^{\frac{1}{2}}\frac{(1-w)^\nu-(w)^\nu}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(w|f'(a)|+(1-w)|f'(b)|)dw\right.$$
$$\left.+\int_{\frac{1}{2}}^1\frac{(w)^\nu-(1-w)^\nu}{\nu p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(w|f'(a)|+(1-w)|f'(b)|)dw\right].$$

On simplification of (34), we get

$$\Big|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right) \tag{35}$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\Big|$$
$$\leq\frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\int_0^1\frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(w|f'(a)|+(1-w)|f'(b)|)dw\right]$$

that is

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|\\&\leq\frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\int_0^1\frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}w|f'(a)|dw\right.\\&\left.+\int_0^1\frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(1-w)|f'(b)|)dw\right].\end{aligned}$$

This completes the proof.

(ii) Proof is similar to (i) by using Lemma 3.1 (ii). □

Corollary 3.4. *If we put $p=-1$ in Theorem 3.3, we get the following Hermite-Hadamard-Fejér type integral inequality for harmonically convex function via generalized fractional integral*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(u\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(u\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fu\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fu\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel u\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(K_1(\nu)|f'(a)|+K_2(\nu)|f'(b)|\right)\end{aligned}$$

where
$K_1(\nu)=\frac{b^{-2}}{(\nu+2)}{}_2F_1(2,1;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;1-\frac{a}{b})+\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;\frac{b-a}{b+a})$
and
$K_2(\nu)=\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,2;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{\nu+2}{}_2F_1(2,\nu+2;\nu+3;1-\frac{a}{b})+\frac{4(a+b)^{-2}}{(\nu+1)}{}_2F_1(2,\nu+1;\nu+2;\frac{b-a}{b+a})-\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(\nu,\nu+1;\nu+3;\frac{b-a}{b+a})$
with $0<\nu\leq 1$.

Remark 3.5. *From Theorem 3.3, we can obtain the following results.*
(i) If we take $\omega=0$, then we get [2, Theorem 10].
(ii) If we take $p=1$ with $\omega=0$, then we get [9, Theorem 2.6].
(iii) If we take $p=1$, $u(x)=1$ with $\omega=0$, then we get [13, Theorem 3].
(vi) If we take $p=1$, $\nu=1$, $u(x)=1$ with $\omega=0$, then we get [3, Theorem 2.2].
(v) If we take $p=1$, $\nu=1$ with $\omega=0$, then we get [2, Corollary 1(1)].
(vi) If we take $p=-1$ and $\omega=0$, then we get [2, Corollary 1(2)].
(vii) If we take $p=-1$, $\nu=1$, $u(x)=1$ with $\omega=0$, then we get [2, Corollary 1(3)].
(viii) If we take $p=-1$, $\nu=1$ with $\omega=0$, then we get [2, Corollary 1(4)].
(ix) If we take $p=-1$, $u(x)=1$ with $\omega=0$, then we get [2, Corollary 1(5)].

Theorem 3.6. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on the interior of I and $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$, $q>0$ is p-convex function on $[a,b]$, $u:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following inequality for generalized fractional integrals holds*

with $p \in \mathbb{R} \setminus \{0\}$
(i) if $p > 0$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq \frac{\| u \|_\infty S\,(b^p-a^p)^{\nu+1}}{\nu} K_5^{1-\frac{1}{q}}(\nu,p)\left(K_1(\nu,p)|f'(a)|^q+K_2(\nu,p)|f'(b)|^q\right)^{\frac{1}{q}},$$

where, $K_1(\nu,p), K_2(\nu,p)$ *are same as in Theorem 3.3 and*

$$K_5(\nu,p)=\int_0^1 \frac{|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}dw,$$

with $h(t)=t^{\frac{1}{p}}$ *for all* $t\in[a^p,b^p]$*,*
(ii) if $p < 0$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(u\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(u\circ h)(b^p)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_+}(fu\circ h)(a^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_-}(fu\circ h)(b^p)\right)\right|$$
$$\leq \frac{\| u \|_\infty S\,(a^p-b^p)^{\nu+1}}{\nu} K_6^{1-\frac{1}{q}}\left(K_3(\nu,p)|f'(a)|^q+K_4(\nu,p)|f'(b)|^q\right)^{\frac{1}{q}},$$

where $K_3(\nu,p), K_4(\nu,p)$ *are same as in Theorem 3.3 and*

$$K_6(\nu,p)=\int_0^1 \frac{-|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}dw,$$

with $h(t)=t^{\frac{1}{p}}$ *for all* $t\in[b^p,a^p]$*.*

Proof. (i) Let $p>0$ by inequality (32) of Theorem 3.3, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right. \tag{36}$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq \frac{\| u \|_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}|f'([wa^p+(1-w)b^p]^{\frac{1}{p}})|dw\right].$$

Using power means, inequality (36) becomes

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right. \tag{37}$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq \frac{\| u \|_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}dw\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}\left|f'([wa^p+(1-w)b^p]^{\frac{1}{p}})\right|^q dw\right)^{\frac{1}{q}}\right].$$

By using the p-convexity of $|f'|^q$ in (37), we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq \frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}dw\right)^{1-\frac{1}{q}}$$
$$\left(\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}wdw\right)|f'(a)|^q\right.$$
$$\left.+\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(1-w)dw\right)|f'(b)|^q\right)^{\frac{1}{q}}.$$

This completes the proof.
(ii) Proof is on the same lines of (i) by using Lemma 3.1 (ii). □

Corollary 3.7. *If we put $p=-1$ in Theorem 3.6, we get the following Hermite-Hadamard-Fejér type integral inequality for harmonically convex function via generalized fractional integral operator,*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(u\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(u\circ h)\left(\frac{1}{b}\right)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fu\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fu\circ h)\left(\frac{1}{b}\right)\right)\right|$$
$$\leq \frac{\parallel u\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[(K_3^{1-\frac{1}{q}}(\nu)\,(K_4(\nu)|f'(a)|^q+K_5(\nu)|f'(b)|^q)^{\frac{1}{q}}\right.$$
$$\left.+(K_6^{1-\frac{1}{q}}(\nu)\,(K_7(\nu)|f'(a)|^q+K_8(\nu)|f'(b)|^q)^{\frac{1}{q}}\right]$$

where
$K_3(\nu)=\frac{2(a+b)^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+3;\frac{b-a}{b+a})$
$K_4(\nu)=\frac{(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2;\nu+1;\nu+3;\frac{b-a}{b+a})$
$K_5(\nu)=K_3(\nu)-K_4(\nu)$
$K_6(\nu)=\frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+1;(1-\frac{a}{b}))-\frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b}))+K_3(\nu)$
$K_7(\nu)=\frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+2;(1-\frac{a}{b}))-\frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b}))+K_4(\nu)$
$K_8(\nu)=K_6(\nu)-K_7(\nu)$
with $0<\nu\leq 1$.

Remark 3.8. *From Theorem 3.6, we can obtain the following results.*
(i) If we take $\omega=0$, then we get [2, Theorem 11].
(ii) If we take $p=1$ with $\omega=0$, then we get [9, Theorem 2.8].
(iii) If we take $p=1$, $\nu=1$, $u(x)=1$ with $\omega=0$, then we get [11, Theorem 1].
(vi) If we take $p=-1$, $u(x)=1$ with $\omega=0$, then we get [9, Theorem 5].
(v) If we take $p=-1$, $\nu=1$, $u(x)=1$ with $\omega=0$, then we get [7, Theorem 2.6].
(vi) If we take $p=1$, $u(x)=1$ and $\omega=0$, then we get [2, Corollary 2(1)].
(vii) If we take $p=1$, $\nu=1$ with $\omega=0$, then we get [2, Corollary 2(2)].
(viii) If we take $p=-1$ with $\omega=0$, then we get [2, Corollary 2(3)].

(ix) If we take $p=-1$, $\nu=1$ with $\omega=0$, then we get [2, Corollary 2(4)].

Theorem 3.9. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on the interior of I and $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^r$, $r>0$ is p-convex function on $[a,b]$, $u:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following inequality for generalized fractional integrals holds with $\frac{1}{r}+\frac{1}{s}=1$, $p\in\mathbb{R}\setminus\{0\}$*
(i) if $p>0$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq\frac{\|u\|_\infty S(b^p-a^p)^{\nu+1}}{\nu}K_7^{\frac{1}{s}}(\nu,p,s)\left(\frac{|f'(a)|^r+|f'(b)|^r}{2}\right)^{\frac{1}{r}},$$

where

$$K_7(\nu,p,s)=\int_0^1\left(\frac{|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}\right)^s dw,$$

with $h(t)=t^{\frac{1}{p}}$ for all $t\in[a^p,b^p]$,
(ii) if $p<0$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq\frac{\|u\|_\infty S(a^p-b^p)^{\nu+1}}{\nu}K_8^{\frac{1}{s}}(\nu,p,s)\left(\frac{|f'(a)|^r+|f'(b)|^r}{2}\right)^{\frac{1}{r}},$$

where

$$K_8(\nu,p,s)=\int_0^1\left(\frac{-|(1-w)^\nu-w^\nu|}{p[wa^p+(1-w)b^p]^{1-\frac{1}{p}}}\right)^s dw,$$

with $h(t)=t^{\frac{1}{p}}$ for all $t\in[b^p,a^p]$.

Proof. (i) By inequality (35) of Theorem 3.6, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right. \tag{38}$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq\frac{\|u\|_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\int_0^1\frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}(w|f'(a)|+(1-w)|f'(b)|)dw\right].$$

By using Hölder inequality and harmonically convexity of $|f'|^r$, (38) follows

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(u\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(u\circ h)(a^p)\right)\right. \tag{39}$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^p_+}(fu\circ h)(b^p)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^p_-}(fu\circ h)(a^p)\right)\right|$$
$$\leq \frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}\right)^s dw\right]^{\frac{1}{s}}$$
$$\left(\int_0^1 |f'(wa^p+(1-w)b^p)^{\frac{1}{p}}|^r dw\right)^{\frac{1}{r}}$$
$$=\frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}\right)^s dw\right]^{\frac{1}{s}}$$
$$\left(\int_0^1 (w|f'(a)|^r+(1-w)|f'(b)|^r)dw\right)^{\frac{1}{r}}$$
$$=\frac{\parallel u\parallel_\infty S(b^p-a^p)^{\nu+1}}{\nu}\left[\left(\int_0^1 \frac{|(1-w)^\nu-(w)^\nu|}{p(wa^p+(1-w)b^p)^{1-\frac{1}{p}}}\right)^s dw\right]^{\frac{1}{s}}$$
$$\left(\frac{|f'(a)|^r+|f'(b)|^r}{2}\right)^{\frac{1}{r}}.$$

This completes the proof.
(ii) The proof is on the same lines of (i). □

Corollary 3.10. *If we put $p=-1$ in Theorem 3.9, we get the following Hermite-Hadamard-Fejér type integral inequality for harmonically convex function via generalized fractional integral operator*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(u\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(u\circ h)\left(\frac{1}{b}\right)\right)\right.$$
$$\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fu\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fu\circ h)\left(\frac{1}{b}\right)\right)\right|$$
$$\leq \frac{\parallel u\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(K_9^{\frac{1}{r}}(\nu,r)\left(\frac{|f'(a)|^s+3|f'(b)|^s}{8}\right)^{\frac{1}{s}}\right.$$
$$\left.+K_{10}^{\frac{1}{r}}(\nu,r)\left(\frac{3|f'(a)|^s+|f'(b)|^s}{8}\right)^{\frac{1}{s}}\right)$$

where
$K_9(\nu,r)=\frac{(a+b)^{-2r}}{2^{-2r+1}(\nu r+1)}{}_2F_1(2r,\nu r+1;\nu r+2;\frac{b-a}{b+a})$
and
$K_{10}(\nu,r)=\frac{b^{-2r}}{2(\nu r+1)}{}_2F_1(2r,1;\nu r+2;\frac{1}{2}(1-\frac{a}{b}))$
with $0\leq\nu<1$.

Remark 3.11. *From Theorem 3.9, we can obtain the following results.*
(i) If we take $\omega=0$, then we get [2, Theorem 12].
(ii) If we take $p=1$ with $\omega=0$, then we get [9, Theorem 2.9(i)].

(iii) If we take $p = 1$, $\nu = 1$, $u(x) = 1$ *with* $\omega = 0$, *then we get* [3, Theorem 2.3].
(vi) If we take $p = 1$, $\nu = 1$ *with* $\omega = 0$, *then we get* [12, Theorem 2.8].
(v) If we take $p = 1$, $u(x) = 1$ *with* $\omega = 0$, *then we get* [7, Corollary 3(1)].
(vi) If we take $p = -1$, $u(x) = 1$ *and* $\omega = 0$, *then we get* [2, Corollary 3(2)].
(vii) If we take $p = -1$, $\nu = 1$, $u(x) = 1$ *and* $\omega = 0$, *then we get* [2, Corollary 3(3)].
(viii) If we take $p = -1$ *with* $\omega = 0$, *then we get* [2, Corollary 3(4)].
(ix) If we take $p = -1$, $\nu = 1$ *with* $\omega = 0$, *then we get* [2, Corollary 3(5)].
(x) If we take $u(x) = 1$, $\nu = 1$ *with* $\omega = 0$, *then we get* [2, Corollary 3(6)].

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