

FRACTIONAL CALCULUS AND GENERALIZED LAGUERRE POLYNOMIALS OF ARBITRARY ORDERS

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ABSTRACT. Fractional calculus has emerged as one of the most important interdisciplinary subjects during the last four decades mainly due to its applications in various fields of science and engineering. In this paper we exploit the fractional calculus to discuss a new class of Laguerre polynomials of two fractional (arbitrary) orders. The properties of this family of polynomials are little known and this paper is devoted to their study to a deeper understanding of their properties and discuss the link with known cases.

1. INTRODUCTION , DEFINITIONS AND GENERAL PROPERTIES

Let $L_1(I)$ be a class of Lebesgue integrable functions on the interval $I = [a, b]$ where $0 \leq a < b < \infty$, and let $\Gamma(\cdot)$ be the gamma function. According to the Riemann-Liouville approach to fractional calculus we present the following definitions of fractional integral and fractional derivative.

Definition 1.1. (see [4]-[10]) Let $f(t) \in L_1, \beta \in \mathfrak{R}^+$. The fractional integral of the function $f(t)$ of order β is defined by

$$I_t^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-s)^{\beta-1} f(s) ds. \quad (1)$$

If $a = 0$, then we write $I_t^\beta f(x) = f(x)\phi_\beta(x)$, where $\phi_\beta(x) = \frac{x^{\beta-1}}{\Gamma\beta}$, for $x > 0$ and $\phi_\beta(x) = 0$, for $x \leq 0$. Note that $\phi_\beta(x) \rightarrow \delta(x)$ as $\beta \rightarrow 0$ in the distributional sense where $\phi_\beta(x) = 0$ is the delta distribution.

Definition 1.2. (see [6] and [8]) The fractional derivative D^α of order $\alpha \in (n-1, n)$, ($n = 1, 2, 3, \dots$) of the function $f(x)$ is given by

$$D_a^\alpha f(x) = I_a^{n-\alpha} D^n f(x), D = \frac{d}{dx}, \quad (2)$$

Some results in the theory of classical polynomials have been extended to fractional polynomials. For example, Laguerre polynomials and Bell polynomials have been

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introduced and studied in [1] and [2] respectively for fractional orders and parameters. In [10] Rida and El-Sayed introduced Laguerre polynomials of fractional order as follows:

Definition 1.3. Let $\alpha, \gamma \in (n - 1, n), n = 1, 2, 3, \dots$, and $\alpha, \beta \in \mathfrak{R}$. We define the generalized Rodrigues formula by the two functions

$$L_{\alpha}^{\beta}(\gamma, a; x) = \frac{e^{ax} x^{-\beta}}{\Gamma(\alpha + 1)} Y_{\alpha}^{\beta}(\gamma, a; x), \gamma + \beta > 0, \tag{3}$$

$$L_{-\alpha}^{\beta}(\gamma, a; x) = \frac{e^{ax} x^{-\beta}}{\Gamma(1 - \alpha)} Y_{-\alpha}^{\beta}(-\gamma, a; x), -\gamma + \beta > 0, \tag{4}$$

where

$$Y_{\alpha}^{\beta}(\gamma, a; x) = D^{\alpha} x^{\gamma + \beta} e^{-ax}, \gamma + \beta > 0, \tag{5}$$

and

$$Y_{-\alpha}^{\beta}(-\gamma, a; x) = I^{\alpha} x^{-\gamma + \beta} e^{-ax}, -\gamma + \beta > 0. \tag{6}$$

Based on the definitions of the fractional calculus mentioned above, we introduced now the following definition of two fractional (arbitrary) orders Laguerre polynomials.

Definition 1.4. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}$. Let

$$Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = D_z^{\nu} \left[e^{z(x-y)} D_z^{\alpha} (z^{\beta + \gamma + \delta} e^{-xz}) \right], \beta + \gamma + \delta > -1, \tag{7}$$

and

$$Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = I_z^{\nu} \left[e^{z(x-y)} I_z^{\alpha} (z^{\beta - \gamma - \delta} e^{-xz}) \right], \beta - \gamma - \delta > -1. \tag{8}$$

We define the Laguerre polynomials of double fractional orders by the formulas:

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{-\beta} e^{yz}}{\Gamma(\alpha + 1)\Gamma(\nu + 1)} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z), \beta + \gamma + \delta > -1, \tag{9}$$

and

$$L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{-\beta} e^{yz}}{\Gamma(1 - \alpha)\Gamma(1 - \nu)} Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z), \beta - \gamma - \delta > -1. \tag{10}$$

In view of the definitions 1.3 and 1.4, the polynomials $L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)$ and $L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)$ can be written in the following more compact forms

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{-\beta} e^{yz}}{\Gamma(\nu + 1)} D_z^{\nu} [e^{-yz} z^{\beta} L_{\alpha}^{\beta}(\gamma + \delta, x; z)], \tag{11}$$

and

$$L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{-\beta} e^{yz}}{\Gamma(1 - \nu)} I_z^{\nu} [e^{-yz} z^{\beta} L_{-\alpha}^{\beta}(\gamma + \delta, x; z)], \tag{12}$$

respectively. Also, for $y = x$ and by using definition 1.3 and the laws of exponents it is easily seen that

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, x; z) = \frac{\Gamma(\alpha + \nu + 1)}{\Gamma(\alpha + 1)\Gamma(\nu + 1)} L_{\alpha + \nu}^{\beta}(\gamma + \delta, x; z), \tag{13}$$

and

$$L_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, x; z) = \frac{\Gamma(1-\alpha-\nu)}{\Gamma(1-\alpha)\Gamma(1-\nu)} L_{-\alpha-\nu}^{\beta}(\gamma + \delta, x; z). \tag{14}$$

Further, according to the formulas (7) to (10) it may of interest to point out that the polynomials $L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$ and $L_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, y; z)$ have the following basic properties

$$L_{\alpha,\nu}^{\beta+2;\gamma,\delta}(x, y; z) = z^{-2} L_{\alpha,\nu}^{\beta;\gamma+1,\delta+1}(x, y; z), \tag{15}$$

$$L_{\alpha,\nu}^{\beta-2;\gamma,\delta}(x, y; z) = z^2 L_{\alpha,\nu}^{\beta;\gamma-1,\delta-1}(x, y; z), \tag{16}$$

$$L_{-\alpha,-\nu}^{\beta-2;\gamma,\delta}(x, y; z) = z^2 L_{-\alpha,-\nu}^{\beta;\gamma-1,\delta-1}(x, y; z), \tag{17}$$

$$L_{\alpha,\nu}^{\beta-1;\gamma,\delta}(x, y; z) = z L_{\alpha,\nu}^{\beta;\gamma-1,\delta}(x, y; z) = z L_{\alpha,\nu}^{\beta;\gamma,\delta-1}(x, y; z), \tag{18}$$

$$L_{-\alpha,-\nu}^{\beta-1;\gamma,\delta}(x, y; z) = z L_{-\alpha,-\nu}^{\beta;\gamma-1,\delta}(x, y; z) = z L_{-\alpha,-\nu}^{\beta;\gamma,\delta-1}(x, y; z), \tag{19}$$

$$L_{\alpha,\nu}^{\beta;1-\gamma,\delta}(x, y; z) = z L_{\alpha,\nu}^{\beta+1;-\gamma,\delta}(x, y; z) = L_{\alpha,\nu}^{\beta;-\gamma,\delta+1}(x, y; z). \tag{20}$$

Furthermore, from the properties of the fractional calculus and the formulas (7) and (8), we can easily prove the following lemma:

Lemma 1.1. Let $\alpha, \gamma, \eta, \nu, \delta, \tau \in (n - 1, n); n = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}$. Then

$$D_z^\eta Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) = Y_{\alpha,\nu+\eta}^{\beta;\gamma,\delta}(x, y; z) = D_z^\nu Y_{\alpha,\eta}^{\beta;\gamma,\delta}(x, y; z), \tag{21}$$

$$I_z^\tau Y_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, y; z) = Y_{-\alpha,-\nu-\tau}^{\beta;\gamma,\delta}(x, y; z) = I_z^\nu Y_{-\alpha,-\tau}^{\beta;\gamma,\delta}(x, y; z), \tag{22}$$

$$D_z^\eta Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y - z^{-1} D_z^\tau; z) = \sum_{k=0}^\infty \frac{1}{k!} Y_{\alpha+\tau k, \nu+\eta}^{\beta;\gamma,\delta}(x, y; z). \tag{23}$$

2. HYPERGEOMETRIC SERIES REPRESENTATIONS

First, we recall the definition of the following confluent hypergeometric function of two variables Φ_2 (see [11,p.25(17)]):

$$\Phi_2(a, b; c; x, y) = \sum_{m,n=0}^\infty \frac{(a)_m (b)_n x^m y^n}{(c)_{m+n} m! n!}, \tag{24}$$

where $(\lambda)_m = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}$, Γ : *Gamma function*.

Theorem 2.1. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in R$. Then

$$Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) = e^{-zy} z^{\beta+\gamma+\delta-\alpha-\nu} \sum_{p=0}^m \sum_{k=0}^n \binom{m}{p} \binom{n}{k} \frac{(-zx)^{n-k} (-zy)^{m-p} \Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - \alpha - \nu + m + n - k - p + 1)} \Phi_2[n - \alpha, m - \nu; \beta + \gamma + \delta - \alpha - \nu + m + n - k - p + 1; zx, zy], \tag{25}$$

$$Y_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, y; z) = e^{-zy} z^{\beta-\gamma-\delta+\alpha+\nu} \frac{\Gamma(\beta - \gamma - \delta + 1)}{\Gamma(\beta + \nu + \alpha - \gamma - \delta + 1)} \Phi_2[\alpha, \nu; \beta + \nu + \alpha - \gamma - \delta + 1; zx, zy]. \tag{26}$$

Proof.

From Leibnitz formula for fractional calculus and (7), we get

$$\begin{aligned}
 Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z) &= D_z^\nu \left[e^{z(x-y)} I_z^{n-\alpha} \left(\sum_{k=0}^n \binom{n}{k} (D_z^k z^{\beta+\gamma+\delta}) (D_z^{n-k} e^{-xz}) \right) \right] \\
 &= \sum_{k=0}^n \binom{n}{k} (-x)^{n-k} \frac{\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\beta+\gamma+\delta-k+1)} D_z^\nu \left[e^{(x-y)z} I_z^{n-\alpha} (z^{\beta+\gamma+\delta-k} e^{-xz}) \right]. \tag{27}
 \end{aligned}$$

On letting $n - \alpha = \sigma$, we get

$$I_z^\sigma (z^{\beta+\gamma+\delta-k} e^{-xz}) = \frac{e^{-xz}}{\Gamma(\sigma)} \int_0^z (z-s)^{\sigma-1} s^{\beta+\gamma+\delta-k} e^{(x-y)s} ds,$$

which on putting $x - z = xt$, gives us

$$\begin{aligned}
 I_z^\sigma (z^{\beta+\gamma+\delta-k} e^{-xz}) &= \frac{e^{-xz} z^{\beta+\gamma+\delta-k}}{\Gamma(\sigma)} \int_0^1 t^{\sigma-1} (1-t)^{\beta+\gamma+\delta-k} e^{xzt} dt. \\
 &= e^{-xz} z^{\beta+\gamma+\delta+\sigma-k} \sum_{p=0}^\infty \frac{\Gamma(\sigma+p)\Gamma(\beta+\gamma+\delta-k+1)(xz)^p}{\Gamma(\sigma)\Gamma(\beta+\gamma+\delta+\sigma+p-k+1)}. \tag{28}
 \end{aligned}$$

Now, substituting from (28) into (27), we get

$$\begin{aligned}
 Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z) &= \sum_{k=0}^n \binom{n}{k} \sum_{p=0}^\infty \frac{(-1)^{n-k} x^{n+p-k} (n-\alpha)_p \Gamma(\beta+\gamma+\delta+1)}{k! \Gamma(\beta+\gamma+\delta-\alpha+n+p-k+1)} \\
 &\quad I_z^{m-\nu} [D_z^m (z^{\beta+\gamma+\delta-\alpha+n+p-k} e^{-yz})]. \tag{29}
 \end{aligned}$$

Again, starting from (29) and exploiting the same procedure leading to (28) and employing the definition (24) of the function Φ_2 , one can derive the formula (25).

Similarly, by starting from (8) and using the fractional integral formula (1), we get

$$\begin{aligned}
 Y_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x,y;z) &= I_z^\nu \left[e^{(x-y)z} \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} s^{\beta-\gamma-\delta} e^{-xs} ds \right] \\
 &= I_z^\nu \left[\frac{e^{-yz} z^{\alpha+\beta-\gamma-\delta}}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(xz)^k}{k!} \int_0^t t^{\alpha+k-1} (1-t)^{\beta-\gamma-\delta} dt \right] \\
 &= \sum_{k=0}^\infty \frac{(xz)^k \Gamma(\alpha+k)\Gamma(\beta-\gamma-\delta+1)}{k! \Gamma(\alpha)\Gamma(\beta+\alpha-\gamma-\delta+k+1)} I_z^\nu [z^{\alpha+\beta+\gamma-\delta+k} e^{-yz}]. \tag{30}
 \end{aligned}$$

Next, by exploiting the same procedure leading to (30) and employing the definition (24) of the function Φ_2 , we can get the result (26). \square

In the same manner one can easily prove the following useful result.

Theorem 2.2. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}$. Then

$$\begin{aligned}
 Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z) &= e^{-zy} z^{\beta+\gamma+\delta-\alpha-\nu} \sum_{p=0}^{m+1} \sum_{k=0}^{n+1} \binom{m+1}{p} \binom{n+1}{k} \\
 &\quad \frac{(-zx)^{n-k+1} (-zy)^{m-p+1} \Gamma(\beta+\gamma+\delta+1)}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+3)} \\
 &\quad \Phi_2 [n - \alpha + 1, m - \nu + 1; \beta + \gamma + \delta - \alpha - \nu + m + n - k - p + 3; zx, zy]. \tag{31}
 \end{aligned}$$

Proof. We infer to the proof of Theorem 2.1.□

Next, in view of the definition of Kampé de Fériets double hypergeometric series [11,p.27(28)]:

$$F_{l;m;n}^{p;q;k} \left[\begin{matrix} (a_p) : (b_q); (c_k); \\ (d_l) : (e_m); (f_n); \end{matrix} \middle| x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s x^r y^s}{\prod_{j=1}^l (d_j)_{r+s} \prod_{j=1}^m (e_j)_r \prod_{j=1}^n (f_j)_s r! s!}, \tag{32}$$

we establish the following series representation for the functions $Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$ and $Y_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, y; z)$.

Theorem 2.3. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $|z(x - y)| < \infty, |-zx| < \infty, \beta \in \Re$. Then

$$Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) = \frac{z^{\beta+\gamma+\delta-\alpha-\nu} \Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - \alpha - \nu + 1)} F_{1;0;1}^{1;0;1} \left[\begin{matrix} \beta + \gamma + \delta - \alpha + 1 : --; \beta + \gamma + \delta + 1; \\ \beta + \gamma + \delta - \alpha - \nu + 1 : --; \beta + \gamma + \delta - \alpha + 1; \end{matrix} \middle| (x - y)z, -xz \right], \tag{33}$$

and

$$Y_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x, y; z) = \frac{z^{\alpha+\beta+\nu-\gamma-\delta} \Gamma(\beta - \gamma - \delta + 1)}{\Gamma(\alpha + \beta + \nu - \gamma - \delta + 1)} F_{1;0;1}^{1;0;1} \left[\begin{matrix} \alpha + \beta - \gamma - \delta + 1 : --; \beta - \gamma - \delta + 1; \\ \alpha + \beta + \nu - \gamma - \delta + 1 : --; \alpha + \beta - \gamma - \delta + 1; \end{matrix} \middle| (x - y)z, -xz \right]. \tag{34}$$

Proof. Starting from the definition of $Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$ expanding the exponential functions $e^{(x-y)z}$ and e^{-xz} in series, applying the elementary fractional derivation, we obtain

$$Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) = \sum_{k,r=0}^{\infty} \frac{(-1)^r (x - y)^k x^r \Gamma(\beta + \gamma + \delta + r + 1)}{k! r! \Gamma(\beta + \gamma + \delta + r - \alpha + 1)} \frac{\Gamma(\beta + \gamma + \delta + r + k - \alpha + 1)}{\Gamma(\beta + \gamma + \delta + r + k - \alpha - \nu + 1)} z^{\beta+\gamma+\delta+r+k-\alpha-\nu}. \tag{35}$$

Now, considering the definition of the double hypergeometric series (32), we get (33). The proof of (34) is similar to that of (33).□

According to the transformation formula [11,p.31(45)]

$$F_{1;0;1}^{1;0;1} \left[\begin{matrix} \alpha : --; \delta; \\ \beta : --; \mu; \end{matrix} \middle| x, -x \right] = {}_2F_2 \left[\begin{matrix} \alpha, \mu - \delta; \\ \beta, \mu; \end{matrix} \middle| x \right],$$

the formulas (33) and (34) can be rewritten in the following interesting forms:

$$Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, 2x; z) = \frac{z^{\beta+\gamma+\delta-\alpha-\nu} \Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - \alpha - \nu + 1)} {}_2F_2 \left[\begin{matrix} \beta + \gamma + \delta - \alpha + 1, -\alpha; \\ \beta + \gamma + \delta - \alpha - \nu + 1, \beta + \gamma + \delta - \alpha + 1; \end{matrix} \middle| xz \right], \tag{36}$$

and

$$Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, 2x; z) = \frac{z^{\alpha+\beta+\nu-\gamma-\delta}\Gamma(\beta-\gamma-\delta+1)}{\Gamma(\alpha+\beta+\nu-\gamma-\delta+1)} {}_2F_2 \left[\begin{matrix} \alpha+\beta-\gamma-\delta+1, \alpha; \\ \alpha+\beta+\nu-\gamma-\delta+1, \alpha+\beta-\gamma-\delta+1; \end{matrix} \quad xz \right]. \tag{37}$$

respectively.

The following results are an immediate consequence of Theorems 2.1, 2.2 and 2.3 respectively.

Corollary 2.1. Let $\alpha, \gamma \in (n-1, n), \nu, \delta \in (m-1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}$. Then

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{\gamma+\delta-\alpha-\nu}\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} \sum_{p=0}^m \sum_{k=0}^n \binom{m}{p} \binom{n}{k} \frac{(-zx)^{n-k}(-zy)^{m-p}}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1)} \times \Phi_2 [n-\alpha, m-\nu; \beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1; zx, zy], \tag{38}$$

$$L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{\alpha+\nu-\gamma-\delta}\Gamma(\beta-\gamma-\delta+1)}{\Gamma(1-\nu)\Gamma(1-\alpha)\Gamma(\beta+\nu+\alpha-\gamma-\delta+1)} \times \Phi_2 [\alpha, \nu; \beta+\nu+\alpha-\gamma-\delta+1; zx, zy]. \tag{39}$$

Corollary 2.2. Let $\alpha, \gamma \in (n-1, n), \nu, \delta \in (m-1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}$. Then

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{\gamma+\delta-\alpha-\nu}}{\Gamma(\alpha+1)\Gamma(\nu+1)} \sum_{p=0}^{m+1} \sum_{k=0}^{n+1} \binom{m+1}{p} \binom{n+1}{k} \frac{(-zx)^{n-k+1}(-zy)^{m-p+1}}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+3)} \Phi_2 [n-\alpha+1, m-\nu+1; \beta+\gamma+\delta-\alpha-\nu+m+n-k-p+3; zx, zy]. \tag{40}$$

Corollary 2.3. Let $\alpha, \gamma \in (n-1, n), \nu, \delta \in (m-1, m); n, m = 1, 2, 3, \dots$, and $|z(x-y)| < \infty, |xz| < \infty, \beta \in \mathfrak{R}$. Then

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{e^{-yz}z^{\gamma+\delta-\alpha-\nu}\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)\Gamma(\beta+\gamma+\delta-\alpha-\nu+1)} F_{1:0;1}^{1:0;1} \left[\begin{matrix} \beta+\gamma+\delta-\alpha+1 : --; \beta+\gamma+\delta+1; \\ \beta+\gamma+\delta-\alpha-\nu+1 : --; \beta+\gamma+\delta-\alpha+1; \end{matrix} \quad (x-y)z, -xz \right], \tag{41}$$

and

$$L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{e^{-yz}z^{\alpha+\nu-\gamma-\delta}\Gamma(\beta-\gamma-\delta+1)}{\Gamma(1-\alpha)\Gamma(1-\nu)\Gamma(\alpha+\beta+\nu-\gamma-\delta+1)} F_{1:0;1}^{1:0;1} \left[\begin{matrix} \alpha+\beta-\gamma-\delta+1 : --; \beta-\gamma-\delta+1; \\ \alpha+\beta+\nu-\gamma-\delta+1 : --; \alpha+\beta-\gamma-\delta+1; \end{matrix} \quad (x-y)z, -xz \right]. \tag{42}$$

Corollary 2.4. Let $x, y, \beta \in \mathfrak{R}$. Then

$$Y_{n,m}^{\beta;\gamma,\delta}(x, y; z) = \sum_{r=0}^m \sum_{k=0}^n \binom{m}{r} \binom{n}{k} e^{-yz} (-x)^{n-k} (-y)^{m-r} \frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - k - r + 1)} z^{\beta+\gamma+\delta-k-r}, \tag{43}$$

and

$$Y_{-n,-m}^{\beta;\gamma,\delta}(x, y; z) = e^{-yz} \frac{\Gamma(\beta - \gamma - \delta + 1)}{\Gamma(\beta - \gamma - \delta + n + m + 1)} z^{\beta-\gamma-\delta+n+m} \times \Phi_2 [n, m; \beta - \gamma - \delta + n + m + 1; zx, zy]. \tag{44}$$

Two special cases of formulas (31) and (42) respectively are of interest. Indeed, by letting $x = \alpha = \gamma = 0$ and $n + 1 = 0$, formulas (31) would reduces to the following known result (see[10,p.35(19)]):

$$Y_{\nu}^{\beta}(\delta, y; z) = e^{-zy} z^{\beta+\delta-\nu} \sum_{p=0}^{m+1} \binom{m+1}{p} \frac{(-zy)^{m-p+1} \Gamma(\beta + \delta + 1)}{\Gamma(\beta + \delta - \nu + m - p + 2)} {}_1F_1 [m - \nu + 1; \beta + \delta - \nu + m - p + 2; zy], \tag{45}$$

while, on letting $y = \gamma = 0$ and $m \mapsto 0$ the assertion (42) yields another known result [10, p.36(24)]

$$Y_n^{\beta}(\delta, x; z) = e^{-zy} \sum_{k=0}^n \binom{n}{k} \frac{(-x)^{n-k+1} \Gamma(\beta + \delta + 1)}{\Gamma(\beta + \delta - p + 1)} z^{\beta+\delta-k}. \tag{46}$$

3. CONTINUATION

Some continuity properties with respect to α and ν of the set of functions $Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$ and $L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$ are given by the following results.

Theorem 3.1. Let $x, y, \beta \in \mathfrak{R}$. Then

$$\lim_{\nu \rightarrow m^-} \left\{ \lim_{\alpha \rightarrow n^-} (Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)) \right\} = Y_{n,m}^{\beta;\gamma,\delta}(x, y; z) = \lim_{\nu \rightarrow m^+} \left\{ \lim_{\alpha \rightarrow n^+} (Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)) \right\}. \tag{47}$$

Proof. First let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$

Since

$$\Phi_2(a, b; c; x, y) = \sum_{n=0}^{\infty} \frac{(b)_n y^n}{(c)_n n!} {}_1F_1 [a; c + n; x],$$

and (see [11,p.322(183)])

$${}_1F_1 [a; c; x] = e^x {}_1F_1 [c - a; c; -x], \tag{48}$$

with the aide of the formula:

$$\lim_{\alpha \rightarrow n^-} \{ {}_1F_1 [n - \alpha; \beta + \gamma + \delta - \alpha - \nu + m + n + s - k - p + 1; -zx] \} = e^{-zx}. \tag{49}$$

gives us

$$\lim_{\alpha \rightarrow n^-} \{ Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) \} = e^{-zy} \sum_{k=0}^n \sum_{p=0}^m \binom{n}{k} \binom{m}{p} (-x)^{n-k} (-y)^{m-p}$$

$$\frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - \nu + m - k - p + 1)} z^{\beta + \gamma + \delta - \nu + m - k - p} {}_1F_1 [m - \nu; \beta + \gamma + \delta - \nu + m - k - p + 1; zy]. \tag{50}$$

Similarly, by starting from(50), taking the limit $\nu \mapsto m^-$ of both sides and making use of (49) and (43), we obtain

$$\lim_{\nu \mapsto m^-} \left\{ \lim_{\alpha \mapsto n^-} (Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)) \right\} = e^{-zy} \sum_{k=0}^n \sum_{p=0}^m \binom{n}{k} \binom{m}{p} (-x)^{n-k} (-y)^{m-p} \frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - k - p + 1)} z^{\beta + \gamma + \delta - k - p} = Y_{n, m}^{\beta; \gamma, \delta}(x, y; z). \tag{51}$$

Secondly, let $\alpha, \gamma \in (n, n + 1), \nu, \delta \in (m, m + 1); n, m = 1, 2, 3, \dots$ According to the formula (44), we can write (30) in the form

$$Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \sum_{k=0}^{m+1} \binom{m+1}{k} e^{(x-y)z} \sum_{p=0}^{\infty} \frac{(m - \nu + 1)_p \Gamma(\beta + \gamma + \delta + 1) (zy)^p (-zy)^{m-k+1}}{p! \Gamma(\beta + \gamma + \delta - \nu + m - k + p + 2)} Y_{\alpha}^{\beta + \delta - \nu + m - k + p + 1}(\gamma, x; z). \tag{52}$$

which on taking the limit $\alpha \mapsto n^+$ and employing the result [10,p.35(23)]

$$\lim_{\alpha \mapsto n^+} Y_{\alpha}^{\beta}(\gamma, a, x) = \sum_{r=0}^n \binom{m+1}{k} (-a)^{n-r} \frac{\Gamma(\beta + \gamma + 1) x^{\beta + \gamma - r}}{\Gamma(\beta + \gamma - r + 1)} e^{-ax}, \tag{53}$$

gives us

$$\lim_{\alpha \mapsto n^+} \{Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)\} = \sum_{r=0}^n \binom{n}{r} (-x)^{n-r} \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{(-y)^{m-\nu+1} \Gamma(\beta + \gamma + \delta + 1) e^{-yz}}{\Gamma(\beta + \gamma + \delta - \nu + m - k - r + 2)} {}_1F_1 [m - \nu + 1; \beta + \gamma + \delta - \nu + m + s - k - r + 2; zy]. \tag{54}$$

Again, starting from (53) and employing equations (44) and (53), we get

$$\lim_{\nu \mapsto m^+} \left\{ \lim_{\alpha \mapsto n^+} (Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)) \right\} = \sum_{r=0}^n \sum_{p=0}^m \binom{n}{r} \binom{m}{p} e^{-yz} (-x)^{n-r} (-y)^{m-p} \frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - r - p + 1)} z^{\beta + \gamma + \delta - k - r} = Y_{n, m}^{\beta; \gamma, \delta}(x, y; z). \tag{55}$$

Now combining (55) with (51) we get (47).□

Theorem 3.2. Let $x, y, \beta \in \mathfrak{R}$. Then

$$\lim_{\nu \mapsto m^-} \left\{ \lim_{\alpha \mapsto n^-} (Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)) \right\} = Y_{-n, -m}^{\beta; \gamma, \delta}(x, y; z) = \lim_{\nu \mapsto m^+} \left\{ \lim_{\alpha \mapsto n^+} (Y_{-\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)) \right\}. \tag{56}$$

Proof. From (26) and (37), we have

$$\lim_{\alpha \mapsto n^-} (Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)) = Y_{-n, -\nu}^{\beta; \gamma, \delta}(x, y; z).$$

Again

$$\lim_{\nu \rightarrow m^-} \left\{ \lim_{\alpha \rightarrow n^-} \left(Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = Y_{-n, -m}^{\beta; \gamma, \delta}(x, y; z). \quad (57)$$

In the same manner, we can show that

$$\lim_{\nu \rightarrow m^+} \left\{ \lim_{\alpha \rightarrow n^+} \left(Y_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = Y_{-n, -m}^{\beta; \gamma, \delta}(x, y; z). \quad (58)$$

Hence from (57) and (58), we obtain (56). \square

The following formulas are direct consequences of Theorems 3.1 and 3.2.

Corollary 3.1. Let $x, y, \beta \in \mathbb{R}$. Then

$$\lim_{\nu \rightarrow m^-} \left\{ \lim_{\alpha \rightarrow n^-} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = L_{n, m}^{\beta; \gamma, \delta}(x, y; z) = \lim_{\nu \rightarrow m^+} \left\{ \lim_{\alpha \rightarrow n^+} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\}. \quad (59)$$

$$\lim_{\nu \rightarrow m^-} \left\{ \lim_{\alpha \rightarrow n^-} \left(L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = L_{-n, -m}^{\beta; \gamma, \delta}(x, y; z) = \lim_{\nu \rightarrow m^+} \left\{ \lim_{\alpha \rightarrow n^+} \left(L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\}. \quad (60)$$

Next, from the properties of the fractional calculus, we can prove the following results.

Lemma 3.1. Let $\alpha, \nu, \gamma, \delta \in (0, 1)$ and $x, y, \beta \in \mathfrak{R}$. Then

$$\lim_{\nu \rightarrow 0^+} \left\{ \lim_{\alpha \rightarrow 0^+} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = z^{\gamma + \delta} = \lim_{\nu \rightarrow 0^-} \left\{ \lim_{\alpha \rightarrow 0^-} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\}. \quad (61)$$

and

$$\lim_{\nu \rightarrow 0^+} \left\{ \lim_{\alpha \rightarrow 0^+} \left(L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = z^{-\gamma - \delta} = \lim_{\nu \rightarrow 0^-} \left\{ \lim_{\alpha \rightarrow 0^-} \left(L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\}. \quad (62)$$

Proof. Since $D^0 = I^0 = I$, where I is the identity operator, we have

$$\lim_{\nu \rightarrow 0^+} \left\{ \lim_{\alpha \rightarrow 0^+} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = \lim_{\nu \rightarrow 0^+} \left\{ \frac{z^{-\beta} e^{yz}}{\Gamma(\nu + 1)} D_z^\nu [e^{-yz} z^{\beta + \gamma + \delta}] \right\} = z^{\gamma + \delta}.$$

Similarly

$$\lim_{\nu \rightarrow 0^-} \left\{ \lim_{\alpha \rightarrow 0^-} \left(L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right) \right\} = z^{\gamma + \delta}.$$

Hence, we have (61). In the same manner we can prove (62). \square

Corollary 3.2. Let $x = y = 1, \alpha = \gamma$ and $\nu = \delta$. Then

$$\lim_{\nu \rightarrow 0^+} \left\{ \lim_{\alpha \rightarrow 0^+} \left(L_{\gamma, \delta}^{\beta; \gamma, \delta}(1, 1; z) \right) \right\} = 1 = \lim_{\nu \rightarrow 0^-} \left\{ \lim_{\gamma \rightarrow 0^-} \left(L_{\gamma, \delta}^{\beta; \gamma, \delta}(1, 1; z) \right) \right\}, \quad (63)$$

and

$$\lim_{\nu \rightarrow 0^+} \left\{ \lim_{\gamma \rightarrow 0^+} \left(L_{-\gamma, -\delta}^{\beta; \gamma, \delta}(1, 1; z) \right) \right\} = 1 = \lim_{\delta \rightarrow 0^-} \left\{ \lim_{\gamma \rightarrow 0^-} \left(L_{-\gamma, \delta}^{\beta; \gamma, \delta}(1, 1; z) \right) \right\}. \quad (64)$$

Corollary 3.3. Let $\beta, x, y \in \mathfrak{R}$ and $\alpha, \gamma, \nu, \delta \in (0, 1)$. Then

$$\lim_{\alpha \rightarrow 0^+} \left\{ L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right\} = L_\nu^{\beta}(\gamma + \delta, y; z) = \lim_{\alpha \rightarrow 0^+} \left\{ L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \right\}, \quad (65)$$

and

$$\lim_{\nu \rightarrow 0^+} \{L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)\} = L_{\alpha}^{\beta}(\gamma + \delta, y; z) = \lim_{\nu \rightarrow 0^+} \{L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)\}. \tag{66}$$

4. RECURRENCE RELATIONS

First, we establish the following pure recurrence relations.

Theorem 4.1. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}, x > 0, y > 0$. Then

$$Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = (\beta + \gamma + \delta)Y_{\alpha-1, \nu}^{\beta-1; \gamma, \delta}(x, y; z) - xY_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y; z), \tag{67}$$

$$= (\beta + \gamma + \delta)Y_{\alpha, \nu-1}^{\beta-1; \gamma, \delta}(x, y; z) - yY_{\alpha, \nu-1}^{\beta; \gamma, \delta}(x, y; z), \tag{68}$$

$$Y_{\alpha+1, \nu}^{\beta; \gamma, \delta}(x, y; z) = (\beta + \gamma + \delta)Y_{\alpha, \nu}^{\beta-1; \gamma, \delta}(x, y; z) - xY_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z), \tag{69}$$

$$Y_{\alpha, \nu+1}^{\beta; \gamma, \delta}(x, y; z) = (\beta + \gamma + \delta)Y_{\alpha, \nu}^{\beta-1; \gamma, \delta}(x, y; z) - yY_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z), \tag{70}$$

$$Y_{\alpha-1, \nu+1}^{\beta; \gamma, \delta}(x, y; z) = Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) + (x - y)Y_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y; z), \tag{71}$$

$$Y_{\alpha+1, \nu-1}^{\beta; \gamma, \delta}(x, y; z) = (\beta + \gamma + \delta)Y_{\alpha, \nu-1}^{\beta-1; \gamma, \delta}(x, y; z) + xY_{\alpha, \nu-1}^{\beta; \gamma, \delta}(x, y; z), \tag{72}$$

$$Y_{\alpha+1, \nu+1}^{\beta; \gamma, \delta}(x, y; z) = xyY_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) - (x + y)(\beta + \gamma + \delta)Y_{\alpha, \nu}^{\beta-1; \gamma, \delta}(x, y; z) + (\beta + \gamma + \delta)(\beta + \gamma + \delta - 1)Y_{\alpha, \nu}^{\beta-2; \gamma, \delta}(x, y; z), \tag{73}$$

$$Y_{1-\alpha, 1-\nu}^{\beta; \gamma, \delta}(x, y; z) = xyY_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) - (x + y)(\beta - \gamma - \delta)Y_{-\alpha, -\nu}^{\beta-1; \gamma, \delta}(x, y; z) + (\beta - \gamma - \delta)(\beta - \gamma - \delta - 1)Y_{-\alpha, -\nu}^{\beta-2; \gamma, \delta}(x, y; z), \tag{74}$$

Proof.

(i) We have

$$Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = D_z^{\nu} [e^{z(x-y)} D_z^{\alpha-1} \{D_z (z^{\beta+\gamma+\delta} e^{-xz})\}] = -x D_z^{\nu} [e^{z(x-y)} D_z^{\alpha-1} (z^{\beta+\gamma+\delta} e^{-xz})] + (\beta + \gamma + \delta) D_z^{\nu} [e^{z(x-y)} D_z^{\alpha-1} (z^{\beta+\gamma+\delta-1} e^{-xz})],$$

which in view of (7) yields formula (67).

(ii) We have

$$Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = D_z^{\nu-1} \{D_z [e^{z(x-y)} D_z^{\alpha} (z^{\beta+\gamma+\delta} e^{-xz})]\} = D_z^{\nu-1} ((x - y)e^{z(x-y)} D_z^{\alpha} (z^{\beta+\gamma+\delta} e^{-xz})) - x D_z^{\nu-1} (e^{x-y} z D_z^{\alpha} (z^{\beta+\gamma+\delta} e^{-xz})) + (\beta + \gamma + \delta) D_z^{\nu-1} (e^{z(x-y)} D_z^{\alpha} (z^{\beta+\gamma+\delta-1} e^{-xz})),$$

which in view of (7) yields formula (68).

(iii) We have

$$Y_{\alpha+1, \nu}^{\beta; \gamma, \delta}(x, y; z) = D_z^{\nu} [e^{z(x-y)} D_z^{\alpha} \{D_z (z^{\beta+\gamma+\delta} e^{-xz})\}] = (\beta + \gamma + \delta) D_z^{\nu} [e^{z(x-y)} D_z^{\alpha} (z^{\beta+\gamma+\delta-1} e^{-xz})] - x D_z^{\nu} [e^{z(x-y)} D_z^{\alpha} (z^{\beta+\gamma+\delta} e^{-xz})],$$

which in view of (7) yields formula (69).

In the same manner as in the proof of assertions (67) to (69), one can prove the results (70) to (74). □

The following results are an immediate consequence of Theorem 4.1.

Corollary 4.1. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}, x > 0, y > 0$. Then

$$L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{(\beta + \gamma + \delta)}{2\alpha(\alpha - 1)} L_{\alpha-1, \nu}^{\beta-1; \gamma, \delta}(x, y; z) - \frac{x}{\alpha(\alpha - 1)} L_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y; z), \tag{75}$$

$$= \frac{(\beta + \gamma + \delta)}{\nu(\nu - 1)} L_{\alpha, \nu-1}^{\beta-1; \gamma, \delta}(x, y, z) - \frac{y}{\nu(\nu - 1)} L_{\alpha, \nu-1}^{\beta; \gamma, \delta}(x, y, z), \quad (76)$$

$$L_{\alpha+1, \nu}^{\beta; \gamma, \delta}(x, y, z) = \frac{(\beta + \gamma + \delta)}{z(\alpha + 1)} L_{\alpha, \nu}^{\beta-1; \gamma, \delta}(x, y, z) - \frac{x}{(\alpha + 1)} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z), \quad (77)$$

$$L_{\alpha, \nu+1}^{\beta; \gamma, \delta}(x, y, z) = \frac{(\beta + \gamma + \delta)}{2(\nu + 1)} L_{\alpha, \nu}^{\beta; \gamma, \delta-1}(x, y, z) - \frac{y}{(\nu + 1)} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z), \quad (78)$$

$$L_{\alpha-1, \nu+1}^{\beta; \gamma, \delta}(x, y, z) = \frac{1}{\alpha(\nu + 1)} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) + \frac{(x - y)}{(\nu + 1)} L_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y, z), \quad (79)$$

$$L_{\alpha+1, \nu-1}^{\beta; \gamma, \delta}(x, y, z) = \frac{(\beta + \gamma + \delta)}{(\alpha + 1)} L_{\alpha, \nu-1}^{\beta-1; \gamma, \delta}(x, y, z) + \frac{x}{(\alpha + 1)} L_{\alpha, \nu-1}^{\beta; \gamma, \delta}(x, y, z), \quad (80)$$

$$\begin{aligned} L_{\alpha+1, \nu+1}^{\beta; \gamma, \delta}(x, y, z) &= \frac{xy}{(\alpha + 1)(\nu + 1)} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) - \frac{(x + y)(\beta + \gamma + \delta)}{z(\alpha + 1)(\nu + 1)} \\ &\times L_{\alpha, \nu}^{\beta-1; \gamma, \delta}(x, y, z) + \frac{(\beta + \gamma + \delta)(\beta + \gamma + \delta - 1)}{z^2(\alpha + 1)(\nu + 1)} L_{\alpha, \nu}^{\beta-2; \gamma, \delta}(x, y, z), \end{aligned} \quad (81)$$

$$\begin{aligned} L_{1-\alpha, 1-\nu}^{\beta; \gamma, \delta}(x, y, z) &= \frac{xy}{(1 - \alpha)(1 - \nu)} L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y, z) - \frac{(x + y)}{z(1 - \alpha)(1 - \nu)} \\ &(\beta - \gamma - \delta) L_{-\alpha, -\nu}^{\beta-1; \gamma, \delta}(x, y, z) + \frac{(\beta - \gamma - \delta)(\beta - \gamma - \delta - 1)}{z^2(1 - \alpha)(1 - \nu)} L_{-\alpha, -\nu}^{\beta-2; \gamma, \delta}(x, y, z), \end{aligned} \quad (82)$$

We now wish to derive some differential recurrence relations

Theorem 4.2. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}, x > 0, y > 0$. Then

$$\begin{aligned} &\frac{\partial}{\partial x} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) \\ &= (\beta + \gamma + \delta - \alpha + 1) Y_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y, z) - x Y_{\alpha-1, \nu}^{\beta+1; \gamma, \delta}(x, y, z) - Y_{\alpha, \nu}^{\beta+1; \gamma, \delta}(x, y, z), \end{aligned} \quad (83)$$

$$\frac{\partial}{\partial y} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) = x Y_{\alpha-1, \nu}^{\beta+1; \gamma, \delta}(x, y, z) - (\beta + \gamma + \delta - \alpha + 1) Y_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y, z), \quad (84)$$

$$\begin{aligned} &z \frac{\partial}{\partial z} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) \\ &= (\beta + \gamma + \delta - \alpha - \nu) Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) - y Y_{\alpha, \nu}^{\beta+1; \gamma, \delta}(x, y, z) - \alpha(x - y) Y_{\alpha-1, \nu}^{\beta; \gamma, \delta}(x, y, z), \end{aligned} \quad (85)$$

$$Y_{\alpha+n, \nu}^{\beta; \gamma, \delta}(x, y, z) = \sum_{k=0}^n (-x)^{n-k} \frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - k + 1)} Y_{\alpha, \nu}^{\beta-k; \gamma, \delta}(x, y, z), \quad (86)$$

$$\begin{aligned} \frac{\partial^n}{\partial z^n} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) &= Y_{\alpha, \nu+n}^{\beta; \gamma, \delta}(x, y, z) = \sum_{k=0}^n \sum_{r=0}^k \binom{n}{k} \binom{n-k}{r} \\ &(x - y)^k (-x)^{n-k-r} \frac{\Gamma(\beta + \gamma + \delta + 1)}{\Gamma(\beta + \gamma + \delta - r + 1)} Y_{\alpha, \nu}^{\beta-k; \gamma, \delta}(x, y, z), \end{aligned} \quad (87)$$

Proof.

From (35), we obtain

$$\begin{aligned} &\frac{\partial}{\partial x} Y_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y, z) \\ &= \sum_{k, r=0}^{\infty} \frac{(-1)^r (x - y)^{k-1} x^r \Gamma(\beta + \gamma + \delta + r + 1) \Gamma(\beta + \gamma + \delta + r + k - \alpha + 1)}{(k - 1)! r! \Gamma(\beta + \gamma + \delta + r - \alpha + 1) \Gamma(\beta + \gamma + \delta + r + k - \alpha - \nu + 1)} \end{aligned}$$

$$\begin{aligned} & \times z^{\beta+\gamma+\delta+r+k-\alpha-\nu} \\ & + \sum_{k,r=0}^{\infty} \frac{(-1)^r (x-y)^k x^{r-1} \Gamma(\beta+\gamma+\delta+r+1) \Gamma(\beta+\gamma+\delta+r+k-\alpha+1)}{(k)!(r-1)! \Gamma(\beta+\gamma+\delta+r-\alpha+1) \Gamma(\beta+\gamma+\delta+r+k-\alpha-\nu+1)} \\ & \times z^{\beta+\gamma+\delta+r+k-\alpha-\nu}. \end{aligned}$$

The result (83) now follows from (35), upon letting $k \mapsto k + 1$ in the first summation and $r \mapsto r + 1$ in the second summation and using the formula $\lambda\Gamma(\lambda) = \Gamma(\lambda + 1)$ for the gamma function. By the same way, we can easily prove the formulas (84) and (85). Also, from the definition of $Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z)$, we have

$$\begin{aligned} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) &= D_z^\nu \left[e^{(x-y)z} D_z^\alpha \left\{ \sum_k^n \binom{n}{k} (D_z^k z^{\beta+\gamma+\delta}) (D_z^{n-k} e^{-zx}) \right\} \right] \\ &= \sum_k^n \binom{n}{k} (-x)^{n-k} \frac{\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\beta+\gamma+\delta-k+1)} D_z^\nu \left[e^{(x-y)z} D_z^\alpha (e^{-xz} z^{\beta+\gamma+\delta}) \right], \end{aligned}$$

from which we get (43). Similarly, one can prove the formulas (44) and (45).
The following results are an immediate consequence of Theorem 4.2.

Corollary 4.2. Let $\alpha, \gamma \in (n - 1, n), \nu, \delta \in (m - 1, m); n, m = 1, 2, 3, \dots$, and $x, y, \beta \in \mathfrak{R}, x > 0, y > 0$. Then

$$\begin{aligned} & \frac{\partial}{\partial x} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) \\ &= (\beta + \gamma + \delta - \alpha + 1) L_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y; z) - x L_{\alpha-1,\nu}^{\beta+1;\gamma,\delta}(x, y; z) - L_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y; z), \end{aligned} \tag{88}$$

$$\begin{aligned} & \frac{\partial}{\partial y} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) \\ &= x L_{\alpha-1,\nu}^{\beta+1;\gamma,\delta}(x, y; z) + y L_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y; z) - (\beta + \gamma + \delta - \alpha + 1) L_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y; z), \end{aligned} \tag{89}$$

$$\begin{aligned} & z \frac{\partial}{\partial z} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) \\ &= (\gamma + \delta - \alpha - \nu + yz) L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) - y L_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y; z) - \alpha(x-y) y L_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y; z), \end{aligned} \tag{90}$$

$$L_{\alpha+n,\nu}^{\beta;\gamma,\delta}(x, y; z) = \sum_{k=0}^n \binom{n}{k} (-x)^{n-k} \frac{\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\beta+\gamma+\delta-k+1)} L_{\alpha,\nu}^{\beta-k;\gamma,\delta}(x, y; z), \tag{91}$$

$$\begin{aligned} \frac{\partial^n}{\partial z^n} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) &= L_{\alpha,\nu+n}^{\beta;\gamma,\delta}(x, y; z) = \sum_{k=0}^n \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \\ & z^{k-n} y^{n-k-r} \frac{\Gamma(1-\beta)}{\Gamma(1-\beta-n+k)} \{ D_z^k L_{\alpha,\nu}^{\beta-k;\gamma,\delta}(x, y; z) \}, \end{aligned} \tag{92}$$

Proof.

We refer to the proof of Theorem 4.2. \square

Certain combinations from the recurrence relations of this section lead to further relations. For example, the subtraction of (78) from (77) yields

$$(x-y) L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y; z) = (\alpha+1) L_{\alpha+1,\nu}^{\beta;\gamma,\delta}(x, y; z) - (\nu+1) L_{\alpha,\nu+1}^{\beta;\gamma,\delta}(x, y; z). \tag{93}$$

Next, for $\alpha \mapsto \alpha - 1$ and $\beta \mapsto \beta + 1$, equation (69) reduces to

$$Y_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y, z) = (\beta + \gamma + \delta + 1)Y_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y, z) - xY_{\alpha-1,\nu}^{\beta+1;\gamma,\delta}(x, y, z). \quad (94)$$

Now, on using (93) in (85) and (86), we obtain

$$\frac{\partial}{\partial x} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = -\alpha Y_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y, z), \quad (95)$$

and

$$\frac{\partial}{\partial y} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = \alpha Y_{\alpha-1,\nu}^{\beta;\gamma,\delta}(x, y, z) - Y_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y, z), \quad (96)$$

respectively. Further, from (95) and (96), we get

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) + Y_{\alpha,\nu}^{\beta+1;\gamma,\delta}(x, y, z) = 0. \quad (97)$$

Furthermore, combining (85), (95) and (96), we obtain

$$\begin{aligned} z \frac{\partial}{\partial z} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - x \frac{\partial}{\partial x} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - y \frac{\partial}{\partial y} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ - (\beta + \gamma + \delta - \alpha - \nu) Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0. \end{aligned} \quad (98)$$

From assertion (98) above, we have easily the following result.

Theorem 4.3. The function $Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z)$ is a particular solution of the differential equations

$$\begin{aligned} z \frac{\partial^2}{\partial z^2} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - x \frac{\partial^2}{\partial x \partial z} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - y \frac{\partial^2}{\partial y \partial z} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ - (\beta + \gamma + \delta - \alpha - \nu) \frac{\partial}{\partial z} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0, \end{aligned} \quad (99)$$

$$\begin{aligned} x \frac{\partial^2}{\partial x^2} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - z \frac{\partial^2}{\partial z \partial x} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) + y \frac{\partial^2}{\partial y \partial x} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ + (\beta + \gamma + \delta - \alpha - \nu) \frac{\partial}{\partial x} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0, \end{aligned} \quad (100)$$

$$\begin{aligned} y \frac{\partial^2}{\partial y^2} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - z \frac{\partial^2}{\partial z \partial y} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) + x \frac{\partial^2}{\partial x \partial y} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ + (\beta + \gamma + \delta - \alpha - \nu) \frac{\partial}{\partial y} Y_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0. \end{aligned} \quad (101)$$

Theorem 4.4. The function $L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z)$ is a particular solution of the differential equations

$$\begin{aligned} z \frac{\partial^2}{\partial z^2} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - x \frac{\partial^2}{\partial x \partial z} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - y \frac{\partial^2}{\partial y \partial z} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ - (\gamma + \delta - \alpha - \nu) \frac{\partial}{\partial z} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0, \end{aligned} \quad (102)$$

$$\begin{aligned} x \frac{\partial^2}{\partial x^2} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - z \frac{\partial^2}{\partial z \partial x} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) + y \frac{\partial^2}{\partial y \partial x} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) \\ + (\gamma + \delta - \alpha - \nu) \frac{\partial}{\partial x} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) = 0, \end{aligned} \quad (103)$$

$$y \frac{\partial^2}{\partial y^2} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) - z \frac{\partial^2}{\partial z \partial y} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z) + x \frac{\partial^2}{\partial x \partial y} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x, y, z)$$

$$+(\gamma + \delta - \alpha - \nu) \frac{\partial}{\partial y} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = 0. \tag{104}$$

Proof. From (1.9) into (4.31), we obtain

$$\begin{aligned} z \frac{\partial}{\partial z} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) - x \frac{\partial}{\partial x} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) - y \frac{\partial}{\partial y} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) \\ - (\gamma + \delta - \alpha - \nu) L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) = 0. \end{aligned} \tag{105}$$

Now, from (105) we obtain Theorem 4.4.□

5. OPERATIONAL IMAGES AND GENERATING FUNCTIONS

First, we obtain the following fractional images:

Theorem 5.1 The following operational images for $L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)$ and $L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)$ hold true:

$$\begin{aligned} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) &= \frac{z^{-\beta} (-x)^n (-y)^m}{\Gamma(\alpha + 1) \Gamma(\nu + 1)} D_z^{\alpha + \nu - m - n} \left(\frac{1 - x^{-1} D_z}{1 - x D_z^{-1}} \right)^n \left(\frac{1 - y^{-1} D_z}{1 - y D_z^{-1}} \right)^m \\ &\quad \times (1 - x D_z^{-1})^\alpha (1 - y D_z^{-1})^\nu \{z^{\beta + \gamma + \delta}\}, \end{aligned} \tag{106}$$

$$L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) = \frac{z^{-\beta} D_z^{-(\alpha + \nu)}}{\Gamma(1 - \alpha) \Gamma(1 - \nu)} (1 - x D_z^{-1})^{-\alpha} (1 - y D_z^{-1})^{-\nu} \{z^{\beta - \gamma - \delta}\}. \tag{107}$$

Proof. To proof (106), let Ω denote the left-hand side of assertion (106), then in view of the binomial expansion

$$(1 - x)^\lambda = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} x^k,$$

one gets

$$\begin{aligned} \Omega &= \frac{z^{-\beta}}{\Gamma(\alpha + 1) \Gamma(\nu + 1)} \sum_{r=0}^m \sum_{k=0}^n \frac{(-1)^{m+n} (-m)_r (-n)_k}{r! k!} y^{m-r} x^{n-k} \\ &\quad \sum_{p, q=0}^{\infty} \frac{(-1)^{m+n} (m - \nu)_p (n - \alpha)_q}{p! q!} y^p x^q D_z^{\beta + \gamma + \delta - \alpha - \nu + p + q + m + n - k - r} \{z^{\beta + \gamma + \delta}\}, \end{aligned}$$

which on applying the fractional derivative formula $D^\lambda x^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \lambda + 1)} x^{\alpha - \lambda}$, using the result

$$(-s)_k = \frac{(-1)^k s!}{(s - k)!},$$

and considering the series representation (25) and definition (9) yields the left hand-side of assertion (106). The proof of (107) is similar to that of (106).□

Next, we derive generating relations for the functions $L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)$ and $L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)$.

Theorem 5.2 The following generating functions for $L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)$ and $L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)$ hold true:

$$\frac{z^{-\beta} D_z^{\alpha + \nu}}{\Gamma(\alpha + 1) \Gamma(\nu + 1)} {}_0F_1 \left[\begin{matrix} -; \\ \alpha + 1; \end{matrix} \middle| -ux(1 - x^{-1}D) \right] {}_0F_1 \left[\begin{matrix} -; \\ \nu + 1; \end{matrix} \middle| -ty(1 - y^{-1}D) \right]$$

$$\times (1 - x^{-1}D)^\alpha (1 - y^{-1}D)^\nu \{z^{\beta+\gamma+\delta}\} = \sum_{n,m=0}^{\infty} L_{\alpha+n,\nu+m}^{\beta;\gamma,\delta}(x,y,z) \frac{u^n t^m}{n!m!}, \quad (108)$$

$$\begin{aligned} & \frac{z^{-\beta}}{\Gamma(\alpha+1)\Gamma(\nu+1)} D_z^{\alpha+\nu} \exp \left[- \left(\frac{x D^{-1}(1-x^{-1}D)}{(1-xD^{-1})} + \frac{y D^{-1}(1-y^{-1}D)}{(1-yD^{-1})} \right) \right] \\ & \times (1 - x^{-1}D)^\alpha (1 - y^{-1}D)^\nu \{z^{\beta+\gamma+\delta}\} \\ & = \sum_{n,m=0}^{\infty} L_{\alpha,\nu}^{\beta-n-m;\gamma+m,\delta+m}(x,y,z) \frac{(u/z)^n (t/z)^m}{n!m!}, \end{aligned} \quad (109)$$

$$\begin{aligned} & \frac{z^{-\beta}(1-xD_z^{-1})^{-\alpha}(1-yD_z^{-1})^{-\nu}}{\Gamma(1-\alpha)\Gamma(1-\nu)} \\ & {}_0F_1 \left[\begin{matrix} -; \\ 1-\alpha; \end{matrix} \middle| u(D_z(1-xD^{-1})) \right] {}_0F_1 \left[\begin{matrix} -; \\ 1-\nu; \end{matrix} \middle| t(D_z(1-yD^{-1})) \right] \\ & = \sum_{n,m=0}^{\infty} L_{-\alpha+n,-\nu+m}^{\beta;\gamma,\delta}(x,y,z) \frac{u^n t^m}{n!m!}, \end{aligned} \quad (110)$$

$$\begin{aligned} & \frac{z^{-\beta} D_z^{-(\alpha+\nu)}}{\Gamma(1-\alpha)\Gamma(1-\nu)} (1-xD^{-1}+uD^{-1})^{-\alpha} (1-yD^{-1}+tD^{-1})^{-\nu} \{z^{\beta-\gamma-\delta}\} \\ & = \sum_{n,m=0}^{\infty} L_{-\alpha-n,-\nu-m}^{\beta;\gamma,\delta}(x,y,z) \frac{u^n t^m}{n!m!}, \end{aligned} \quad (111)$$

Proof. By starting from (106) replacing α and ν by $\alpha+n$ and $\nu+m$ respectively, multiplying both sides by $\frac{u^n t^m}{n!m!}$ and then taking the double sum, we obtain (108). By following similar procedure we can proof (109), (110) and (111). \square

Now, according to the fact that [11,p.50(11) and p.325(203)]

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k! \Gamma(k+p+1)} \quad \text{and} \quad I_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{k! \Gamma(k-p+1)},$$

where $J_p(x)$ is Bessel functions of non-integral order $p, p \geq 0$ and $I_{-p}(x)$ is the modified Bessel functions of first kind of negative order $p, p > 0$ and in view of Theorem 5.2, we get the following generating cases.

Corollary 5.1. The following generating functions for $L_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y,z)$ and $L_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x,y,z)$ hold true:

$$\begin{aligned} & z^{-\beta} D^{\beta+\nu} J_\alpha(2\sqrt{ux(1-x^{-1}D)}) J_\nu(2\sqrt{ty(1-y^{-1}D)}) \\ & \times \left[\frac{(1-x^{-1}D)}{\sqrt{ux(1-x^{-1}D)}} \right]^\alpha \left[\frac{(1-y^{-1}D)}{\sqrt{ty(1-y^{-1}D)}} \right]^\nu \{z^{\beta+\gamma+\delta}\} \\ & = \sum_{n,m=0}^{\infty} L_{\alpha+n,\nu+m}^{\beta;\gamma,\delta}(x,y,z) \frac{u^n t^m}{n!m!}, \end{aligned} \quad (112)$$

$$\begin{aligned} & z^{-\beta} I_{-\alpha}(2\sqrt{ux(1-x^{-1}D)}) \times I_{-\nu}(2\sqrt{ty(1-y^{-1}D)}) \{(\sqrt{u})^\alpha (\sqrt{t})^\nu z^{\beta+\gamma+\delta}\} \\ & = \sum_{n,m=0}^{\infty} L_{-\alpha+n,-\nu+m}^{\beta;\gamma,\delta}(x,y,z) \frac{u^n t^m}{n!m!}, \end{aligned} \quad (113)$$

Proof. From the definitions of $J_\nu(x)$ and $I_{-\nu}(x)$ and the assertions (108) and (110), we get the result. \square

6. INTEGRAL TRANSFORMS

By using the Eulerian integral of second kind (see e.g. [3]):

$$a^{-z}\Gamma(z) = \int_0^\infty e^{-at}t^{z-1}dt, \Re(z) > 0, \Re(a) > 0, \tag{114}$$

we first derive the following Laplace transform formulas

Theorem 6.1. Let $\Re(\sigma) > 0, \Re(a) > 0$. Then

$$\int_0^\infty e^{-az}z^{\sigma-1}L_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z)dz = \frac{a^{-(\sigma+\gamma+\delta-\alpha-\nu)}\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} \\ \sum_p^m \sum_k^n \binom{m}{p} \binom{n}{k} \frac{\Gamma(\sigma+\gamma+\delta-\alpha-\nu+m+n-p-k)}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-p-k+1)} \left(\frac{-x}{a}\right)^{n-k} \left(\frac{-y}{a}\right)^{m-p} \\ \times F_1 \left[\sigma+\gamma+\delta-\alpha-\nu+m+n-p-k, n-\alpha, m-\nu \right. \\ \left. ; \beta+\gamma+\delta-\alpha-\nu+m+n-p-k+1; \frac{x}{a}, \frac{y}{a} \right], \tag{115}$$

$$\max \{ |x/a|, |y/a| \} < 1;$$

$$\int_0^\infty e^{-az}z^{\sigma-1}L_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x,y;z)dz = \frac{a^{(\sigma-\gamma-\delta+\alpha+\nu)}\Gamma(\beta-\gamma-\delta+1)\Gamma(\sigma-\gamma-\delta+\alpha+\nu)}{\Gamma(1-\alpha)\Gamma(1-\nu)\Gamma(\beta-\gamma-\delta+\alpha+\nu+1)} \\ \times F_1 \left[\sigma-\gamma-\delta+\alpha+\nu, \alpha, \nu; \beta-\gamma-\delta+\alpha+\nu+1; \frac{x}{a}, \frac{y}{a} \right], \tag{116}$$

$$\max \{ |x/a|, |y/a| \} < 1;$$

where F_1 is Appell's function [11,p.22(2)].

Proof. Denote for convenience, the left -hand side of assertion (115) by I . Then in view of (38) it is easily seen that:

$$I = \frac{\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} \sum_{p=0}^m \sum_{k=0}^n \binom{m}{p} \binom{n}{k} \frac{(-x)^{n-k}(-y)^{m-p}}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1)} \\ \times \sum_{r,s=0}^\infty \frac{(n-\alpha)_r(m-\nu)_s x^r y^s}{r!s!(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1)_{r+s}} \\ \times \int_0^\infty e^{-az}z^{\sigma+\gamma+\delta-\alpha-\nu+m+n-k-p+r+s} dz.$$

Upon using the integral formula (114) and considering the definition of Appell's function F_1 [11,p.22(2)], we led finally to the right-hand side of formula (115). In the light of the expressions (39), it is equally straightforward in the same manner to derive the formula (116). \square

By setting $\sigma = \beta + 1$ equations (115) and (116) would reduce to the following interesting special cases:

$$\int_0^\infty e^{-az}z^\beta L_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z)dz = \frac{a^{-(\beta+\gamma+\delta+1)}\Gamma(\beta+\gamma+\delta+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} (a-x)^\alpha (a-y)^\nu, \tag{117}$$

and

$$\int_0^\infty e^{-az} z^\beta L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) dz = \frac{a^{(\beta-\gamma-\delta+1)} \Gamma(\beta-\gamma-\delta+1)}{\Gamma(1-\alpha)\Gamma(1-\nu)} (a-x)^{-\alpha} (a-y)^{-\nu}, \tag{118}$$

respectively. Another particular case of (116) would occur when $a = y$ and make use of the transformation formula

$$F_1 [a, b, c; e; x, 1] = \frac{\Gamma(e)\Gamma(e-a-c)}{\Gamma(e-a)\Gamma(e-c)} {}_2F_1 [a, b; e-c; x].$$

Thus, we get

$$\begin{aligned} & \int_0^\infty e^{-yz} z^{\sigma-1} L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) dz \\ &= \frac{y^{(\sigma-\gamma-\delta+\alpha+\nu)} \Gamma(\beta-\gamma-\delta+1) \Gamma(\sigma-\gamma-\delta+\alpha+\nu) \Gamma(\beta-\sigma-\nu+1)}{\Gamma(1-\alpha)\Gamma(1-\nu)\Gamma(\beta-\gamma-\delta+\alpha+\nu+1)\Gamma(\beta-\sigma+1)} \\ & \quad \times {}_2F_1 \left[\alpha, \sigma-\gamma-\delta+\alpha+\nu; \beta-\gamma-\delta+\alpha+1; \frac{x}{y} \right], \end{aligned} \tag{119}$$

where ${}_2F_1$ is Gaussian hypergeometric series [11,p.18(17)]. Next, we establish the following double integral transforms for the functions $L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z)$ and $L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z)$ in the term of Appell's function F_3 [11,p.23(4)].

Theorem 6.2. Let $\Re(\sigma) > 0, \Re(\lambda) > 0, \Re(a) > 0, \Re(b) > 0$. Then

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-(ax+by)} x^{\sigma-1} y^{\lambda-1} L_{\alpha, \nu}^{\beta; \gamma, \delta}(x, y; z) dx dy = \\ & \quad \frac{a^{-\sigma} b^{-\lambda} \Gamma(\beta+\gamma+\delta+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} z^{\gamma+\delta-\alpha-\nu} \\ & \quad \sum_p^m \sum_k^n \binom{m}{p} \binom{n}{k} \frac{\Gamma(\sigma+n-k)\Gamma(\lambda+m-p)(-z/a)^{n-k}(-z/b)^{m-p}}{\Gamma(\beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1)} \\ & \times F_3 \left[n-\alpha, \sigma+n-k, m-\nu, \lambda+m-p; \beta+\gamma+\delta-\alpha-\nu+m+n-k-p+1; \frac{z}{a}, \frac{z}{b} \right], \end{aligned} \tag{120}$$

$$\begin{aligned} & \max \{ |z/a|, |z/b| \} < 1; \\ & \int_0^\infty \int_0^\infty e^{-(ax+by)} x^{\sigma-1} y^{\lambda-1} L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) dx dy = \\ & \quad \frac{a^{-\sigma} b^{-\lambda} \Gamma(\beta-\gamma-\delta+1) \Gamma(\sigma) \Gamma(\lambda)}{\Gamma(\beta+\alpha+\nu-\gamma-\delta+1) \Gamma(1-\alpha) \Gamma(1-\nu)} z^{\alpha+\nu-\gamma-\delta} \\ & \quad \times F_3 \left[\alpha, \sigma, \nu, \lambda; \beta-\gamma-\delta+\alpha+\nu+1; \frac{z}{a}, \frac{z}{b} \right]. \end{aligned} \tag{121}$$

$$\max \{ |z/a|, |z/b| \} < 1;$$

Proof. We refer to the proof of Theorem 6.1.

Now, projection integral transforms would occur if we use the definition of fractional integrals (1) and this asserts.

Theorem 6.3. Let $\Re(\lambda) > 0$. Then

$$\frac{t^{\lambda+\beta} L_{-\alpha, -\nu}^{\lambda+\beta; \gamma, \delta}(x, y; t)}{\Gamma(\lambda+\beta-\gamma-\delta+1)} = \frac{1}{\Gamma(\lambda)} \int_0^t \frac{(t-z)^{\lambda-1} z^\beta}{\Gamma(\beta-\gamma-\delta+1)} L_{-\alpha, -\nu}^{\beta; \gamma, \delta}(x, y; z) dz. \tag{122}$$

$$\frac{t^{\lambda+\beta} L_{\alpha,\nu}^{\lambda+\beta;\gamma,\delta}(x,y;t)}{\Gamma(\lambda+\beta+\gamma+\delta+1)} = \frac{1}{\Gamma(\lambda)} \int_0^t \frac{(t-z)^{\lambda-1} z^\beta}{\Gamma(\beta+\gamma+\delta+1)} L_{\alpha,\nu}^{\beta;\gamma,\delta}(x,y;z) dz, \quad (123)$$

Proof. From (39), we have

$$\begin{aligned} & \frac{1}{\Gamma(\lambda)} \int_0^t (t-z)^{\lambda-1} z^\beta L_{-\alpha,-\nu}^{\beta;\gamma,\delta}(x,y;z) dz \\ &= \frac{\Gamma(\beta-\gamma-\delta+1)}{\Gamma(\beta-\gamma-\delta+\alpha+\nu+1)\Gamma(1-\alpha)\Gamma(1-\nu)} \sum_{r,s=0}^{\infty} \frac{(\alpha)_r (\nu)_s x^r y^s}{r!s!(\beta+\alpha+\nu-\gamma-\delta+1)_{r+s}} \\ & \quad \times \int_0^t (t-z)^{\lambda-1} z^{\beta-\gamma-\delta+\alpha+\nu+r+s} dz. \end{aligned} \quad (124)$$

putting $t-z = t(1-p)$, we get

$$\begin{aligned} & \int_0^t (t-z)^{\lambda-1} z^{\beta-\gamma-\delta+\alpha+\nu+r+s} dz \\ &= t^{\lambda+\beta-\gamma-\delta+\alpha+\nu+r+s} \int_0^1 (1-p)^{\lambda-1} p^{\beta-\gamma-\delta+\alpha+\nu+r+s} dp. \end{aligned} \quad (125)$$

Now, by substituting (125) into (124) and applying the definition of Beta function

$$\int_0^1 (1-s)^{y-1} s^{x-1} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we get (6.9). The proof of (123) is similar to that of (122). \square

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