COMMENT ON: "DYNAMICAL ANALYSIS OF A PREY-PREDATOR FRACTIONAL ORDER MODEL" BY ELETTREBY ET AL. [JOURNAL OF FRACTIONAL CALCULUS AND APPLICATIONS 8 (2017) 237–245]

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ABSTRACT. This comment concerns an article recently published by Elettreby et al. (2017). We explore in this comment what we believe to be an error in the study of stable equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ for the ordinary differential equation form. The authors claim in the conclusion that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is a stable equilibrium point under some conditions for the ordinary differential equation form and stable without any conditions for the fractional form. By using the Routh–Hurwitz criteria, we show that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is a stable equilibrium point without any conditions for the ordinary differential equation form and fractional form. By using suitable Lyapunov function we show that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is globally asymptotically stable for the ordinary differential equation form.

The purpose of this comment is to point out what we believe to be a mathematical error in the study of stable equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the paper by Elettreby et al. [1].

Elettreby et al. (2017) presented a fractional order two-prey one-predator model as follows

$$
\begin{aligned}
D_*^\alpha x_1(t) &= ax_1(t)(1 - x_1(t)) - x_1(t)x_3(t), \\
D_*^\alpha x_2(t) &= bx_2(t)(1 - x_2(t)) - x_2(t)x_3(t), \\
D_*^\alpha x_3(t) &= -cx_3^2(t) + dx_1(t)x_3(t) + ex_2(t)x_3(t),
\end{aligned}
\tag{1}
$$

where a, b, c, d and e are positive constants.

The authors [1] studied the existence and uniqueness of the model as well as the stability of the equilibrium points and numerical solutions of the fractional order model (1).

The authors claim in the conclusion that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is a stable equilibrium point under some conditions for the ordinary differential equation form. But in the fractional form, they found that the same point is stable without any conditions. They also claim that this is an example of the equilibrium

point which is a centre for the integer order system but locally asymptotically stable for its fractional-order counterpart.

But, on page 241, the authors indicate that the eigenvalues of the Jacobian matrix evaluated at $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ are the roots of the following characteristic polynomial:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = a\bar{x_1} + b\bar{x_2} + c\bar{x_3},$$
$$a_2 = ab\bar{x_1}\bar{x_2} + ac\bar{x_1}\bar{x_3} + bc\bar{x_2}\bar{x_3} + e\bar{x_2}\bar{x_3} + d\bar{x_1}\bar{x_3},$$
$$a_3 = \bar{x_1}\bar{x_2}\bar{x_3}(abc + ae + bd).$$

According to Routh–Hurwitz criteria, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$, which ensures that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is a stable equilibrium point without any conditions for the ordinary differential equation form and fractional form. Hence, the fractional order system (1) is locally asymptotically stable around the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ for the ordinary differential equation form and fractional form. This is inconsistent with the conclusion by [1].

One can prove that the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is globally asymptotically stable for the ordinary differential equation form as in the following theorem.

Theorem 1. *The equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is globally asymptotically stable for the ordinary differential equation form.*

Proof. To study the globally asymptotically stable of the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ for the ordinary differential equation form, we consider the following positive definite Lyapunov function

$$V(x_1, x_2, x_3) = \left(x_1 - \bar{x_1} - \bar{x_1}\ln\frac{x_1}{\bar{x_1}}\right) + \frac{e}{d}\left(x_2 - \bar{x_2} - \bar{x_2}\ln\frac{x_2}{\bar{x_2}}\right) + \frac{1}{d}\left(x_3 - \bar{x_3} - \bar{x_3}\ln\frac{x_3}{\bar{x_3}}\right),$$

by taking the time derivative of $V(x_1, x_2, x_3)$ along the solution of the ordinary differential equation form. One has

$$\dot{V}(x_1, x_2, x_3) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \frac{\partial V}{\partial x_3}\dot{x_3}$$

$$= \left(1 - \frac{\bar{x_1}}{x_1}\right)(ax_1(1 - x_1) - x_1x_3)$$

$$\quad + \frac{e}{d}\left(1 - \frac{\bar{x_2}}{x_2}\right)(bx_2(1 - x_2) - x_2x_3)$$

$$\quad + \frac{1}{d}\left(1 - \frac{\bar{x_3}}{x_3}\right)(-cx_3^2 + dx_1x_3 + ex_2x_3)$$

$$= (x_1 - \bar{x_1})(a\bar{x_1} + \bar{x_3} - ax_1 - x_3)$$

$$\quad + \frac{e}{d}(x_2 - \bar{x_2})(b\bar{x_2} + \bar{x_3} - bx_2 - x_3)$$

$$\quad + \frac{1}{d}(x_3 - \bar{x_3})(-cx_3 + c\bar{x_3} + dx_1 - d\bar{x_1} + ex_2 - e\bar{x_2})$$

$$= -a(x_1 - \bar{x_1})^2 - \frac{eb}{d}(x_2 - \bar{x_2})^2 - \frac{c}{d}(x_3 - \bar{x_3})^2.$$

Thus, $\dot{V}(x_1, x_2, x_3) < 0$, and $\dot{V}(x_1, x_2, x_3) = 0$ if and only if $x_1 = \bar{x_1}$, $x_2 = \bar{x_2}$ and $x_3 = \bar{x_3}$. By LaSalle's invariance principle [2, 3], the equilibrium point $E_6(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is globally asymptotically stable for the ordinary differential equation form.   □

References

[1] M. F. Elettreby, T. Nabil, and A. A. Al-Raezah, Dynamical analysis of a prey-predator fractional order model, Journal of Fractional Calculus and Applications Vol. 8, 237-245, 2017.

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[3] X. Tian, and R. Xu, Global dynamics of a predator-prey system with Holling type II functional response, Lithuanian Association of Nonlinear Analysts (LANA) Vol. 16, 242–253, 2011.

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