Journal of Fractional Calculus and Applications
Vol. 10(1) Jan. 2019, pp. 1-9
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON THE STABILITY ANALYSIS OF THE FRACTIONAL NONLINEAR SYSTEMS WITH HURWITZ STATE MATRIX

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Abstract. This paper deals with the stability analysis of the fractional nonlinear systems. It treats the fractional exponential stability and the asymptotic stability of the fractional nonlinear systems with Hurwitz state matrix, using the Lyapunov direct method. We give algebraic conditions under which the fractional nonlinear systems are fractional exponentially stable. Two numerical examples are provided to illustrate the proposed theoretical results.

1. Introduction

In the last years many papers appeared and gave some results and role of the fractional calculus in physics, control engineering and signal processing [4, 5, 10, 11]. Fractional calculus received attention due to its important role in modeling the anomalous dynamics of various processes related to complex systems in the most areas of science an engineering. In 1965, l'Hospital asked a remarkable question, what does it mean $\frac{d^n f}{dx^n}$ when $n = \frac{1}{2}$ [8]. This problem has now an explicit answer. The limit definition of a fractional derivative was introduced to answer to this question [1, 8]. Fractional calculus is a generalization of ordinary differential and integration to arbitrary non integer order. In [8], Khalil gives a definition of a derivative called conformable derivative mathematically expressed by

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{1}$$

In [3], Almeida introduced a similar limit definition of fractional derivative of a function if we do not know the kernel as follows

$$f^\alpha(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon k(t)^{1-\alpha}) - f(t)}{\epsilon}. \tag{2}$$

If the $k(t) = t$, we recover the conformable derivative given by Khalil. If expanding the function $te^{\epsilon t^{1-\alpha}}$ at neighborhood of $\epsilon = 0$ it follows that $te^{\epsilon t^{1-\alpha}} = t + \epsilon t^{1-\alpha} + o(\epsilon)$,

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2010 *Mathematics Subject Classification.* 26A33, 34D20, 93D05.
Key words and phrases. Conformable derivative, Fractional nonlinear systems, Fractional exponential stable.
Submitted Feb. 25, 2018. Revised March 7, 2018.

we recover the Katugampola idea of fractional derivative given in [6] expressed as

$$D_{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{f(te^{\epsilon t^{1-\alpha}})-f(t)}{\epsilon}. \tag{3}$$

All these definitions of the idea of the fractional derivative are clearly equivalent. Last generalizations of these definitions were given in the literature (see [2, 13]). In this paper, we will use Khalil idea of the fractional derivative given by the condition (1) due to its practical and simplest use. Stability of nonlinear systems received increased attention due to its important role in areas of science and engineering. A large number of monograph and papers are devoted to the fractional nonlinear systems [4, 9, 14]. This paper deals with the stability analysis of the fractional nonlinear systems. It treats the stability of a particular class of the fractional dynamical nonlinear systems. Many results in stability analysis of the fractional nonlinear systems are consigned in [14]. In [9], author study stability analysis of particular class of the fractional nonlinear systems using Lyapunov direct method.

It is well known that the nonlinear dynamic systems that share a Lyapunov function is global asymptotically stability [7]. For the fractional nonlinear systems, we have the asymptotic stability or the fractional exponential stability. For linear time invariant systems, the global asymptotic stability is guaranteed if the state matrix is Hurwitz. For fractional linear system, we have the fractional exponential stability. Providing conditions under which a fractional nonlinear system is fractional exponentially stable has been the object of intense research.

Modeling phenomena using dynamical system is not very easy. There are many types of errors in the modeling. In this paper, we will gather these errors in the perturbation term. The perturbation term could result from modeling errors, aging, uncertainties and disturbances which exist in many realistic problems. Another purpose of this paper is to find algebraic conditions under which the fractional nonlinear systems with perturbation term are asymptotically stable and fractional exponentially stable. In many practical systems, the system behaves well when it is not too much disturbed, but may have more complicated behaviors when the disturbance is too strong. These reasons, motive the works of this paper. We use the Lyapunov direct method.

The paper is organized as follows : in section 2, after recalling some necessary definitions, we will describe the classes of the fractional nonlinear systems, and will provide the main results. In section 3, we will give two numerical examples to illustrate our main results. And then we are going to end this paper by giving our proofs, conclusions and remarks in section 4.

Notation. $\mathcal{PD}$ denotes the set of all continuous functions $\chi:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ satisfying $\chi(0)=0$ and $\chi(s)>0$ for all $s>0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_{\infty}$ denotes the set of all unbounded $\mathcal{K}$ function. A continuous function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to be class $\mathcal{KL}$ if $\beta(.,t)\in\mathcal{K}$ for any $t\geq 0$ and $\beta(s,.)$ is non increasing and tends to zero as its arguments tends to infinity. Given $x\in\mathbb{R}^n$, $\|x\|$ stands for its Euclidean norm: $\|x\|:=\sqrt{x_1^2+\ldots+x_n^2}$. For a matrix A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. If the condition $Re\left(\lambda_i\right)<0,\forall i=1,2,...,n$, holds then the matrix A is said Hurwitz .

2. Preliminaries definitions and main results

In this section, we introduce some definitions of the fractional calculus and several lemmas.

Definition 1. [8] *Given a function* $f : [0, +\infty[\longrightarrow \mathbb{R}$. *Then the conformable derivative of* f *of order* α *is defined by*

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{4}$$

all $t > 0$, $\alpha \in (0, 1)$. *If* f *is* α*-differentiable in* $(0, a)$, $a > 0$, *and* $\lim_{\epsilon \to 0^+} f^{(\alpha)}(t)$ *exists, then define*

$$f^{(\alpha)}(0) = \lim_{\epsilon \to 0^+} f^{(\alpha)}(t).$$

Definition 2. [14] *We denote by* $C_\infty((0, +\infty), \mathbb{R}^n)$ *the set of function* $x \in C_\infty((0, +\infty), \mathbb{R}^n)$ *such that* $T_\alpha x(t)$ *exists and is continuous on* $(0, +\infty)$.

Lemma 1. [8] *Let* $\alpha \in (0, 1)$ *and* f *is* α*-differentiable at point* $t > 0$. *If* f *is differentiable, then*

$$T_\alpha f(t) = t^{1-\alpha} \frac{df}{dt}. \tag{5}$$

From expression given by (5), it is clear if $\alpha = 1$, we recover the classical derivative. Khalil definition of fractional derivative satisfies the following properties (see [8] for details):

Lemma 2. *Let* $\alpha \in (0, 1)$ *and* f, g *be* α*-differentiable at point* $t > 0$. *Then*

(1) $T_\alpha [af + bg] = aT_\alpha [f] + bT_\alpha [g]$ *for all constant* $a, b \in \mathbb{R}$.
(2) $T_\alpha [\lambda] = 0$, *for all constant function* $f(t) = \lambda$.
(3) $T_\alpha [fg] = fT_\alpha [g] + gT_\alpha [f]$.
(4) $T_\alpha \left[\frac{f}{g}\right] = \frac{fT_\alpha[g] - gT_\alpha[f]}{g^2}$.
(5) *The triangular inequality :*

$$T_\alpha [|f + g|] \leq T_\alpha [|f|] + T_\alpha [|g|] \tag{6}$$

is not in hold general.

We give the following counterexample to illustrate the items (5). The proof of the other items can be found in [8].

Counterexample: To see that, let the function $f(t) = t^2$ and $g(t) = t$ on interval $[0, 1]$, we have that $|f| = f \leq g \leq |g|$. But $T_\alpha [|f|](1) = 2$ and $T_\alpha [|g|](1) = 1$. And remark that $T_\alpha [|g|](1) \leq T_\alpha [|f|](1)$. Then T_α is not a monotone operator. On this condition the triangular inequality is not hold.

Definition 3. [8] *The conformable integral starting from* a *of a function* f *of order* $\alpha \in (0, 1]$ *is defined by*

$$I_\alpha^a f(t) = \int_a^t x^{\alpha - 1} f(x) dx \tag{7}$$

Lemma 3. [8] *Let* $\alpha \in (0, 1]$ *and* f *is any continuous in a domain of* I_α, *for* $t > a$ *we have*

$$T_\alpha I_\alpha^a f(t) = f(t). \tag{8}$$

Generally the fractional nonlinear systems which we consider in this paper is mathematically represented by the following form

$$T_\alpha x(t) = f(t, x(t)) \tag{9}$$

where $x(t) \in \mathbb{R}^n$ is state variable and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ continuous locally Lipschitz function satisfying $f(t, 0) = 0$ for all $t > 0$.

We introduce some definitions of the fractional nonlinear system (9).

Definition 4. [14] *The trivial solution of system (9) is said to be stable if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that for any initial condition $\|x(t_0\| < \delta$, the solution $x(t)$ of the system (9) satisfies inequality $\|x(t)\| < \epsilon$ for all $t > t_0$.*

The trivial solution of system (9) is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to+\infty} x(t) = 0$.

Definition 5. [14] *The conformable exponential function is defined for every $s \geq 0$ by*

$$E_\alpha(\lambda, s) = \exp\left(\lambda \frac{s^\alpha}{\alpha}\right) \tag{10}$$

where $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}$.

Definition 6. [14] *The origin of the fractional nonlinear system (9) is said to be fractional exponentially stable if*

$$\|x(t)\| \leq K \|x(t_0)\| E_\alpha(-\lambda, t - t_0) \tag{11}$$

with $t > t_0$ and $\lambda, K > 0$.

We introduce some several important lemmas and assumption.

Lemma 4. [14] *Let $x = 0$ be an equilibrium point for the fractional nonlinear system (9) and $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Suppose that c_1, c_2 and c_3 are arbitrary positive constants. If the following conditions are satisfied :*

(1) $c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$.
(2) *$V(t, x)$ has conformable fractional derivate of order α for all $t_0 \geq 0$*
(3) $T_\alpha V(t, x(t)) \leq -c_3 \|x\|^2$.

Then the origin of the fractional nonlinear system (9) is fractional exponentially stable.

Lemma 5. [14] *Let $x = 0$ be an equilibrium point for the fractional nonlinear system (9) and $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous function and class $\mathcal{K}$ function χ_1 satisfying following condition :*

(1) $\chi_1(\|x\|) \leq V(t, x)$
(2) *$V(t, x)$ has conformable fractional derivate of order α for all $t_0 \geq 0$*
(3) $T_\alpha V(t, x(t)) \leq 0$.

Then the origin of the fractional nonlinear system (9) is stable.

Lemma 6. *Let $x = 0$ be an equilibrium point for the fractional nonlinear system (9) and there exist $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous Lyapunov candidate and class $\mathcal{K}$ function χ_4 satisfying following conditions :*

(1) *$V(t, x)$ has conformable fractional derivate of order α for all $t_0 \geq 0$*
(2) $T_\alpha V(t, x(t)) \leq -\chi_4(\|x\|)$.

Then the origin of the fractional nonlinear system (9) is asymptotically stable.

For the proof of this theorem we can remark that, if there exists $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous Lyapunov candidate function, then there exist classes $\mathcal{K}$ functions χ_2, χ_3 satisfying the condition $\chi_2(\|x\|) \leq V(t,x) \leq \chi_3(\|x\|)$. The rest of the proof is given by Theorem 3 in [14].

Referring to these above lemmas we make the following lemma.

Lemma 7. *If the equilibrium point $x = 0$ for the fractional nonlinear system (9) is fractional exponentially stable then there exists class $\mathcal{K}_\infty$ function χ_5 such that*

$$\|x(t)\| \leq \chi_5(\|x_0\|). \tag{12}$$

Proof : Let $x = 0$ be the equilibrium point of the system (9). If the origin of the system (9) is fractional exponentially stable then

$$\|x(t)\| \leq K \|x(t_0)\| E_\alpha(-\lambda, t - t_0).$$

Observe that, the conformable fractional exponential $E_\alpha(-\lambda, t - t_0)$ decreases on $t - t_0$, and we have in particular

$$E_\alpha(-\lambda, t - t_0) \leq E_\alpha(-\lambda, 0) = 1.$$

We obtain that $\|x(t)\| \leq K \|x(t_0)\| E_\alpha(-\lambda, t - t_0) \leq K \|x(t_0)\|$. Let that $\chi_5(s) = Ks \in \mathcal{K}_\infty$. Finally we have $\|x(t)\| \leq \chi_5(\|x_0\|)$.

Assumption 1. *The function $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with Lipschitz constant L, that is*

$$\|g(t,x) - g(t,y)\| \leq L \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and all $t \in \mathbb{R}^+$.

Lemma 8. [12, 15] *Given any scalar $\epsilon \geq 0$, $u, v \in \mathbb{R}^n$, it holds that*

$$u^T v + v^T u \leq \epsilon^{-1} u^T u + \epsilon v^T v.$$

We now ready to state the main results of this paper which are provided in section 4. Generally the fractional nonlinear systems which we consider in this paper is mathematically represented by the following form

$$T_\alpha x(t) = f(t,x) = Ax(t) + g(t, x(t)) \tag{13}$$

where $x(t) \in \mathbb{R}^n$ is state variable, A is an matrix in $\mathbb{R}^{n \times n}$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption 1 and $g(t, 0) = 0$.

Let $g = 0$, then we obtain the following particular fractional nonlinear systems expressed by

$$T_\alpha x(t) = Ax(t) \tag{14}$$

The fractional systems define by (14) are called the fractional linear systems. We have the following results.

Theorem 1. *Let $x = 0$ be an equilibrium point of the system (14). If the state matrix A is Hurwitz then the trivial solution of the fractional linear system (14) is fractional exponentially stable.*

Theorem 2. *Let $x = 0$ be an equilibrium point of the system (14). If the state matrix A is Hurwitz then the trivial solution of the fractional linear system (14) is asymptotically stable.*

We give the main results with the fractional nonlinear system (13). For that, we consider the perturbation term $g(t,x) \neq 0$ for all $x \neq 0$, and with $g(t,0) = 0$.

Theorem 3. *Let $x = 0$ be an equilibrium point of the system (13). Let that the state matrix A is Hurwitz, and the condition $\|g(t, x)\| < \gamma \|x\|$ holds. If there exist a positive definite matrix P such that the following inequality holds*

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}.$$

where $-Q = A^T P + PA$, then the trivial solution of the fractional nonlinear system (13) is fractional exponentially stable.

Theorem 4. *If Assumption 1 holds, there exist a positive definite matrix P, a scalar $\epsilon \geq 0$, such that the following inequality holds*

$$\lambda_{min}(R) < \epsilon L^2 \tag{15}$$

where L is Lipschitz constant of g and $-R = A^T P + PA + \epsilon^{-1} P^2$, then the trivial solution of the system (13) is fractional exponentially stable.

3. Two numerical examples

In this section, two examples are provided to illustrate the proposed theorems in Section 2.

• For illustration of Theorem 1, let that the following fractional linear system defined as

$$T_\alpha x(t) = Ax(t) \tag{16}$$

where $x(t) = (x_1(t), x_2(t))$, $A = \begin{pmatrix} -5 & 1 \\ 1 & -4 \end{pmatrix}$.

By simple calculation, the eigenvalues of the state matrix A are $\lambda_1 = -5,618$ and $\lambda_2 = -3.382$. Thus the condition $Re(\lambda_i) < 0, \forall i = 1, 2$, is hold, then the state matrix A is Hurwitz. Using the Theorem 1, we conclude that the trivial solution of the fractional linear system (16) is fractional exponentially stable.

• For illustration of Theorem 3, let that the following fractional nonlinear system defined as

$$T_\alpha x(t) = Ax(t) + g(t, x) \tag{17}$$

where $x(t) = (x_1(t), x_2(t))$, $A = \begin{pmatrix} -5 & 1 \\ 1 & -4 \end{pmatrix}$, and $g(t, x) = (\sin x_1(t), \sin x_2(t))$.

We choose a Lyapunov candidate function $V(t, x) = x(t)^T Px(t)$ where $P = I_2$. The α derivative of V along the trajectories of (17) is given by

$$\begin{aligned} T_\alpha V(t, x(t)) \leq 2x^T PT_\alpha x &= [Ax + g(t, x)]^T Px + x^T P [Ax + g(t, x)] \\ &= -10x_1^2 + 4x_1x_2 - 8x_2^2 + 2x_1 \sin x_1 + 2x_2 \sin x_2 \\ &\leq -8\left(x_1 - \frac{1}{8}x_2\right)^2 - \frac{47}{8}x_2^2 \end{aligned}$$

Hence $T_\alpha V(t, x(t))$ is negative definite which implies the trivial solution of the fractional nonlinear system $T_\alpha x(t) = Ax(t) + g(t, x)$ is fractional exponentially stable. This conclusion can be obtained by applying the Theorem 3. To see that, we can remark the state matrix A is Hurwitz and the condition $\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$ is hold, with $\gamma = 1$, $\lambda_{min}(Q) = 3.382$ and $\lambda_{max}(P) = 1$, thus the trivial solution of the fractional nonlinear system $T_\alpha x(t) = Ax(t) + g(t, x)$ is fractional exponentially stable.

4. Proofs

4.1. **Proof of Theorem 1.** We choose a Lyapunov candidate function $V(t,x(t)) = x(t)^T Px(t)$ where $A^T P + PA = -Q$. The α derivative of V along the trajectories of (14) is given by

$$\begin{aligned} T_\alpha V(t,x(t)) \leq 2x^T P T_\alpha x &= [Ax]^T Px + x^T P[Ax] \\ &= x^T A^T Px + x^T PAx \\ &= x^T \left(A^T P + PA\right) x \end{aligned}$$

By the assumption that the state matrix A of the fractional linear system (14) is Hurwitz, it follows the matrix $Q = -A^T P - PA$ is positive definite and we have

$$T_\alpha V(t,x(t)) \leq -\lambda_{min}(Q) \|x\|^2 ,$$

where $\lambda_{min}(Q)$ is the minimum eigenvalue of the matrix Q, which is positive. Using Lemma 4, we conclude that the trivial solution of the fractional system (14) is fractional exponentially stable.

4.2. **Proof of Theorem 2.** Using the proof of the Theorem 1, the trivial solution of the fractional linear system is fractional exponentially stable. Using the Lemma 7 then there exists class $\mathcal{K}$ function α_5 such that $x(t)) \leq \alpha_5(\|x_0\|)$. That make the trivial solution of the fractional linear system (14) stable. To prove the attractive, we can use the solution of the fractional linear system given by

$$x(t) = x(t_0)e^{A\frac{(t-t_0)^\alpha}{\alpha}} = x(t_0)E_\alpha\left(A, t-t_0\right).$$

We know that the fractional exponential $E_\alpha(A, t-t_0)$ tends to 0 if t tends to ∞ if and only if the condition $Re(\lambda_i) < 0, \forall i = 1, 2, ..., n,$ holds [7], where λ_i are the eigenvalues of the matrix A. This condition is satisfied because the state matrix A is Hurwitz [7]. Then the origin of the fractional linear system (14) is attractive. Finally, by attractive and stability, it follows that the trivial solution of the fractional linear system (14) is asymptotically stable.

4.3. **Proof of Theorem 3.** We choose a Lyapunov candidate function $V(t,x(t)) = x^T Px$ where $A^T P + PA = -Q$. The α derivative of V along the trajectories of (13) is given by

$$\begin{aligned} T_\alpha V(t,x(t)) \leq 2x^T P T_\alpha x &= [Ax + g(t,x)]^T Px + x^T P[Ax + g(t,x)] \\ &= x^T A^T Px + g^T(t,x)Px + x^T PAx + x^T Pg(t,x) \\ &= x^T \left(A^T P + PA\right) x + g^T(t,x)Px + x^T Pg(t,x) \\ &\leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)\gamma \|x\|^2 \\ &= -\left[\lambda_{min}(Q) - 2\lambda_{max}(P)\gamma\right] \|x\|^2 \end{aligned}$$

Clearly, if $\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$, it follows that $T_\alpha V(t,x(t)) < 0$. Using the Lemma (5), we conclude that the trivial solution of the fractional nonlinear system (13) is fractional exponentially stable.

4.4. **Proof of Theorem 4.** We choose a Lyapunov candidate function $V(t, x(t)) = x(t)^T P x(t)$ where $A^T P + PA = -Q$. The α derivative of V along the trajectories of (13) is given by

$$\begin{aligned} T_\alpha V(t, x(t)) \le 2x^T P T_\alpha x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \\ &= [Ax + g(t, x)]^T Px + x^T P [Ax + g(t, x)] \\ &= x^T A^T Px + g^T(t, x)Px + x^T PAx + x^T Pg(t, x) \\ &= x^T \left(A^T P + PA\right) x + g^T(t, x)Px + x^T Pg(t, x) \end{aligned}$$

By Lemma 8, it follows that there exists positive constant $\epsilon \ge 0$, such that

$$g^T(t, x)Px + x^T Pg(t, x) \le \epsilon^{-1}(Px)^T(Px) + \epsilon g^T(t, x)g(t, x).$$

Under Assumptions 1, the function g is locally Lipschitz and continuous then $g^T(t, x)g(t, x) - L^2 x^T x \le 0$. Replacing in above, we have that

$$g^T(t, x)Px + x^T Pg(t, x) \le \epsilon^{-1}(Px)^T(Px) + \epsilon L^2 x^T Ix.$$

Replacing in above inequality, we obtain that

$$\begin{aligned} T_\alpha V(t, x(t)) &\le x^T \left(A^T P + PA\right) x + \epsilon^{-1}(Px)^T(Px) + \epsilon L^2 x^T Ix \\ &\le x^T \left(A^T P + PA\right) x + \epsilon^{-1} x^T P^2 x + \epsilon L^2 x^T Ix \\ &\le x^T \left(A^T P + PA + \epsilon^{-1} P^2 + \epsilon L^2 I\right) x \end{aligned}$$

$T_\alpha V(t, x(t))$ is negative definite if the matrix $A^T P + PA + \epsilon^{-1}P^2 + \epsilon L^2 I < 0$. Hence the system (13) is fractional exponential stable with Lyapunov function V if $A^T P + PA + \epsilon^{-1}P^2 + \epsilon L^2 I < 0$. Furthermore let that $-R = A^T P + PA + \epsilon^{-1}P^2$ which is negative definite (assumption)

$$\begin{aligned} T_\alpha V(t, x(t)) &\le x^T \left(A^T P + PA + \epsilon^{-1}P^2 + \epsilon L^2 \|d\|^2 I\right) x \\ &\le -\lambda_{min}(R) \|x\|^2 + \epsilon L^2 \|x\|^2 \end{aligned}$$

By assuming $\lambda_{min}(R) < \epsilon L^2$, it follows that $T_\alpha V(t, x(t)) < 0$, which implies the fractional exponential stability of the trivial solution of the system (13).

Acknowledgments : The author thank both referees for their careful reading of the manuscript, for their comments, and their corrections. The author would like to thank Antoine Chaillet and Moussa Balde my professor on stability analysis of the nonlinear systems. The author would like to thank Mme Sow Aissatou V. Dieye.

5. Conclusion

We have discussed in this paper the asymptotic stability and the fractional exponential stability of the fractional nonlinear system with Hurwitz state matrix. It contributes to give a practical conditions under which the fractional nonlinear systems with perturbation term are asymptotically stable and fractional exponentially stable.

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