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INTEGRALS INVOLVING JACOBI POLYNOMIALS AND M-SERIES

D.L. SUTHAR, HAGOS TADESSE AND KELELAW TILAHUN

Abstract. In this paper, we establish certain integrals formula involving generalized Jacobi polynomials associated with M-series. The results are presented in generalized Gauss hypergeometric function and beta function. Some out of the ordinary special cases of our main result are also considered and shown to be associated with certain known ones.

1. Introduction

Recently, Sharma and Jain [16] introduced the generalized M-series as the function and defined in term of power series:

$$
\begin{aligned}
{}_{p}^{\vartheta}M_{q}^{\tau}(z) &= {}_{p}^{\vartheta}M_{q}^{\tau}(a_{1,\ldots,}a_{p}; b_{1,\ldots,}b_{q}; z) \\
&= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{\Gamma(\vartheta n + \tau)}, \quad z, \vartheta, \tau \in \mathbb{C}, \Re(\vartheta) > 0. \qquad (1)
\end{aligned}
$$

where $(a_j)_n$ and $(b_j)_n$ are known as Pochhammer symbols. The series terminates to a polynomial in z, if any numerator parameter a_j is a negative integer or zero. The series (1) is defined when none of parameters $b_j's, j = 1, 2, \ldots q$; is a negative integer or zero. From the ratio test it is evident that the series is convergent for all z if $p \leq q$. It is convergent for $|z| < \delta$ the series can converge on conditions depending on the parameters (see Kilbas et al. [1]). The summation of the convergent series is denoted by the symbol ${}_{p}^{\vartheta}M_{q}^{\tau}(.)$, therefore, throughout the residual work we will called this generalized M-series as generalized M-function.

Some key special cases of generalized M-function are itemized below:

(i) For $\tau = 1$, the M-series from Sharma ([15], p.188. eq.(3)) as:

$$
{}_{p}^{\vartheta}M_{q}(z) = {}_{p}^{\vartheta}M_{q}^{1}(a_{1,\ldots,}a_{p}; b_{1,\ldots,}b_{q}; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{\Gamma(\vartheta\, n + 1)}, \qquad (2)
$$

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(ii) For $\vartheta = \tau = 1$, the generalized M-function is the generalized hypergeometric function:

$$ {}_pF_q \left[\begin{matrix} a_{1,\ldots,}a_p \\ b_{1,\ldots,}b_q \end{matrix} ; z \right] = {}^{1}_{p}M^{1}_{q} \left(a_{1,\ldots,}a_p ; b_{1,\ldots,}b_q ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{3} $$

(iii) When $\vartheta = \tau = 1$, and $p = q = 0$, then we find Mittag-Leffler function [6]:

$$ E(z) = {}^{1}_{0}M^{1}_{0} (-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)}, (\vartheta > 0) \tag{4} $$

(iv) Again, for $p = q = 0$, then we get two index Mittag-Leffler function introduced by Wiman [18]:

$$ E_{\vartheta,\tau}(z) = {}^{\vartheta}_{0}M^{\tau}_{0} (-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\vartheta n + \tau)}, (\vartheta > 0, \tau > 0) \tag{5} $$

(v) Further, If we put $p = q = 1, a_1 = \gamma, b_1 = 1$, the generalized M-function reduces to generalized Mittag-Leffler introduced by Prabhakar [7] as below:

$$ E^{\gamma}_{\vartheta,\tau}(z) = {}^{\vartheta}_{1}M^{\tau}_{1} (\gamma;1;z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\vartheta n + \tau)} \frac{z^n}{n!}, \tag{6} $$

where $\vartheta, \tau, \gamma \in \mathbb{C}$; $\Re(\vartheta) > 0$, $\Re(\tau) > 0$, $\Re(\gamma) > 0$.

A number of properties and applications of M-Series (1) and its special cases have studied by Kilbas *et al.*[1], Miller [4], Mishra *et al.* [5], Purohit *et al.* [8], Saigo and Kilbas [10], Saxena *et al.* [12]. Generalized M-function plays an important role in the solution of integral equations and fractional order differential.

For the present investigation, we also consider the following required definitions:

The generalized Jacobi polynomial $P_{\upsilon}^{(\alpha,\beta,c,d)}$ studied by Sarabia and Kalla ([11], p. 386) (see also [13]), as:

$$ P_{\upsilon}^{(\alpha,\beta,c,d)}(x) = \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)} {}_3F_2 \left[\begin{matrix} -\upsilon, \upsilon+\alpha+\beta+1, c; \\ \alpha+1, d; \end{matrix} \quad \frac{1-x}{2} \right], \tag{7} $$

where $d \in \mathbb{C} - \mathbb{Z}^- \cup \{0\} : \alpha, \upsilon \in \mathbb{C} - \mathbb{Z}^-$; $\beta \in \mathbb{C}; \Re(d - \beta - c) > 0$.

For $c = d$, the generalized Jacobi polynomial reduces in to Jacobi polynomial $P_{\upsilon}^{(\alpha,\beta)}(x)$ which is defined in Rainville [9] as:

$$ P_{\upsilon}^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_{\upsilon}}{\upsilon!} {}_2F_1 \left[\begin{matrix} -\upsilon, \upsilon+\alpha+\beta+1; \\ \alpha+1; \end{matrix} \quad \frac{1-x}{2} \right], \tag{8} $$

If we substitute $\alpha = \beta = 0$ the Jacobi polynomial in (8), reduces to the Legendre polynomial [9].

For the value $x = 1$ in equation (8), $P_{\upsilon}^{(\alpha,\beta)}(x)$ is also a polynomial of degree n. that is

$$ P_{\upsilon}^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_{\upsilon}}{\upsilon!}. \tag{9} $$

In the Jacobi function given above, we have to apply the concept of Gauss hypergeometric functions by Rainville ([9], p.45).

In the present paper, we established integrals with M-Series, add one more dimension to this study by introducing certain integral for the generalized Jacobi polynomials. The integral established in this paper are believed to be a new contribution in the theory of fractional calculus.

2. Main Integral formulas

In this section, we establish four general integrals, which are expressed in terms of generalized hypergeometric function by inserting product of M-series (1) and Jacobi polynomial (7) with suitable arguments into the integrand.

Lemma: Let $\alpha, \upsilon \in \mathbb{C} - \mathbb{Z}^{-}$;$\beta \in \mathbb{C}$;$\Re(d-\beta-c) > 0$, $P \in \mathbb{C} - \mathbb{Z}^{-} \cup \{0\}$, then the following integral holds true.

$$\int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x)dx$$

$$= 2^{\lambda+\delta+1} \frac{(\alpha+1)_{\upsilon} B(\lambda+1,\delta+1)}{\Gamma(\upsilon+1)} {}_4F_3 \left[\begin{array}{c} -\upsilon, \upsilon+\alpha+\beta+1, c, \lambda+1; \\ \\ \alpha+1, d, \lambda+\delta+2; \end{array} 1 \right]. \quad (10)$$

Proof. Let ℓ be the left-hand side of (10), using (7), we have

$$\ell = \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x)dx$$

$$= \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)} \sum_{s=0}^{\infty} \frac{(-\upsilon)_s (\upsilon+\alpha+\beta+1)_s (c)_s}{(\alpha+1)_s (d)_s 2^s s!} \int_{-1}^{+1} (1-x)^{\lambda+s} (1+x)^{\delta} dx, \quad (11)$$

Now, using the known result by Rainville ([9], p.261)

$$\int_{-1}^{+1} (1-x)^{\alpha+n} (1+x)^{\beta+n} dx = 2^{2n+\alpha+\beta+1} B(\alpha+n+1, \beta+n+1), \quad (12)$$

We arrive at

$$= 2^{\lambda+\delta+1} \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)} \sum_{s=0}^{\infty} \frac{(-\upsilon)_s (\upsilon+\alpha+\beta+1)_s (c)_s}{(\alpha+1)_s (d)_s s!} \frac{\Gamma(\lambda+s+1)\Gamma(\delta+1)}{\Gamma(\lambda+\delta+s+2)}, \quad (13)$$

Interpreting the right-hand side of (13), in view of the definition (3), we obtain the result (10). □

Now, we establish four general integrals as follows:

First Integral:

$$I_1 = \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) {}_p^{\vartheta}M_q^{\tau} \left[z (1+x)^h \right] dx$$

$$= 2^{\lambda+\delta+1} \frac{(\alpha+1)_{\upsilon}\Gamma(\lambda+1)}{\Gamma(\upsilon+1)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+hk+1)}{\Gamma(\lambda+\delta+hk+2)} {}_p^{\vartheta}M_q^{\tau} \left[2^h z\right]$$

$$\times {}_4F_3 \left[\begin{array}{c} -\upsilon, \upsilon+\alpha+\beta+1, c, \lambda+1; \\ \\ \alpha+1, d, \lambda+\delta+hk+2; \end{array} 1 \right], \quad (14)$$

which provided that $z, \vartheta, \tau \in \mathbb{C}, \Re(\vartheta) > 0$, $\alpha > -1$ and $\beta > -1$.

Proof. Let ℓ be the left-hand side of (14), using (1), we have

$$\ell = \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) {}_{p}^{\vartheta}M_{q}^{\tau}\left[z(1+x)^{h}\right] dx$$

$$= \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{\left(z(1+x)^{h}\right)^{k}}{\Gamma(\vartheta k+\tau)} dx, \quad (15)$$

Interchanging the order of integration and summation within the integrand can be justified by the absolute convergence of the integral and the uniform convergence of the series involved under the given condition, then the above expression becomes

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\vartheta k+\tau)} \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta+hk} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) dx, \quad (16)$$

Now, using above Lemma and (1), right side of (16), we obtain the result (14). □

Second Integral:

$$I_2 = \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) P_{\omega}^{(\rho,\sigma,e,f)}(x) {}_{p}^{\vartheta}M_{q}^{\tau}\left[z(1-x)^{h}\right] dx$$

$$= 2^{\lambda+\delta+1} \frac{(\rho+1)_{\omega}(\alpha+1)_{\upsilon}}{\Gamma(\omega+1)\Gamma(\upsilon+1)} \Gamma(\delta+1) \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+h\,k+1)}{\Gamma(\lambda+\delta+h\,k+2)} {}_{p}^{\vartheta}M_{q}^{\tau}\left(2^{h}z\right)$$

$$\times F_{1:2;2}^{1:3;3}\left[\begin{array}{c} (\lambda+hk+1):(-\omega,\omega+\rho+\sigma+1,e);(-\upsilon,\upsilon+\alpha+\beta+1,c) \\ (\lambda+\delta+hk+2):(\rho+1,f);(\alpha+1,d,) \end{array} |1,1\right] \quad (17)$$

which provided that $\vartheta, \tau, \beta, \sigma \in \mathbb{C}, P \geq 0, \Re(\vartheta) > 0,, \ \alpha > -1$ and $\beta > -1$.

Proof. Let $\wp$ be the left-hand side of (18), using (1), we have

$$\wp = \int_{-1}^{+1} (1-x)^{\lambda} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) P_{\omega}^{(\rho,\sigma,e,f)}(x) {}_{p}^{\vartheta}M_{q}^{\tau}\left[z(1-x)^{h}\right] dx$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\vartheta k+\tau)}$$

$$\int_{-1}^{+1} (1-x)^{\lambda+hk} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) P_{\omega}^{(\rho,\sigma,e,f)}(x) dx, \quad (18)$$

Now, substituting from (7) in above equation, we get

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\vartheta\,k+\tau)} \frac{(\rho+1)_{\omega}}{\Gamma(\omega+1)} \sum_{t=0}^{\infty} \frac{(-\omega)_t (\omega+\rho+\sigma+1)_t (e)_t}{(\rho+1)_t (f)_t 2^t t!}$$

$$\times \int_{-1}^{+1} (1-x)^{\lambda+t+hk} (1+x)^{\delta} P_{\upsilon}^{(\alpha,\beta,c,d)}(x) dx, \quad (19)$$

and again from equation (7), we arrive

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\vartheta\,k+\tau)} \frac{(\rho+1)_{\omega}}{\Gamma(\omega+1)} \frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)}$$

$$\times \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-\omega)_t(-\upsilon)_s(\omega+\rho+\sigma+1)_t(\upsilon+\alpha+\beta+1)_s(c)_t(e)_s}{(\rho+1)_t\,(\alpha+1)_s\,(d)_s(f)_t 2^{s+t}(s!)(t!)}$$
$$\times \int_{-1}^{1} (1-x)^{\lambda+hk+s+t}\,(1+x)^{\delta}\,dx, \tag{20}$$

Now apply equation (12) , we obtain

$$= 2^{\lambda+\delta+1}\frac{(\rho+1)_\omega}{\Gamma(\omega+1)}\frac{(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty} {}_p^{\vartheta}M_q^{\tau}\left(2^h z\right)$$
$$\times \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-\omega)_t(-\upsilon)_s(\omega+\rho+\sigma+1)_t(\upsilon+\alpha+\beta+1)_s(c)_t(e)_s}{(\rho+1)_t\,(\alpha+1)_s\,(d)_s(f)_t 2^{s+t}(s!)(t!)}$$
$$\times \frac{\Gamma\left(\lambda+h\,k+s+t+1\right)\Gamma\left(\delta+1\right)}{\Gamma\left(\lambda+\delta+hk+s+t+2\right)},$$
$$= 2^{\lambda+\delta+1}\frac{(\rho+1)_\omega}{\Gamma(\omega+1)}\frac{(\alpha+1)_\upsilon\,\Gamma\left(\delta+1\right)}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\lambda+hk+1\right)}{\Gamma\left(\lambda+\delta+hk+2\right)} {}_p^{\vartheta}M_q^{\tau}\left(2^h z\right)$$
$$\times \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(\lambda+hk+1)_{s+t}\,(-\omega)_t(-\upsilon)_s(\omega+\rho+\sigma+1)_t(\upsilon+\alpha+\beta+1)_s(c)_t(e)_s}{(\lambda+\delta+hk+2)_{s+t}\,(\rho+1)_t\,(\alpha+1)_s\,(f)_t(d)_s 2^{s+t}(s!)(t!)}, \tag{21}$$

Expressing the above result (21) in terms of Kampé de Fériet's hypergeometric function (see [3], p. 160) it follow that, we obtain the result (18). □

Third Integral:

$$I_3 = \int_{-1}^{+1} (1-x)^{\lambda}\,(1+x)^{\delta}\,P_\upsilon^{(\alpha,\beta,c,d)}(x) {}_p^{\vartheta}M_q^{\tau}\left[z\,(1-x)^h\,(1+x)^r\right]\,dx$$
$$= 2^{\lambda+\delta+1}\frac{(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\lambda+hk+1\right)\Gamma\left(\delta+r\,k+1\right)}{\Gamma\left(\lambda+\delta+(h+r)k+2\right)} {}_p^{\vartheta}M_q^{\tau}\left(2^{h+r}z\right)$$
$$\times\, {}_4F_3\left[\begin{array}{c} -\upsilon,\upsilon+\alpha+\beta+1,c,\lambda+hk+1; \\ \\ \alpha+1,d,\lambda+\delta+(h+r)k+2; \end{array} 1\right]. \tag{22}$$

which provided that $z,\vartheta,\tau\in\mathbb{C}, \Re\left(\vartheta\right)>0$, $\alpha>-1$ and $\beta>-1$.

Proof. Let Θ be the left-hand side of (22), using (1), we have

$$\Theta = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma\left(\vartheta k+\tau\right)}\int_{-1}^{+1}(1-x)^{\lambda+hk}\,(1+x)^{\delta+rk}\,P_\upsilon^{(\alpha,\beta,c,d)}(x)dx, \tag{23}$$

Applying Lemma, then we arrive right hand side of (22). □

Forth Integral:

$$I_4 = \int_{-1}^{+1} (1-x)^{\lambda}\,(1+x)^{\delta}\,P_\upsilon^{(\alpha,\beta,c,d)}(x) {}_p^{\vartheta}M_q^{\tau}\left[z\,(1-x)^h\,(1+x)^{-r}\right]dx$$
$$= 2^{\lambda+\delta+1}\frac{(\alpha+1)_\upsilon}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\lambda+hk+1\right)\Gamma\left(\delta-r\,k+1\right)}{\Gamma\left(\lambda+\delta+(h-r)k+2\right)} {}_p^{\vartheta}M_q^{\tau}\left(2^{h-r}z\right)$$

$$\times\, {}_4F_3\left[\begin{array}{c}-\upsilon, \upsilon+\alpha+\beta+1, c, \lambda+hk+1;\\ \\ \alpha+1, d, \lambda+\delta+(h-r)k+2;\end{array} 1\right]. \tag{24}$$

which provided that $z, \vartheta, \tau \in \mathbb{C}, \Re(\vartheta)>0$, $\alpha>-1$ and $\beta>-1$.

Proof. Similar manner of (22), we arrive at the result (24). □

3. Special cases

In this section, we consider other variations of integral formulas for one to four. In detail on account of the general nature of the Jacobi polynomial occurring in our main integrals, a huge number of integrals involving straightforward functions of one can simply be obtained as their special cases. We present here some special cases.

(I). On setting $c=d$, in integral $(I_1)-(I_4)$ and $e=f$ in integral (I_2), we obtain the following new interesting results:

$$C_1=\int_{-1}^{+1}(1-x)^{\lambda}(1+x)^{\delta}P_{\upsilon}^{(\alpha,\beta)}(x){}_p^{\vartheta}M_q^{\tau}\left[z(1+x)^h\right]dx$$

$$=2^{\lambda+\delta+1}\frac{(\alpha+1)_{\upsilon}\Gamma(\lambda+1)}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma(\delta+hk+1)}{\Gamma(\lambda+\delta+hk+2)}{}_p^{\vartheta}M_q^{\tau}\left[2^hz\right]$$

$$\times\, {}_3F_2\left[\begin{array}{c}-\upsilon, \upsilon+\alpha+\beta+1, \lambda+1;\\ \\ \alpha+1, \lambda+\delta+hk+2;\end{array} 1\right]. \tag{25}$$

$$C_2=\int_{-1}^{+1}(1-x)^{\lambda}(1+x)^{\delta}P_{\upsilon}^{(\alpha,\beta,)}(x)P_{\omega}^{(\rho,\sigma)}(x){}_p^{\vartheta}M_q^{\tau}\left[z(1-x)^h\right]dx$$

$$=2^{\lambda+\delta+1}\frac{(\rho+1)_{\omega}(\alpha+1)_{\upsilon}}{\Gamma(\omega+1)\Gamma(\upsilon+1)}\Gamma(\delta+1)\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+hk+1)}{\Gamma(\lambda+\delta+hk+2)}{}_p^{\vartheta}M_q^{\tau}\left(2^hz\right)$$

$$\times\, F_{1:1;1}^{1:2;2}\left[\begin{array}{c}(\lambda+hk+1):(-\omega,\omega+\rho+\sigma+1);(-\upsilon,\upsilon+\alpha+\beta+1)\\ (\lambda+\delta+hk+2):(\rho+1);(\alpha+1)\end{array}\Big|1,1\right] \tag{26}$$

$$C_3=\int_{-1}^{+1}(1-x)^{\lambda}(1+x)^{\delta}P_{\upsilon}^{(\alpha,\beta)}(x){}_p^{\vartheta}M_q^{\tau}\left[z(1-x)^h(1+x)^r\right]dx$$

$$=2^{\lambda+\delta+1}\frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+h\,k+1)\,\Gamma(\delta+r\,k+1)}{\Gamma(\lambda+\delta+(h+r)k+2)}{}_p^{\vartheta}M_q^{\tau}\left(2^{h+r}z\right)$$

$$\times\, {}_3F_2\left[\begin{array}{c}-\upsilon, \upsilon+\alpha+\beta+1, \lambda+hk+1;\\ \\ \alpha+1, \lambda+\delta+(h+r)k+2;\end{array} 1\right]. \tag{27}$$

$$C_4=\int_{-1}^{+1}(1-x)^{\lambda}(1+x)^{\delta}P_{\upsilon}^{(\alpha,\beta)}(x){}_p^{\vartheta}M_q^{\tau}\left[z(1-x)^h(1+x)^{-r}\right]dx$$

$$=2^{\lambda+\delta+1}\frac{(\alpha+1)_{\upsilon}}{\Gamma(\upsilon+1)}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+hk+1)\,\Gamma(\delta-rk+1)}{\Gamma(\lambda+\delta+(h-r)k+2)}{}_p^{\vartheta}M_q^{\tau}\left(2^{h-r}z\right)$$

$$\times\, {}_3F_2\left[\begin{array}{c} -\upsilon, \upsilon+\alpha+\beta+1, \lambda+hk+1; \\ \\ \alpha+1, \lambda+\delta+(h-r)k+2; \end{array} 1\right]. \tag{28}$$

(II). If we replace $\lambda = \lambda - 1$ and $\alpha = \beta = \rho = \sigma = \delta = 0$, in integral C_2, the following integral involving Legendre polynomials.

$$C_5 = \int_{-1}^{+1} (1-x)^{\lambda} P_{\upsilon}(x) P_{\omega}(x) {}_p^{\vartheta}M_q^{\tau}\left[z\,(1-x)^h\right] dx$$

$$= 2^{\lambda} \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-\omega)_t(-\upsilon)_s(\omega+1)_t(\upsilon+1)_s}{2^{s+t}(s!)^2(t!)^2} {}_p^{\vartheta}M_q^{\tau}\left(2^h z\right) B\left(\lambda+hk+s+t, 1\right) \tag{29}$$

(III). If we set $\lambda = \lambda - 1, \delta = \delta - 1$ and $\alpha = \beta = 0$, in integral C_3, the following integral involving Legendre polynomials.

$$C_6 = \int_{-1}^{+1} (1-x)^{\lambda-1}(1+x)^{\delta-1} P_{\upsilon}(x) {}_p^{\vartheta}M_q^{\tau}\left[z\,(1-x)^h\,(1+x)^r\right] dx$$

$$= 2^{\lambda+\delta-1} \sum_{s=0}^{\infty} \frac{(-\upsilon)_s(\upsilon+1)_s}{(s!)^2} {}_p^{\vartheta}M_q^{\tau}\left(2^{h+r} z\right) B\left(\lambda+hk+s, \delta+rk\right) \tag{30}$$

(IV). If we set $\lambda = \lambda - 1, \delta = \delta - 1$ and $\alpha = \beta = 0$, in integral C_4, takes the following integral involving Legendre polynomials.

$$C_7 = \int_{-1}^{+1} (1-x)^{\lambda-1}(1+x)^{\delta-1} P_{\upsilon}(x) {}_p^{\vartheta}M_q^{\tau}\left[z\,(1-x)^h\,(1+x)^{-r}\right] dx$$

$$= 2^{\lambda+\delta-1} \sum_{s=0}^{\infty} \frac{(-\upsilon)_s(\upsilon+1)_s}{(s!)^2} {}_p^{\vartheta}M_q^{\tau}\left(2^{h-r} z\right) B\left(\lambda+hk+s, \delta-rk\right) \tag{31}$$

4. Concluding remark

We have established four new integral relations involving the generalized M-function, in terms of the hypergeometric function, Kampé de Fériet's hypergeometric function and beta function. Some special cases of integrals involving the generalized Mittag -Leffler function have been investigated in the literature by a many authors with different arguments. It is interesting to observe that the results given by Singh and Rawat [14] and Suthar and Haile [17] follow from the special cases results derived in this paper, if we use (6) and some suitable parametric replacements.

References

[1] A.A. Kilbas, M. Saigo and R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms Spec. Funct., Vol. 15, 31-49, 2004.

[2] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations. Elsevier, North Holland Math. Studies 204, Amsterdam, 2006.

[3] A.M. Mathai, and R.K. Saxena, The H-function with applications in statistics and other disciplines, Wiley Eastern, New Delhi, 1978.

[4] K.S. Miller, The Mittag-Leffler and Related functions. Integral transforms and special functions, Vol.1, issue 1, 41-49, 1993.

[5] V.N. Mishra, D.L. Suthar and S.D. Purohit, Marichev-Saigo-Maeda fractional calculus operators, Srivastava polynomials and generalized Mittag-Leffler function. Cogent Math., Vol. 4, 1-11, 2017.

[6] G.M. Mittag-Leffier, Sur la nouvelle function E_α(x). C.R. Acad. Sd. Paris (Ser. II), Vol. 137, 554-558, 1903.

[7] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., Vol. 19, 7-15, 1971.

[8] S.D. Purohit, D.L. Suthar and S.L. Kalla, Some results on fractional calculus operators associated with the M-function. Hadronic J., Vol. 33, Issue 3, 225-236, 2010.

[9] E.D. Rainville, Special function, The Macmillan Co., New York, 1960.

[10] M. Saigo and A.A. Kilbas, On Mittag-Leffler type function and applications. Integral Transform. Spec. Funct., Vol. 7, Issue (1-2), 97-112, 1998.

[11] J. Sarabia, and S.L. Kalla, On a generalized Jacobi transform. J. Appl. Math. Stochastic Anal., Vol. 15, Issue 4, 385-398, 2002.

[12] R.K. Saxena, J. Ram and D.L. Suthar, Generalized fractional calculus of the generalized Mittag- Leffler functions. J. Indian Acad. Math., Vol. 31, Issue 1, 165-172, 2009.

[13] R.K. Saxena, J. Ram, D.L. Suthar and S.L. Kalla, On a generalized Wright transform. Algebras Groups Geom., Vol. 23, Issue 2, 25-42, 2006.

[14] D.K. Singh and R. Rawat, Integrals invoiving generalized Mittag-Leffler function. J. Fract. Calc. Appl., Vol. 4, Issue 2, 234-244, 2013.

[15] M. Sharma, Fractional integration and fractional differentiation of the M-series. Fract. Calc. Appl Anal, Vol. 11, Issue 2, 187-191, 2008.

[16] M. Sharma and R. Jain, A note on a generalized M -series as a special function of fractional calculus. Fract. Calc. Appl Anal, Vol. 12, Issue 4, 449-452, 2009.

[17] D.L. Suthar and Haile Habenom, Integrals involving generalized Bessel-Maitland function. J. Sci. Arts, Vol. 37, Issue 4, 357-362, 2016.

[18] A. Wiman, Uber de Fundamental Satz in der Theorie derFunktionen $E_\alpha(x)$. Acta Mathematica, Vol. 29, Issue 1, 191-201, 1905.

D.L. Suthar
Department of Mathematics, Wollo University, P.O. Box:1145, Dessie, ETHIOPIA
E-mail address: `dlsuthar@gmail.com`

Hagos Tadesse
Department of Mathematics, Wollo University, P.O. Box:1145, Dessie, ETHIOPIA
E-mail address: `hagos.tadesse2@gmail.com`

Kelelaw Tilahun
Department of Mathematics, Wollo University, P.O. Box:1145, Dessie, ETHIOPIA
E-mail address: `kta3151@gmail.com`