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ON SOLUTIONS TO FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this article, we prove the sufficient conditions for the existence, uniqueness and continuous dependence of mild solutions to fractional neutral differential equations in a Banach space with infinite delay. The results are obtained by using the theory of semigroup of almost sectorial operators and the fixed point theorems. Example is discussed to illustrate the results.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a complex Banach space. We study the existence, uniqueness and continuous dependence of mild solutions to the following problem in X:

$$\left.\begin{aligned} {}_cD_t^{\alpha}[u(t)+g(t,u_t)]+Au(t) &= f(t,u_t), \quad t\in J=[0,a], 0\leq\alpha\leq 1 \\ u(t) &= \phi(t), \quad (-\infty,0], \end{aligned}\right\} \tag{1.1}$$

where the functions $f\colon J\times\mathcal{B}\to X$, $f\colon J\times\mathcal{B}\to X$ are non-linear and satisfy some appropriate conditions. Here ${}_cD_t^{\eta}$ denotes the Caputo fractional derivative of order η with respect to t and $A : D(A)\subset X\to X$ is a linear operator. The resolvent of A satisfies a growth of order $-\gamma$, $-1<\gamma<0$ in a sector of the complex plane. Let $u_t(\cdot)$ denotes an element of the abstract phase space $\mathcal{B}$ defined as $u_t(\theta)=u(t+\theta), \theta\in(-\infty,0]$.

A systems in which the information is never transferred from the input to the output is called the system with infinite delay. This system can be modeled as a state space model where output is disconnected with the states. The initial value problem for fractional differential equations with infinite delay also describes models in some scientific areas, such as population dynamics, biology and epidemiology [20, 19]. The plentiful application of fractional differential equations with delay motivates the rapid development and gained much attention in the recent years. We refer to [2, 24, 25, 33, 29, 30, 32] for more details.

We consider the following fractional Cauchy problem in X :

$$\left.\begin{aligned} {}_cD_t^{\eta}u(t)+Au(t) &= f(t)\ , t\in[0,\infty), 0\leq\eta\leq 1, \\ u(0) &= u_0, \end{aligned}\right\} \tag{1.2}$$

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where $u_0 \in X$, $f : [0, \infty) \to X$ and $A : D(A) \subset X \to X$ is a linear operator. Here ${}_cD_t^\eta$ denotes the Caputo fractional derivative of order η with respect to t. Problem (1.2) has studied by many authors [10, 2, 37, 17, 26, 27] using the theory of analytic semigroup as well as the theory of sectorial operators.

Recently, the concept of almost sectorial operator has introduced by Wahl [38]. Subsequently, lots of results have established for the abstract theory of Cauchy problems for linear and non-linear differential equations with almost sectorial operators for integer and fractional order derivatives [4, 5, 6, 12, 16, 31, 37]. Hernández [23] has established the existence of mild solution to following problem in $(X, \|\cdot\|)$ with finite delay:

$$\left.\begin{array}{rcl} u'(t) &=& Au(t) + f(t, u_t), t \in [0, a], \\ u(0) &=& \phi \in \Omega, \end{array}\right\} \tag{1.3}$$

where $A : D(A) \subset X \to X$ is an almost sectorial operator, $\Omega \subset \mathcal{B}$, $\mathcal{B}$ is the phase space and $f : [0, a] \times \Omega \to X$ is an appropriate function.

Further, Wang *et al.*[37] have established the existence theorems of solutions to the following semi-linear Cauchy problem in X :

$$\left.\begin{array}{rcl} {}_cD_t^\eta u(t) &=& Au(t) + f(t, u(t)), t \in [0, \infty), 0 \leq \eta \leq 1, \\ u(0) &=& u_0, \end{array}\right\} \tag{1.4}$$

where ${}_cD_t^\eta$ denotes the Caputo fractional derivative of order η, $A : D(A) \subset X \to X$ is a almost sectorial operator and $f : [0, \infty) \times X \to X$ satisfies some appropriate conditions. The results can be proved under a weaker assumption on A, for more details, we refer the readers to Favini and Yagi [17, Section 3]. It is to be mentioned that Kostić [27] had studied the results of Wang *et. al.*[37] for abstract degenerate differential equations.

As delay differential equations occur in many processes including biology and engineering, the qualitative study of solutions for integer-order as well as fractional order are carried out by many authors [8, 36, 3, 18, 1, 2, 7, 9, 13, 14, 15, 35, 22, 28]. Ye *et al.*[36] have studied the following problem in a Banach space X,

$$\begin{array}{rcl} {}_cD_t^\eta[u(t) - g(t, u_t)] &=& Au(t) + f(t, u_t, \int_0^t k(t, s, u_s)ds), \quad t \in J = [0, a], \\ u(t) &=& \phi(t), \quad (-\infty, 0], \end{array}$$

where the functions $f\colon J \times \mathcal{B} \to X$, $f\colon J \times \mathcal{B} \times \mathcal{B} \to X$ are non-linear and satisfy some appropriate conditions. Here ${}_cD_t^\eta$ denotes the Caputo fractional derivative of order $\eta \in [0, 1]$ with respect to t and $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0}$ on X and $u_t(\cdot)$ denotes an element of the abstract phase space $\mathcal{B}$ defined as $u_t(\theta) = u(t+\theta), \theta \in (-\infty, 0]$. The existence results are established by the resolvent operator technique and Krasnoselskii's fixed point theorem. With this motivation from Ye *et al.*[36] and Wang *et al*[37] , we study the existence, uniqueness and continuous dependence of mild solutions to Problem (1.1) when A is an almost sectorial operator. The main results generalizes some results in [36, 13].

The article is organized as follows. The definition of the Caputo fractional derivative, Riemann-Liouville integral, the theory of semigroup of bounded a linear operators and some lemmas are recalled in Section 2. The existence and uniqueness of mild solutions to Problem are proved in Section 3. The continuous dependence on the initial data of the solution has established in Section 4. Finally, we discuss the results by an example.

2. Preliminaries and assumptions

In this section, we collect the basic definitions, notations, Lemmas that will be used in the remaining part of the article. We use $D(A)$ for the domain of a operator, $\sigma(A)$ for its spectrum, $\rho(A) : \mathbb{C} \setminus \sigma(A)$ for its resolvent, $R(\lambda; A), \lambda \in \rho(A)$ for the resolvent operator, $\mathcal{L}(Y, Z)$ for the space of all bounded linear operators between two normed spaces Y and Z, $\mathcal{L}(Y)$ for $Y = Z$.

The choice of the phase space $\mathcal{B}$ plays an importnat role in problems with infinite delay. We deal with all spaces satisfying a given set of axioms rather than working in a fixed phase space which is known as classical axiomatic approach. The approach for functional differential equations with infinite delay has nicely described in [21]. The space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-normed space consisting of functions $v : (-\infty, 0] \to X$ satisfying the following assumptions:

(A1) If $v : (-\infty, a] \to X$ is continuous on $J = [0, a]$ and $v_0 \in \mathcal{B}$, then for every $t \in J$, the following conditions hold:
  (a) $v_t \in \mathcal{B}$,
  (b) $\|v(t)\| \leq L\|v_t\|_{\mathcal{B}}$,
  (c) $\|v_t\|_{\mathcal{B}} \leq p_1(t) \sup_{0\leq s\leq t} \|v(s)\| + p_2(t)\|v_0\|_{\mathcal{B}}$,
  where $L \geq 0$ is a constant, $p_1 : [0, \infty) \to [0, \infty)$ is continuous, $p_2 : [0, \infty) \to [0, \infty)$ is locally bounded which are independent of v. Let $k_1 = \sup_{t\in J} p_1(t)$ and $k_2 = \sup_{t\in J} p_2(t)$.

(A2) For the function v defined in (A1), the function v_t is a $\mathcal{B}-$valued continuous function on J.

(A3) The phase space $\mathcal{B}$ is complete.

Next we recall the following definition of almost sectorial operator was introduced by Wahl [38].

Definition 2.1. *For $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, we say that a closed linear operator $A : D(A) \subset X \to X$ is an almost sectorial operator on X if*

(1) $\sigma(A) \subset \Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq w\} \cup \{0\}$
(2) *for every $\omega < \mu < \pi$, there exists a positive constant C_μ such that*

$$\|R(z, A)\| \leq C_\mu |z|^\gamma \quad \textit{for all} \quad z \in \mathbb{C} \setminus \Sigma_\mu. \tag{2.5}$$

We denote the family of all almost sectorial operators by $\mathcal{F}^\gamma_\omega(X)$.

Example 2.2. *We consider the following example from [37, 4]. Let Ω be the union of two bounded domain in $\mathbb{R}^n, n \geq 2$ with smooth boundaries. Let $B(\cdot, \cdot)$ be defined as*

$$B(u, v) \;=\; (-\Delta u + u, -\frac{1}{g}(gv')') \quad \textit{for} \quad (u, v) \in D(B), \tag{2.6}$$

where $D(B)$ is a dense subset of $L^p(\Omega) \oplus L^p_g(0, 1)(1 \leq p < \infty)$ for a smooth function $g : [0, 1] \to (0, \infty)$ and Δ is the Laplacian with Neumann boundary condition. The domain $D(B)$ is endowed with the norm

$$\|(u, v)\| = \left(\int_\Omega |u|^p + \int_0^1 g|v|^p \right)^{1/p}.$$

If $p > n/2$, then B is a closed linear operator with compact resolvent. We note that B is not sectorial and the resolvent operator $R(z;-B)$ satisfies

$$\|R(z;-B)\| \leq \frac{C}{|z|^{\gamma}} \quad \textit{for} \quad z \in \Sigma_{\mu} \setminus \{0\},$$

where $\Sigma_{\mu} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi - \mu\} \cup \{0\}\} \subset \rho(-B)$ for $\mu \in (0, \pi/2)$, $0 < \gamma < 1 - n/2p$ and some positive constant C. The operator B is almost sectorial. Details can be found in [37, 4, 5, 6].

We make the following assumption on A.

($B1$) The operator $A : D(A) \subset X \to X$ is almost sectorial and $A \in \mathcal{F}_{\omega}^{\gamma}(X)$ for $-1 < \gamma < 0$. Further, we assume that $R(\lambda; -A)$ is compact for each $\lambda > 0$.

It follows from the assumptions ($B1$) that A generates an analytic semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators on X with growth $1 + \gamma$ in an open sector of the complex plane $\mathbb{C}$ (see Lemma 2.3). We note that $T(t)$ is discontinuous at $t = 0$ in the strong operator topology [31, 37].

For $0 < \mu < \pi$, let $\Sigma_{\mu}^{0} = \left\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\right\}$ be the open sector. If $t \in \Sigma_{\pi/2-\omega}^{0}$ and $\omega < \phi < \mu < \frac{\pi}{2} - |\arg t|$, then the family

$$T(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\gamma_{\phi}} e^{-tz} R(z; A) dz, \tag{2.7}$$

where $\gamma_{\phi} = \{\mathbb{R}_{+} e^{i\phi}\} \cup \{\mathbb{R}_{+} e^{-i\phi}\}$, forms an analytic semigroup of growth order $-\gamma - 1$. For $\beta > 1 + \gamma$, $A^{-\beta}$ is a bounded linear operator on X. We define $X_{\beta} = D(A^{\beta})$ for $\beta > 1 + \gamma$, endowed with the norm

$$\|x\|_{\beta} = \|A^{\beta} x\| \quad \text{for } x \in X_{\beta}.$$

Then X_{β} is a Banach space endowed with the norm $\|\cdot\|_{\beta}$. For more on analytic semigroups, we refer the readers to Tanaka [34]. The following properties of $T(t)$ will be used.

Lemma 2.3. [31, 37] *For $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$, let $A \in \mathcal{F}_{\omega}^{\gamma}(X)$.*

(i) *The operator $T(t)$ is analytic in $\Sigma_{\pi/2-\omega}^{0}$ and $\frac{d^n}{dt^n} T(t) = (-A)^n T(t)$ ($t \in \Sigma_{\pi/2-\omega}^{0}$);*

(ii) *$T(s+t) = T(s)T(t)$ for all $s, t \in \Sigma_{\pi/2-\omega}^{0}$;*

(iii) *There exists a constants $C(\gamma) > 0$ such that $\|T(t)\| \leq C(\gamma) t^{-\gamma-1}$ for $t > 0$;*

(iv) *For $t \in \Sigma_{\pi/2-\omega}^{0}$, the range $R(T(t)) \subset D(A^{\beta})$ for all $\beta \in \mathbb{C}$ with $Re\beta > 0$, we have*

$$A^{\beta} T(t) x = \frac{1}{2\pi i} \int_{\gamma_{\phi}} z^{\beta} e^{-tz} R(z; A) x dz \quad \textit{for all } x \in X$$

and there exists a constant $C^*(\gamma, \beta) > 0$ such that

$$\|A^{\beta} T(t)\| \leq C^* t^{-\gamma - Re\beta - 1} \quad t > 0;$$

(v) *For $\beta > 1 + \gamma$, we have $D(A^{\beta}) = \{x \in X : \lim_{t \to 0+} T(t) x = x\}$.*

The generalized Mittag-Leffler function $E_{\alpha,\beta}$ is defined as

$$E_{\alpha,\beta} := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\chi} \frac{\lambda^{\alpha-\beta} e^{\lambda}}{\lambda^{\alpha} - z} d\lambda \quad \text{for} \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where χ is a contour starts and ends at $-\infty$ and encircles the disc $|\lambda| \leq |z|^{1/\alpha}$ counterclockwise. We denote

$$E_\alpha(z) := E_{\alpha,1}(z), \quad e_\alpha(z) := E_{\alpha,\alpha}(z).$$

The function of Wright-type is defined as

$$\Psi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha) \text{ for } \ z \in \mathbb{C}$$

if $0 < \alpha < 1$.

For $t \in \Sigma^0_{\pi/2-\omega}$ and $\omega < \phi < \mu < \pi/2 - |\arg t|$, we define

$$\mathcal{S}_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma_\phi} E_\alpha(-zt^\alpha) R(z; A) dz,$$

$$\mathcal{P}_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma_\phi} e_\alpha(-zt^\alpha) R(z; A) dz,$$

where $\gamma_\phi = \{\mathbb{R}_+ e^{i\phi}\} \cup \{\mathbb{R}_+ e^{-i\phi}\}$ is oriented counter-clockwise. We note that

$$\mathcal{S}_\alpha(t) = \int_0^\infty \Psi_\alpha(\alpha) T(st^\alpha) x \text{ ds}, \ x \in X, \tag{2.8}$$

$$\mathcal{P}_\alpha(t) = \int_0^\infty \alpha s \Psi_\alpha(\alpha) T(st^\alpha) x \text{ ds}, \ x \in X. \tag{2.9}$$

Lemma 2.4. [37, Theorem 3.1] *If $t \in \Sigma^0_{\pi/2-\omega}$ and $\omega < \pi/2 - |\arg t|$, then $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are bounded linear operators on X. Furthermore,*

$$\|\mathcal{S}_\alpha(t)x\| \leq k_s(\alpha,\gamma) t^{-\alpha(1+\gamma)} \|x\|, \quad \forall t > 0, \forall x \in X, \tag{2.10}$$

$$\|\mathcal{P}_\alpha(t)x\| \leq k_p(\alpha,\gamma) t^{-\alpha(1+\gamma)} \|x\|, \quad \forall t > 0, \forall x \in X \tag{2.11}$$

where $k_s(\alpha,\gamma) = C_0 \frac{\Gamma(-\gamma)}{\Gamma(1-\alpha(1+\gamma))}$ and $k_p(\alpha,\gamma) = \alpha C_0 \frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha\gamma)}$ for some positive constant C_0.

Lemma 2.5. [37, Theorem 3.2] *For $t > 0$, the operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology. Further, the continuity is uniform on $[r, \infty)$ for every $r > 0$.*

Lemma 2.6. [37, Theorem 3.5] *If $R(\lambda; -A)$ is compact for every $\lambda > 0$, then $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are compact for every $t > 0$.*

We recall the definition of fractional integral and derivative of a function.

Definition 2.7. *The Riemann-Liouville fractional integral of order η of $h \in L^1(I; X)$ with the lower limit zero is defined as*

$$J_t^\eta h(t) = \frac{1}{\Gamma(\eta)} \int_0^t \frac{h(s)}{(t-s)^{1-\eta}} ds, \ t > 0, \ \eta > 0$$

provided that the right hand side is defined pointwise on $[0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.8. *Let $h \in C^{m-1}(I; X)$ and $(J_t^\eta h)^{(m)} \in L^1(I; X)$. The Caputo derivative of order η of h is defined as*

$${}_cD_t^\eta h(t) = D_t^m J_t^{m-\eta} \left(h(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} h^{(k)}(0) \right), \ t > 0, \ m - 1 < \eta < m,$$

where $D_t^m = \frac{d^m}{dt^m}$.

Lemma 2.9. [37, Theorem 3.4] *The following properties hold.*

(i) *Let* $\beta > 1+\gamma$. *For all* $x \in D(A^\beta)$, $\lim_{t\to 0+} \mathcal{S}_\alpha(t)x = x$;

(ii) *For all* $x \in D(A), t > 0$, $D_t^\alpha \mathcal{S}_\alpha(t)x = -A\mathcal{S}_\alpha(t)x$;

We consider the following fractional Cauchy problem:

$$\left.\begin{array}{rcl} {}_cD_t^\alpha u(t) + Au(t) &=& f(t), t > 0, \\ u(0) &=& u_0, \end{array}\right\} \tag{2.12}$$

where $u_0 \in X$ and $f : (0,\infty) \to X$.

Definition 2.10. *A continuous function* $u : (0,\infty) \to X$ *is said to be a mild solution of problem (2.12) if* u *satisfies the following integral equation*

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds.$$

Theorem 2.11. *Let* $A \in \mathcal{F}_\omega^\gamma(X)$, *where* $0 < \omega < \frac{\pi}{2}$. *Suppose that* $f \in D(A)$ *and* $Af(t) \in L^\infty((0,T];X)$. *Then for each* $u_0 \in X$, *Problem (2.12) has a unique mild solution on* $(0,t_0]$ *for some* $0 < t_0 \le T$.

For a proof of the theorem we refer to Wang *et. al* [37, Theorem 4.1]. Further, we remark that [37, Theorem 4.1] has been exteded in Kostić [27, Theorem 4.3]. We use the following notion.

Let S be the set defined by

$$S = \{u |\ u : (-\infty,a] \to X, u|_{(-\infty,0]} \in \mathcal{B}, u|_J \in C(J,X)\}.$$

Definition 2.12. *A function* $u \in S$ *is called a mild solution to problem (1.1) if*

(i) $u_0 = \phi \in \mathcal{B}$ *on* $(-\infty,0]$,

(ii) *the function* $\mathcal{P}_\alpha(t-s)Ag(s,u_s)$ *is integrable for each* $s \in [0,t]$,

(iii) u *satisfies the following integral equation*

$$\begin{aligned} u(t) &= \mathcal{S}_\alpha(t)[\phi(0) + g(0,\phi)] - g(t,u_t) \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)Ag(s,u_s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s,u_s)ds, \quad t \in [0,a]. \end{aligned} \tag{2.13}$$

We make the following assumptions on f and g.

$(B2)$ Let $f : J \times \mathcal{B} \to X$ be a Carathéodory function and for any $r > 0$ there exist functions $m_r(t) \in L^p(J;\mathbb{R}^+)$ such that

$$\|f(t,x)\| \le m_r(t) \text{ and } \lim_{r\to+\infty} \frac{\|m_r(t)\|_{L^p(J)}}{r} = \rho < \infty \tag{2.14}$$

$$\|f(t,x)\|_{\mathcal{B}} \le c_f(\|x\|_{\mathcal{B}} + 1). \tag{2.15}$$

$(B3)$ Let $g : J \times \mathcal{B} \to X_1$ be a continuous map such that

$$\|g(t,x)\|_1 \le c_g(\|x\|_{\mathcal{B}} + 1), \tag{2.16}$$

$$\|g(t,x_1) - g(s,x_2)\|_1 \le L_g\|x_1 - x_2\|_{\mathcal{B}} \tag{2.17}$$

for all $x_1, x_2 \in \mathcal{B}$ and $s,t \in J$ and for some positive constants c_g and L_g.

We also recall the Krasnoselskii's fixed point theorem. We refer the reader for proof to Burton [11].

Theorem 2.13. *Let P be a map from a closed bounded convex subset S of X into S. Suppose that $Px = P_1x + P_2x$ for $x \in S$ and $P_1u + p_2v \in S$ for every pair $u, v \in S$. If P_1 is contraction and P_2 is compact, then the equation $P_1u + p_2u = u$ has a solution in S.*

3. Existence and uniqueness of mild solution

The following theorem gives the existence of a mild solutions to Problem (1.1).

Theorem 3.1. *Let the assumptions $(B1)$-$(B3)$ hold and $\phi(0) \in D(A)$ with $\|\phi(0)\|_{D(A)} \leq L_{\mathcal{B}}\|\phi\|_{\mathcal{B}}$. Then Problem (1.1) has a mild solution on $(-\infty, t_0]$ for some $0 < t_0 \leq a$ if*

$$c_g k_1\big(1 + k_p(\alpha,\gamma)a^{-\alpha\gamma}\big) + k_p(\alpha,\gamma)\rho\left\{\frac{a^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right\}^{\frac{1}{q}} < 1, \tag{3.18}$$

$$\left(k_1 L_g\|A^{-1}\| + L_g k_p(\alpha,\gamma)k_1\frac{a^{-\alpha\gamma}}{-\alpha\gamma}\right) < 1 \tag{3.19}$$

Proof. We define a map $z : (-\infty, a] \to X$ defined as

$$z(t) = \begin{cases} \phi(t) & \text{for } t \in (-\infty, 0], \\ \mathcal{S}_\alpha(t)\phi(0) & \text{for } t \in J. \end{cases}$$

This implies that $z_0 = \phi$. Let $u(t) = y(t) + z(t),\ -\infty < t \leq a$. Then u satisfies (2.13) if and only if y satisfies $y_0 = 0$ and

$$\begin{aligned} y(t) = {} & \mathcal{S}_\alpha(t)g(0,\phi) - g(t, y_t + z_t) \\ & + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)Ag(s, y_s + z_s)ds \\ & + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, y_s + z_s)ds, \quad t \in J. \end{aligned} \tag{3.20}$$

Let S_0 be the set defined by

$$S_0 = \{y \in S : y_0 = 0\}.$$

Then $(S_0, \|\cdot\|_{S_0})$ is a Banach space equipped with the seminorm $\|\cdot\|_{S_0}$ defined as

$$\|y\|_{S_0} = \|y_0\|_{\mathcal{B}} + \sup_{t\in J}\|y(t)\| = \sup_{t\in J}\|y(t)\| \text{ for } y \in S_0.$$

For $r \geq 0$, let $B_r = \{u \in S_0 : \|u\| \leq r\}$. Then B_r is uniformly bounded and for $y \in B_r$, we have

$$\begin{aligned} \|y_t + z_t\|_{\mathcal{B}} & \leq \|y_t\|_{\mathcal{B}} + \|z_t\|_{\mathcal{B}} \\ & \leq k_1 \sup_{s\in[0,t]}\|y(s)\| + k_2\|y_0\|_{\mathcal{B}} + k_1\sup_{s\in[0,t]}\|z(s)\| + k_2\|z_0\|_{\mathcal{B}} \\ & \leq k_1 r + \|A^{-1}\|k_s(\alpha,\gamma)t^{-\alpha(1+\gamma)}\|\phi(0)\|_{D(A)} + \|\phi\|_{\mathcal{B}} \\ & \leq k_1 r + \|A^{-1}\|k_s(\alpha,\gamma)t_0^{-\alpha(1+\gamma)}L_{\mathcal{B}}\|\phi\|_{\mathcal{B}} + \|\phi\|_{\mathcal{B}} = r'(say), \end{aligned}$$

where $0 < t_0 \leq a$. Next, we define a map $Q : S_0 \to S_0$ by

$$Qy(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_\alpha(t)g(0,\phi) - g(t, y_t + z_t) & \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)Ag(s, y_s + z_s)ds & \\ \quad\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, y_s + z_s)ds, & t \in J. \end{cases}$$

We claim that $Q(B_r) \subset B_r$ for some $r > 0$. Suppose this is not the case, then for each $r > 0$, there exists $\widetilde{y} \in B_r$ and $\widetilde{t} \in J$ such that $\|Q\widetilde{y}(\widetilde{t})\| > r$. By assumption $(B2) - (B3)$ and the estimates (2.10), (2.11) (see Lemma 2.4), we have

$$\begin{aligned} r &< \|Q\widetilde{y}(\widetilde{t})\| \\ &\leq \|\mathcal{S}_\alpha(\widetilde{t})g(0,\phi)\| + \|g(\widetilde{t}, y_{\widetilde{t}} + z_{\widetilde{t}})\| + \int_0^{\widetilde{t}} \|(\widetilde{t}-s)^{\alpha-1}\mathcal{P}_\alpha(\widetilde{t}-s)Ag(s, y_s + z_s)\|ds \\ &\quad + \int_0^{\widetilde{t}} \|(\widetilde{t}-s)^{\alpha-1}\mathcal{P}_\alpha(\widetilde{t}-s)f(s, y_s + z_s)\|ds \\ &\leq k\sup_{t\in J}\|\mathcal{S}_\alpha(\widetilde{t})\|c_g(\|\phi\|_{\mathcal{B}} + 1) + c_g\Big(k_1 r + \|A^{-1}\|k_s(\alpha,\gamma)t_0^{-\alpha(1+\gamma)}L_{\mathcal{B}}\|\phi\|_{\mathcal{B}} + \|z\|_{\mathcal{B}}\Big) \\ &+ c_g k_p(\alpha,\gamma)\int_0^{\widetilde{t}}(\widetilde{t}-s)^{-1-\alpha\gamma}(1 + \|y_s + z_s\|_{\mathcal{B}})ds + k_p(\alpha,\gamma)\int_0^{\widetilde{t}}(\widetilde{t}-s)^{-1-\alpha\gamma}m_r(s)ds \\ &\leq k\sup_{t\in J}\|\mathcal{S}_\alpha(\widetilde{t})\|c_g(\|\phi\|_{\mathcal{B}} + 1) + c_g\Big(k_1 r + \|A^{-1}\|k_s(\alpha,\gamma)t_0^{-\alpha(1+\gamma)}L_{\mathcal{B}}\|\phi\|_{\mathcal{B}} + \|z\|_{\mathcal{B}}\Big) \\ &\quad + \Big[k_1 r + \|A^{-1}\|k_s(\alpha,\gamma)t_0^{-\alpha(1+\gamma)}L_{\mathcal{B}}\|\phi\|_{\mathcal{B}} + \|z\|_{\mathcal{B}}\Big]c_g k_p(\alpha,\gamma)a^{-\alpha\gamma} \\ &\quad + k_p(\alpha,\gamma)\|m_r\|_{L^p(0,a)}\left\{\frac{a^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right\}^{\frac{1}{q}}, \end{aligned}$$

where $q = \frac{p}{p-1}$ and k is some constant. Making $r \to \infty$, we obtain that

$$1 < c_g k_1\big(1 + k_p(\alpha,\gamma)a^{-\alpha\gamma}\big) + k_p(\alpha,\gamma)\rho\left\{\frac{a^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right\}^{\frac{1}{q}},$$

which gives a contradiction to (3.18). Thus for $r > 0$, we have $Q(B_r) \subset B_r$. Now we decompose the map Q as $Q = Q_1 + Q_2$, where

$$\begin{aligned} Q_1 y(t) &= \mathcal{S}_\alpha(t)g(0,\phi) - g(t, y_t + z_t) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)Ag(s, y_s + z_s)ds, \\ Q_2 y(t) &= \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, y_s + z_s)ds. \end{aligned}$$

We claim that the operator equation $y = Q_1 y + Q_2 y$ has solution in B_r. **Step I**: We show that Q_1 is contraction on B_r. For any $u, v \in B_r$, we have

$$\begin{aligned}
&\|Q_1 u(t) - Q_1 v(t)\| \\
&\leq \|g(t, u_t + z_t) - g(t, v_t + z_t)\| \\
&\qquad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|g(s, u_s + z_s) - g(s, v_s + z_s)\|_1 ds \\
&\leq L_g \|A^{-1}\| \|u_t - v_t\|_{\mathcal{B}} + L_g k_p(\alpha,\gamma) \|u_t - v_t\|_{\mathcal{B}} \int_0^t (t-s)^{\alpha-1-\alpha(1+\gamma)} \|\mathcal{P}_\alpha(t-s)\| ds \\
&\leq k_1 L_g \|A^{-1}\| \sup_{0\leq s\leq t} \|u(s) - v(s)\| + L_g k_p(\alpha,\gamma) k_1 \sup_{0\leq s\leq t} \|u(s) - v(s)\| \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \\
&= \left(k_1 L_g \|A^{-1}\| + L_g k_p(\alpha,\gamma) k_1 \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) \sup_{0\leq s\leq t} \|u(s) - v(s)\|.
\end{aligned}$$

Taking the supremum over $t \in J$, we obtain that

$$\|Q_1 u - Q_1 v\|_{S_0} \leq \mathcal{C} \|u - v\|_{S_0},$$

where $\mathcal{C} = \left(k_1 L_g \|A^{-1}\| + L_g k_p(\alpha,\gamma) k_1 \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right)$. By assumption (3.19), Q_1 is a contraction on B_r.

Step II: To show that the operator Q_2 is completely continuous on B_r. We begin with showing that the operator Q_2 is continuous. Let $\{y^{(n)}(t)\}$ be sequence in B_r such that $y^{(n)} \to y$ as $n \to \infty$ in S_0 for some $y \in B_r$. By hypothesis $(B2)$, we have

$$\|f(t, y_t^{(n)} + z_t) - f(t, y_t + z_t)\| \to 0 \text{ as } t \to \infty$$

and

$$\|f(t, y_t^{(n)} + z_t) - f(t, y_t + z_t)\| \leq 2m_r(t)$$

for a. e. $t \in J$. Thus

$$(t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s) f(t, y_t^{(n)} + z_t)\| \leq k_p(\alpha,\gamma)(t-s)^{-1-\alpha\gamma} m_r(t) \in L^1(J).$$

It follows from the dominated convergence theorem that

$$\begin{aligned}
&\|Q_2 y^{(n)}(t) - Q_2 y(t)\| \\
&\leq k_p(\alpha,\gamma) \int_\tau^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|f(t, y_t^{(n)} + z_t) - f(t, y_t + z_t)\| ds \to 0
\end{aligned}$$

as $n \to \infty$. That is $\lim_{n\to\infty} \|Q_2 y^{(n)} - Q_2 y\| = 0$. So Q_2 is continuous in B_r.

Step III: We show that the set $\{Q_2 u(t) : t \in J,\ u \in B_r\}$ is equicontinuous. Let $0 < \tau < t \leq a$ and $\delta > 0$ small enough. Form Lemma 2.5 and hypothesis $(B2)$, it follows that

$$\begin{aligned}
&\|Q_2y(t) - Q_2y(\tau)\| \\
&\leq \int_\tau^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)f(s,y_s+z_s)\|ds \\
&\quad + \int_0^{\tau-\delta} (t-s)^{\alpha-1}\|[\mathcal{P}_\alpha(t-s) - \mathcal{P}_\alpha(\tau-s)]f(s,y_s+z_s)\|ds \\
&\quad + \int_{\tau-\delta}^{\tau} (t-s)^{\alpha-1}\|[\mathcal{P}_\alpha(t-s) - \mathcal{P}_\alpha(\tau-s)]f(s,y_s+z_s)\|ds \\
&\quad + \int_0^{\tau} |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}|\|\mathcal{P}_\alpha(t-s)f(s,y_s+z_s)\|ds \\
&\leq k_p(\alpha,\gamma)\Big(\frac{(t-\tau)^{(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad + \sup_{s\in[0,\tau-\delta]}\|\mathcal{P}_\alpha(t-s) - \mathcal{P}_\alpha(\tau-s)\|\Big(\int_0^{\tau-\delta}(\tau-s)^{q\alpha-q}ds\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad + k_p(\alpha,\gamma)\int_{\tau-\delta}^{\tau}(\tau-s)^{\alpha-1}2(\tau-s)^{-\alpha(\gamma+1)}m_r(s)ds \\
&\quad + k_p(\alpha,\gamma)\Big(\int_0^{\tau}(\tau-s)^{-q(\alpha\gamma+1)} - (t-s)^{-q(\alpha\gamma+1)}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\leq k_p(\alpha,\gamma)\Big(\frac{(t-\tau)^{(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad + \sup_{s\in[0,\tau-\delta]}\|\mathcal{P}_\alpha(t-s) - \mathcal{P}_\alpha(\tau-s)\|\Big(\frac{\tau^{1+q(\alpha-1)} - \delta^{1+q(\alpha-1)}}{1+q(\alpha-1)}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad + k_p(\alpha,\gamma)\Big(\frac{\delta^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad + k_p(\alpha,\gamma)\Big(\frac{(t-\tau)^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q} + \frac{\tau^{1-(\alpha\gamma+1)q} - t^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)} \\
&\quad \to 0
\end{aligned}$$

as $t \to \tau$ and $\delta \to 0$. Thus for $y \in B_r$, we obtain

$$\|Q_2y(t) - Q_2y(\tau)\| \to 0 \text{ as } t \to \tau.$$

By Lemma 2.4 and assumption $(B2)$, we have

$$\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)f(s,y_s+z_s)\|ds \leq k_p(\alpha,\gamma)\Big(\frac{t^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q}\Big)^{\frac{1}{q}}\|m_r\|_{L^p(J)}.$$

Hence

$$\|Q_2y(t)\| \to 0$$

as $t \to 0$, where the limit is independent of $y \in B_r$. Finally, we show that the set $S_1 = \{Q_2y(t) : y \in B_r,\ t \in [0,a]\}$ is precompact in X. Let $t \in (0,a]$ be fixed and $\epsilon, \eta > 0$. We define the following map

$$P_{\epsilon,\eta}y(t) = \int_0^{t-\epsilon}\int_\delta^\infty \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s,y_s+z_s)ds$$

for $y \in B_r$. We note that hypothesis $(B1)$ and Lemma 2.6 imply that $\{T(t) : t > 0\}$ is compact. Thus for each $t \in (0, a]$ and $0 < \epsilon < t$, the set S_1 is precompact in X. Further, it follows from hypothesis $(B2)$ and (2.8), (2.9) that

$$\begin{aligned}
&\|Q_2y(t) - P_{\epsilon,\eta}y(t)\| \\
&\leq \left\| \int_0^t \int_0^\delta \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s, y_s + z_s)ds \right\| \\
&\quad + \left\| \int_{t-\epsilon}^t \int_\delta^\infty \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s, y_s + z_s)ds \right\| \\
&\leq k_p(\alpha, \gamma) \int_0^t (t-s)^{-1-\alpha\gamma} m_r(s)ds \int_0^\delta \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\quad + k_p(\alpha, \gamma) \int_{t-\epsilon}^t (t-s)^{-1-\alpha\gamma} m_r(s)ds \int_\delta^\infty \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\leq k_p(\alpha, \gamma) \left(\frac{a^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q} \right)^{\frac{1}{q}} \|m_r\|_{L^p(J)} \int_0^\delta \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\quad + k_p(\alpha, \gamma) \left(\frac{\epsilon^{1-(\alpha\gamma+1)q}}{1-(\alpha\gamma+1)q} \right)^{\frac{1}{q}} \|m_r\|_{L^p(J)} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma\alpha)}.
\end{aligned}$$

It follws that

$$\|Q_2y(t) - P_{\epsilon,\eta}y(t)\| \to 0 \text{ as } \epsilon, \delta \to 0^+.$$

Thus there are precompact sets arbitrarily close to the set S_1. Hence for each $t \in [0, a]$, the set $\{Q_2y(t) : y \in B_r\}$ is precompact in X. By the Arzela-Ascoli theorem, Q_2 is compact operator.

By the Krasnoselskii's fixed point theorem, $Q = Q_1 + Q_2$ has a fixed point on B_r. This completes the proof. □

The uniqueness of mild solution is proved with a stronger condition on the function f.

$(B2)'$ There exists a positive constant L_f such that the continuous map $f : J \times \mathcal{B} \to X$ satisfies

$$\|f(t, x_1) - f(t, x_2)\| \leq L_f(\|x_1 - x_2\|) \tag{3.21}$$

for all $x_1, x_2 \in \mathcal{B}$, and $t \in J$.

Theorem 3.2. *Let the assumptions $(B1)$, $(B2)'$ and $(B3)$ hold. Then Problem (1.1) has a*

unique mild solution on $[0, t_0]$ with $0 < t_0 \leq a$ for each $\phi(0) \in D(A)$ if

$$\left(k_1L_g\|A^{-1}\| + k_1L_gk_p(\alpha, \gamma)\frac{a^{-\alpha\gamma}}{-\alpha\gamma} + k_1L_fk_p(\alpha, \gamma)\frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) < 1. \tag{3.22}$$

Proof. As in Theorem 3.1, we define a map $Q : S_0 \to S_0$ by

$$Qy(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_\alpha(t)g(0, \phi) - g(t, y_t + z_t) & \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)Ag(s, y_s + z_s)ds & \\ \quad\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, y_s + z_s)ds, & t \in J. \end{cases}$$

For $u, v \in S_0$ and $t \in J$, we have

$$
\begin{aligned}
&\|Qu(t) - Qv(t)\| \\
&\le \|g(t, u_t + z_t) - g(t, v_t + z_t)\| \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|Ag(s, u_s + z_s) - Ag(s, v_s + z_s)\| ds \\
&\qquad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|f(s, u_s + z_s) - f(s, v_s + z_s)\| ds \\
&\le L_g \|A^{-1}\| (\|u_t - v_t\|_{\mathcal{B}}) + L_g k_p(\alpha, \gamma) \int_0^t (t-s)^{-\alpha\gamma-1} \|(\|u_s - v_s\|_{\mathcal{B}}) ds \\
&\quad + L_f k_p(\alpha, \gamma) \int_0^t (t-s)^{-\alpha\gamma-1} (\|u_s - v_s\|_{\mathcal{B}}) ds \\
&\le L_g \|A^{-1}\| k_1 (\|u - v\|_{S_0}) + L_g k_p(\alpha, \gamma) \int_0^t (t-s)^{-\alpha\gamma-1} k_1 (\|u - v\|_{S_0}) ds \\
&\quad + L_f k_p(\alpha, \gamma) \int_0^t (t-s)^{-\alpha\gamma-1} k_1 (\|u - v\|_{S_0}) ds \\
&\le \left(k_1 L_g \|A^{-1}\| + k_1 L_g k_p(\alpha, \gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} + k_1 L_f k_p(\alpha, \gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) \|u - v\|_{S_0}.
\end{aligned}
$$

By the condition (3.22), the map Q is a contraction on S_0. By the Banach fixed point theorem, Q has a fixed point on S_0. Thus Problem (1.1) has a unique mild solution. □

4. Continuous dependence of solutions

The section is devoted to show the continuous dependence of the mild solution to the phase space.

Theorem 4.1. *Let $\phi_1, \phi_2 \in \mathcal{B}$ such that $\|\phi_1(0) - \phi_2(0)\|_{D(A)} \le L_{\mathcal{B}} \|\phi_1 - \phi_2\|_{\mathcal{B}}$ for some constant $L_{\mathcal{B}} > 0$. Suppose that hypotheses of Theorem 3.2 are satisfied. If $u_1(t)$ and $u_2(t)$ are two solutions to the problem*

$$
\left.
\begin{aligned}
{}_cD_t^\alpha [u(t) + g(t, u_t)] + Au(t) &= f(t, u_t), \quad t \in J = [0, a], \\
u(t) &= \phi_i(t), \quad (-\infty, 0], i = 1, 2,
\end{aligned}
\right\} \tag{4.23}
$$

then we have

$$
\|u_{1t} - u_{2t}\|_{\mathcal{B}} \le \left(k_1 \frac{C}{1 - D} + k_s(\alpha, \gamma) t_0^{-\alpha(1+\gamma)} \right) \|\phi_1 - \phi_2\|_{\mathcal{B}}.
$$

Proof. As in the proof of Theorem 3.1, we write u_i as $u_i = y_i + z_i,\ 0 \le t \le a$, where y_i and z_i are defined as in Theorem 3.1. Furthermore we have $u_{it} = y_{it} + z_{it},\ 0 \le t \le a$, and y_i satisfies

$$
y_i(t) = \begin{cases}
0, & t \in (-\infty, 0], \\
\mathcal{S}_\alpha(t) g(0, \phi_i) - g(t, y_{it} + z_{it}) \\
\quad + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) A g(s, y_{is} + z_{is}) ds \\
\qquad + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, y_{is} + z_{is}) ds, & t \in J.
\end{cases}
$$

For $t \in [0, a]$, we have

$$
\begin{aligned}
&\|y_1(t) - y_2(t)\| \\
&\le \|\mathcal{S}_\alpha(t)[g(0,\phi_1) - g(0,\phi_2)]\| + \|[g(t, y_{1t} + z_{1t}) - g(t, y_{2t} + z_{2t})]\| \\
&\qquad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|[Ag(s, y_{1s} + z_{1s}) - Ag(s, y_{2s} + z_{2s})]\| ds \\
&\qquad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|[f(s, y_{1s} + z_{1s}) - f(s, y_{2s} + z_{2s})]\| ds \\
&\le L_g k_s(\alpha,\gamma) \|A^{-1}\| t_0^{-\alpha(1+\gamma)} \|\phi_1 - \phi_2\|_{\mathcal{B}} + L_g \|A^{-1}\| \big(\|y_{1t} - y_{2t}\|_{\mathcal{B}} + \|z_{1t} - z_{2t}\|_{\mathcal{B}} \big) \\
&\qquad + (L_g + L_f) k_p(\alpha,\gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \big(\|y_{1t} - y_{2t}\|_{\mathcal{B}} + \|z_{1t} - z_{2t}\|_{\mathcal{B}} \big) \\
&\le L_g k_s(\alpha,\gamma) \|A^{-1}\| t_0^{-\alpha(1+\gamma)} \|\phi_1 - \phi_2\|_{\mathcal{B}} + \left(L_g \|A^{-1}\| + (L_g + L_f) k_p(\alpha,\gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) \\
&\qquad \times \left(k_1 \|y_1 - y_2\|_{S_0} + k_1 \sup_{t \in J} \|z_1(t) - z_2(t)\| + k_2 \|\phi_1 - \phi_2\|_{\mathcal{B}} \right) \\
&\le L_g k_s(\alpha,\gamma) \|A^{-1}\| t_0^{-\alpha(1+\gamma)} \|\phi_1 - \phi_2\|_{\mathcal{B}} + \left(L_g \|A^{-1}\| + (L_g + L_f) k_p(\alpha,\gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) \\
&\qquad\quad \times \left(k_1 \|y_1 - y_2\|_{S_0} + k_1 k_s(\alpha,\gamma) t_0^{-\alpha(1+\gamma)} \|\phi_1 - \phi_2\|_{\mathcal{B}} + k_2 \|\phi_1 - \phi_2\|_{\mathcal{B}} \right) \\
&= C \|\phi_1 - \phi_2\|_{\mathcal{B}} + D \|y_1 - y_2\|_{S_0},
\end{aligned}
$$

where

$$
\begin{aligned}
C = {} & L_g k_s(\alpha,\gamma) \|A^{-1}\| t_0^{-\alpha(1+\gamma)} \\
& \qquad + \left(L_g \|A^{-1}\| + (L_g + L_f) k_p(\alpha,\gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) (k_1 k_s(\alpha,\gamma) t_0^{-\alpha(1+\gamma)} + k_2)
\end{aligned}
$$

and $D = \left(L_g \|A^{-1}\| + (L_g + L_f) k_p(\alpha,\gamma) \frac{a^{-\alpha\gamma}}{-\alpha\gamma} \right) k_1$. It follows that

$$
\|y_1 - y_2\|_{S_0} \le \frac{C}{1-D} \|\phi_1 - \phi_2\|_{\mathcal{B}}. \tag{4.24}
$$

Again we have $u_{it} = y_{it} + z_{it},\ t \in J$. Using (4.24), we obtain,

$$
\begin{aligned}
\|u_{1t} - u_{2t}\|_{\mathcal{B}} &\le \|y_{1t} - y_{2t}\|_{\mathcal{B}} + \|z_{1t} - z_{2t}\|_{\mathcal{B}} \\
&\le k_1 \|y_1 - y_2\|_{S_0} + k_s(\alpha,\gamma) t_0^{-\alpha(1+\gamma)} \|\phi_1 - \phi_2\|_{\mathcal{B}} \\
&\le \left(k_1 \frac{C}{1-D} + k_s(\alpha,\gamma) t_0^{-\alpha(1+\gamma)} \right) \|\phi_1 - \phi_2\|_{\mathcal{B}}
\end{aligned}
$$

which completes the proof. □

5. Application

To apply the results obtained in the previous section, we consider the following problem in $X = C^l([0,1])$(the space of all complex valued Hölder continuous functions on $[0,1]$, $0 < l < 1$). For $0 < T < 1$ and $(x,t) \in (0,1) \times (0,T)$, we consider the following problem in X [1],

$$\left.\begin{array}{rcl}\frac{\partial^\alpha}{\partial t^\alpha}\left[u(t,x)-\int_{-\infty}^{0} R_1(\theta,u(t+\theta,x))d\theta\right]-\frac{\partial^2 u}{\partial x^2} & = & \int_{-\infty}^{0} R_2(\theta,u(t+\theta,x))d\theta,\\ u(t,0) & = & u(t,1)=0,\\ u(\theta,x) & = & u_0(\theta,x),\ -\infty<\theta\leq 0,\end{array}\right\}\quad(5.25)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order $\alpha, 0<\alpha<1$, $R_1, R_2 : (-\infty,0]\times\mathbb{R}\to\mathbb{R}$, $v_0 : (-\infty,0]\times[0,1]\to\mathbb{R}$ are continuous functions.

Let us define the set B_η as

$$B_\eta=\left\{\phi \mid \phi:(-\infty,0]\to X \text{ is continuous and } \lim_{\theta\to-\infty} e^{\theta\eta}\phi(\theta)\text{exists}\right\}.$$

Then B_η is a Banach space endowed with the norm $\|\phi\|_\eta=\sup_{\theta\leq 0}(e^{\theta\eta}||\phi(\theta)||)$ and $(A1),(A2),(A3)$ are satisfied [1]. Let $u(t,\cdot)=v(t)(\cdot)$,

$$Av \quad = \quad -\frac{d^2u}{dx^2}, \quad v\in D(A), \tag{5.26}$$

where $D(A)=\{u\in C^{2+l}[0,1] : u(0)=u(1)=0\}$. We note that [37]

(a) the domain of A is not dense X;
(b) there exists $\theta,\delta>0$ such that

$$\sigma(A+\theta)\subset\Sigma_{\frac{\pi}{2}-\delta}=\{\lambda\in\mathbb{C}\setminus\{0\} : |\arg\lambda|\leq\frac{\pi}{2}-\delta\}\cup\{0\}$$
$$\|R(\lambda;A+\theta)\|_{\mathcal{L}(C^l[0,1])}\leq\frac{C}{\lambda^{1-l/2}},\quad \lambda\in\mathbb{C}\setminus\Sigma_{\frac{\pi}{2}-\delta}$$

for some positive constant C.

This implies that the operator $A\in\mathcal{F}^{-1+l/2}_{\frac{\pi}{2}-\delta}(X)$. Then Problem (5.25) can be put in the following form

$$\left.\begin{array}{rcl}{}_cD_t^\alpha[v(t)+g(t,v_t)]+Av(t) & = & f(t,v_t),\\ v(0) & = & \phi\in B_\eta,\end{array}\right\}\tag{5.27}$$

where $g(t,\phi)(x)=-\int_{-\infty}^{0} R_1(\theta,v(\theta)(x))d\theta$ and $f(t,\phi)(x)=-\int_{-\infty}^{0} R_2(\theta,v(\theta)(x))d\theta$. Furthermore, we assume the following conditions

(i) R_1 and R_2 are nonnegative integrable functions functions on $(-\infty,0]$ such that

$$|R_i(\theta,\xi_1)-R_i(\theta,\xi_1)|\leq c\|\xi_1-\xi_2\|$$

for some constant $c>0$.
(ii) $\lim_{\theta\to-\infty} e^{\theta\eta}v_0(\theta,x)$ exists uniformly for $x\in[0,1]$.
(iii) $v_0(0,\cdot)\in D(A)$. Assumption (i) implies that, for

It follows from assumption (i) that f and g satisfy assumptions $(B2)'$ and $(B3)$.

Theorem 5.1. *For each $v_0\in D(A)$, Problem (5.27) has a unique mild solution.*

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