Journal of Fractional Calculus and Applications
Vol. 10(1) Jan. 2019, pp. 46-54.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

GLOBAL ATTRACTIVITY FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH HADAMARD FRACTIONAL DERIVATIVE

ASLI B. ÖZAYDIN, FATMA KARAKOÇ

Abstract. This paper deals with a nonlinear fractional differential equation with Hadamard fractional derivative. By using comparision results sufficient conditions are obtained for the global attractivity of the solutions for nonlinear fractional differential equations in weighted spaces.

1. Introduction

In recent years, fractional order differential equations have been gain importance and studied systematically [6, 8, 9, 11, 13, 14, 16]. Moreover, application fields of fractional order differential equations show increase in engineering and other science [7, 15, 17]. However, it is quite difficult to find analytical solutions of fractional differential equations. Therefore, examining solutions of equations with qualitative methods carry great importance. In this context, doing the stability analysis will help to determine the behaviour of solutions [1, 2, 3, 4, 5, 10, 18].

Let $a \leq x \leq b \leq \infty$ and $0 < \alpha < 1$. We consider the following nonlinear fractional differential equation

$$(D^{\alpha}_{a+}y)(x) = f(x, y(x)),\ x > a, \tag{1}$$

with the initial condition

$$(J^{1-\alpha}_{a+}y)(x)\mid_{x=a} = y_o, \tag{2}$$

where D^{α}_{a+} is the Hadamard fractional derivative operator of order α, $J^{1-\alpha}_{a+}$ is the Hadamard fractional integral operator of order $1-\alpha$, $f(x, y(x)) \in C_{\gamma,\log}[a, b]$ for any $y \in G, G$ is an open set in $\mathbb{R}$ and $y_0 \in \mathbb{R}$. For $0 \leq \gamma < 1$, $\gamma \geq 1-\alpha$, $C_{\gamma,\log}[a, b]$ is the weighted space of functions g such that $(\log \frac{x}{a})^{\gamma} g(x) \in C[a, b]$, where C denotes the spaces of the continuous functions.

2010 *Mathematics Subject Classification.* 34A08, 26A33.
Key words and phrases. Fractional differential equation, Global attractivity, Nonlinear, Hadamard fractional derivative.
Submitted Feb. 11, 2018. Revised March 22, 2018.

2. Preliminaries

In this section, we present some definitions and lemmas which will be used later.

Definition 1 [8]. The left-sided Hadamard fractional integral $J^{\alpha}_{a+}f$ of order $\alpha \in \mathbb{R}$ is defined by

$$(J^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\log\frac{x}{t})^{\alpha-1}\frac{f(t)}{t}dt \ (x > a, \alpha > 0). \tag{3}$$

Definition 2 [8]. Let $\delta = xD$ ($D = \frac{d}{dx}$) be the δ-derivative. The left-sided Hadamard fractional derivative D^{α}_{a+} of order $\alpha \in \mathbb{R}^+$ is defined by

$$\begin{aligned}(D^{\alpha}_{a+}f)(x) &= \delta^{(n)}(J^{n-\alpha}_{a+}f)(x)\\ &= (x\frac{d}{dx})^n\frac{1}{\Gamma(n-\alpha)}\int_a^x (\log\frac{x}{t})^{n-\alpha-1}\frac{f(t)}{t}dt, x > a,\end{aligned} \tag{4}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of α.

Property 1 [8]. If $\alpha > 0, \beta > 0$ and $0 < a < \infty$, then

$$(J^{\alpha}_{a+}(\log\frac{t}{a})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\log\frac{x}{a})^{\beta+\alpha-1} \tag{5}$$

Definition 3 [12]. One-parameter Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

where $\alpha > 0$, $z \in \mathbb{C}$.
The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{6}$$

where $\alpha > 0$ and $\beta, z \in \mathbb{C}$.

Definition 4. The constant y_{eq} is an equilibrium of the fractional differential equation $(D^{\alpha}_{a^+}y)(x) = f(x, y(x))$ if and only if $f(x, y_{eq}) = (D^{\alpha}_{t_0+}y)(x)\ |_{y(x)=y_{eq}}$ for all $x > a$, where $D^{\alpha}_{a^+}$ denotes Hadamard fractional derivative operator.

In this paper we assume that $y_{eq} = 0$.

Definition 5. The zero solution of the equation (1) is called globally attractive if every solution of (1) tends to zero as $x \to \infty$.

Definition 6 [10]. A function $\theta(r)$ is said to belong to class-K if $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\theta(0) = 0$ and it is strictly increasing.

The following theorem will be used in the proof of main results.
Theorem 1[8, *Theorem* 4.5]. Let $\lambda \in \mathbb{R}$, $\alpha > 0, n = -[-\alpha]$ and γ $(0 \leq \gamma < 1)$ be such that $\gamma \geq n - \alpha$. If $f \in C_{\gamma,\log}[a, b]$, then the Cauchy type problem

$$\begin{cases} (D^{\alpha}_{a^+}y)(x) - \lambda y(x) = f(x),\ a < x \leq b,\\ (D^{\alpha-k}_{a^+}y)(a+) = b_k\ (b_k \in \mathbb{R};\ k = 1, ..., n)\end{cases}$$

has a unique solution $y(x) \in C^{\alpha}_{\delta;n-\alpha,\gamma}[a,b]$ and this solution is given by

$$\begin{aligned} y(x) &= \sum_{j=1}^{n} b_j (\log \frac{x}{a})^{\alpha-j} E_{\alpha,\alpha-j+1}\left[\lambda(\log \frac{x}{a})^{\alpha}\right] \\ &+ \int_a^x (\log \frac{x}{t})^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(\log \frac{x}{a})^{\alpha}\right] f(t)\frac{dt}{t}, \end{aligned}$$

where $C^{\alpha}_{\delta;n-\alpha,\gamma}[a,b] = \left\{ y(x) \in C_{n-\alpha,\log}[a,b] : (D^{\alpha}_{a^+}y)(x) \in C_{\gamma,\log}[a,b] \right\}$.
The following lemma is a generalization of Lemma 3.5 in [8].

Lemma 1. Let $0 < \alpha < 1$ and let $y(x) \in C_{1-\alpha,\log}[a,b]$. If

$$\lim_{x\to a^+} \left[(\log \frac{x}{a})^{1-\alpha} y(x)\right] = c,\ \ c \in \mathbb{R}, \tag{7}$$

then

$$(J^{1-\alpha}_{a^+}y)(a+) = \lim_{x\to a^+} (J^{1-\alpha}_{a^+}y)(x) = c\Gamma(\alpha). \tag{8}$$

Proof. Choose an arbitrary $\varepsilon > 0$. By (7), there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left|(\log \frac{t}{a})^{1-\alpha} y(t) - c\right| < \frac{\varepsilon}{\Gamma(\alpha)} \tag{9}$$

for $a < t < a + \delta$. From (5), we get

$$\Gamma(\alpha) = (J^{1-\alpha}_{a^+}(\log \frac{t}{a})^{\alpha-1})(x). \tag{10}$$

Using this equality, and from (3), we have

$$\begin{aligned} \left|(J^{1-\alpha}_{a^+}y)(x) - c\Gamma(\alpha)\right| &= \left|(J^{1-\alpha}_{a^+}y)(x) - c(J^{1-\alpha}_{a^+}(\log \frac{t}{a})^{\alpha-1})(x)\right| \\ &= \left|\frac{1}{\Gamma(1-\alpha)}\int_a^x (\log \frac{x}{t})^{-\alpha}\frac{y(t)}{t}dt - c\frac{1}{\Gamma(1-\alpha)}\int_a^x (\log \frac{x}{t})^{-\alpha}(\log \frac{t}{a})^{\alpha-1}\frac{dt}{t}\right| \\ &\leq \frac{1}{\Gamma(1-\alpha)}\int_a^x (\log \frac{x}{t})^{-\alpha}\left|y(t) - c(\log \frac{t}{a})^{\alpha-1}\right|\frac{dt}{t} \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^x (\log \frac{x}{t})^{-\alpha}(\log \frac{t}{a})^{\alpha-1}\left|y(t)(\log \frac{t}{a})^{1-\alpha} - c\right|\frac{dt}{t}. \end{aligned}$$

Applying the estimate (9) and using (3) and (5), for $a < t < x < a + \delta$ we obtain that

$$\begin{aligned} \left|(J^{1-\alpha}_{a^+}y)(x) - c\Gamma(\alpha)\right| &< \frac{\varepsilon}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_a^x (\log \frac{x}{t})^{-\alpha}(\log \frac{t}{a})^{\alpha-1}\frac{dt}{t} \\ &= \frac{\varepsilon}{\Gamma(\alpha)}(J^{1-\alpha}_{a^+}(\log \frac{t}{a})^{\alpha-1})(x) \\ &= \frac{\varepsilon}{\Gamma(\alpha)}\Gamma(\alpha) \\ &= \varepsilon \end{aligned}$$

which proves (8).

The following lemma gives a formula for the fractional derivative of Mittag-Leffler functions (6).

Lemma 2. Let $A > 0,\ 0 < \alpha < 1$. Then the following relation holds:

$$\left(D^{\alpha}_{a+}\left[(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left(A(\log\frac{x}{a})^{\alpha}\right)\right]\right)(x) = A(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left(A(\log\frac{x}{a})^{\alpha}\right),\ x > a,$$

where D^{α}_{a+} is the Hadamard fractional derivative operator of order α.

Proof. From (4) we have

$$\left(D^{\alpha}_{a+}\left[(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left(A(\log\frac{x}{a})^{\alpha}\right)\right]\right)(x) = x\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x(\log\frac{x}{t})^{-\alpha}(\log\frac{t}{a})^{\alpha-1}E_{\alpha,\alpha}(A(\log\frac{t}{a})^{\alpha})\frac{dt}{t}$$

Note that the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is an entire function. So, using (6) and interchanging $u = \dfrac{\log t - \log a}{\log x - \log a}$ we obtain that

$$\begin{aligned}\left(D^{\alpha}_{a+}\left[(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left(A(\log\frac{x}{a})^{\alpha}\right)\right]\right)(x) &= x\frac{d}{dx}\left[1+\frac{A}{\Gamma(\alpha+1)}(\log\frac{x}{a})^{\alpha}\right.\\ &\quad\left.+\frac{A^2}{\Gamma(2\alpha+1)}(\log\frac{x}{a})^{2\alpha}+...\right]\\ &= A(\log\frac{x}{a})^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}+\frac{A}{\Gamma(2\alpha)}(\log\frac{x}{a})^{\alpha}+...\right]\\ &= A(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}(A(\log\frac{x}{a})^{\alpha}).\end{aligned}$$

3. Main Results

In this section, first we give auxiliary results. Then the main results are proved. The following lemma is a generalization of Lemma 2.3.1 in [9].

Lemma 3. Let $m(x) \in C_{1-\alpha,\log}[a, b]$ and satisfies the following inequality

$$\left|(\log\frac{x}{a})^{1-\alpha}m(x) - (\log\frac{y}{a})^{1-\alpha}m(y)\right| \le M\left|\log\frac{x}{y}\right|^{\lambda}, \tag{11}$$

where $M > 0, 0 < \alpha < \lambda < 1$. For any $x_1 \in (a, b]$, if

$$m(x_1) = 0 \text{ and } m(x) \le 0,\ a < x \le x_1, \tag{12}$$

then

$$(D^{\alpha}_{a+}m)(x_1) \ge 0,\ 0 < \alpha < 1.$$

Proof. From (4), we have

$$(D^{\alpha}_{a+}m)(x) = x\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x(\log\frac{x}{t})^{-\alpha}\frac{m(t)}{t}dt,\ x > a.$$

Define $H(x) = \int_a^x(\log\frac{x}{t})^{-\alpha}\frac{m(t)}{t}dt$. For small $h > 0$, it is obtained that

$$\begin{aligned}H(x_1) - H(x_1-h) &= \int_a^{x_1}(\log\frac{x_1}{t})^{-\alpha}\frac{m(t)}{t}dt - \int_a^{x_1-h}(\log\frac{x_1-h}{t})^{-\alpha}\frac{m(t)}{t}dt\\ &= \int_a^{x_1-h}\left[(\log\frac{x_1}{t})^{-\alpha} - (\log\frac{x_1-h}{t})^{-\alpha}\right]\frac{m(t)}{t}dt\\ &\quad+\int_{x_1-h}^{x_1}(\log\frac{x_1}{t})^{-\alpha}\frac{m(t)}{t}dt.\end{aligned}$$

Since $\left(\log\dfrac{x_1}{t}\right)^{-\alpha}-\left(\log\dfrac{x_1-h}{t}\right)^{-\alpha}\leq 0$ and $m(t)\leq 0$ for $a<x_1-h<t<x_1$, we have

$$H(x_1)-H(x_1-h)\geq\int_{x_1-h}^{x_1}(\log\frac{x_1}{t})^{-\alpha}\frac{m(t)}{t}dt. \tag{13}$$

From (11), there exists a constant $M=k(x_1)>0$ such that

$$\left|(\log\frac{x_1}{a})^{1-\alpha}m(x_1)-(\log\frac{t}{a})^{1-\alpha}m(t)\right|\leq k(x_1)\left|\log\frac{x_1}{t}\right|^{\lambda}.$$

Since $m(x_1)=0$, from the above inequality we have

$$-k(x_1)(\log\frac{x_1}{t})^{\lambda}\leq-(\log\frac{t}{a})^{1-\alpha}m(t)\leq k(x_1)(\log\frac{x_1}{t})^{\lambda}$$

Using this inequality, we obtained that

$$\begin{aligned}H(x_1)-H(x_1-h)&\geq -k(x_1)\int_{x_1-h}^{x_1}(\log\frac{x_1}{t})^{-\alpha+\lambda}(\log\frac{t}{a})^{\alpha-1}\frac{dt}{t}\\&\geq -k(x_1)\left(\log\frac{x_1-h}{a}\right)^{\alpha-1}\int_{x_1-h}^{x_1}(\log\frac{x_1}{t})^{-\alpha+\lambda}\frac{dt}{t}\\&= \frac{-k(x_1)}{1-\alpha+\lambda}\left(\log\frac{x_1-h}{a}\right)^{\alpha-1}\left(\log\frac{x_1}{x_1-h}\right)^{1-\alpha+\lambda}.\end{aligned} \tag{14}$$

For sufficiently small $h>0$, from (14), we have

$$\frac{H(x_1)-H(x_1-h)}{h}+\frac{k(x_1)}{1-\alpha+\lambda}\left(\log\frac{x_1-h}{a}\right)^{\alpha-1}\frac{1}{h}\left(\log\frac{x_1}{x_1-h}\right)^{1-\alpha+\lambda}\geq 0.$$

Letting $h\to 0$, we get

$$\frac{d}{dx}H(x)\mid_{x=x_1}\geq 0.$$

So, since $x>0$ and $1-\alpha>0$, from (4) we have

$$(D^{\alpha}_{a^+}m)(x_1)\geq 0.$$

Hence the proof is completed.

The following theorem is a generalization of Theorem 2.3.1 in [9].

Theorem 2. Assume that $v,w\in C_{1-\alpha,\log}[a,b],\ 0<\alpha<1$, satisfy the following conditions:
(i) For $0<\alpha<\lambda<1$ and $M_1,M_2>0$

$$\left|(\log\frac{x}{a})^{1-\alpha}v(x)-(\log\frac{y}{a})^{1-\alpha}v(y)\right|\leq M_1\left|\log\frac{x}{y}\right|^{\lambda} \tag{15}$$

$$\left|(\log\frac{x}{a})^{1-\alpha}w(x)-(\log\frac{y}{a})^{1-\alpha}w(y)\right|\leq M_2\left|\log\frac{x}{y}\right|^{\lambda} \tag{16}$$

(ii) For $f\in C_{\gamma,\log}[a,b]$, $0\leq\gamma<1$, $\gamma\geq 1-\alpha$

$$(D^{\alpha}_{a^+}v)(x)\leq f(x,v(x)) \tag{17}$$

$$(D^{\alpha}_{a^+}w)(x)\geq f(x,w(x)) \tag{18}$$

one of the inequalities (17) or (18) being strict. Then

$$v_0<w_0$$

implies

$$v(x) < w(x),\ a < x \leq b, \tag{19}$$

where $v_0 = (J_{a^+}^{1-\alpha} v)(x) \mid_{x=a}, w_0 = (J_{a^+}^{1-\alpha} w)(x) \mid_{x=a}$.

Proof. Suppose that $v_0 < w_0$ be satisfied and the conclusion (19) is not true. Then, from the definition of v_0, w_0 and continuity of $\Gamma(\alpha)v(x)(\log \frac{x}{a})^{1-\alpha}$ and $\Gamma(\alpha)w(x)(\log \frac{x}{a})^{1-\alpha}$, there exists x_1 such that, for $a < x_1 \leq b$

$$v(x_1) = w(x_1) \text{ and } v(x) < w(x),\ a < x < x_1.$$

Setting $m(x) = v(x) - w(x)$, $a < x \leq x_1$, we find that $m(x) < 0,\ a < x < x_1$ and $m(x_1) = 0$. Moreover, it is clear that $m(x) \in C_{1-\alpha,\log}[a, b]$ satisfy (11). Then by Lemma 3, we get $(D_{a^+}^{\alpha} m)(x_1) \geq 0$.
Let us suppose that the inequality (18) is strict. Then from (17), we have

$$f(x_1, v(x_1)) \geq (D_{a^+}^{\alpha} v)(x_1) \geq (D_{a^+}^{\alpha} w)(x_1) > f(x_1, w(x_1)).$$

This is a contradiction since $v(x_1) = w(x_1)$. If inequality (17) is strict, then we obtain similar contradiction. Hence the conclusion (19) is valid and the proof is complete.

The following lemma is the main tool for the proof of main results.

Lemma 4. Assume that the conditions of Theorem 2 hold with nonstrict inequalities (17) and (18). Suppose further that f satisfies the following Lipschitz condition

$$f(x, u_1) - f(x, u_2) \leq L(u_1 - u_2),\ u_1 \geq u_2,\ L > 0. \tag{20}$$

Then

$$v_0 \leq w_0$$

implies

$$v(x) \leq w(x),\ a < x \leq b, \tag{21}$$

where $v_0 = (J_{a^+}^{1-\alpha} v)(x) \mid_{x=a},\ w_0 = (J_{a^+}^{1-\alpha} w)(x) \mid_{x=a}$.

Proof. For small $\varepsilon > 0$ let the function w_ε is defined as

$$w_\varepsilon(x) = w(x) + \varepsilon\lambda(x),$$

where

$$\lambda(x) = (\log \frac{x}{a})^{\alpha-1} E_{\alpha,\alpha}\left[2L(\log \frac{x}{a})^{\alpha}\right]. \tag{22}$$

Using Lemma 1, from the definition of $w_\varepsilon(x)$ we get

$$w_{\varepsilon_0} = w_0 + \varepsilon\lambda_0, \tag{23}$$

where $w_{\varepsilon_0} = (J_{a^+}^{1-\alpha} w_\varepsilon)(x) \mid_{x=a}$, $w_0 = (J_{a^+}^{1-\alpha} w)(x) \mid_{x=a}$, and $\lambda_0 = (J_{a^+}^{1-\alpha} \lambda)(x) \mid_{x=a}$.
On the other hand, taking into account (6), (22), and Lemma 1 it is obtained that $\lambda_0 = 1$. So, since $v_0 \leq w_0$, from (23) we have $w_{\varepsilon_0} > w_0 \geq v_0$.
Now, using (18) and (20) we get

$$\begin{aligned} (D_{a^+}^{\alpha} w_\varepsilon)(x) &= (D_{a^+}^{\alpha} w)(x) + \varepsilon(D_{a^+}^{\alpha} \lambda)(x) \\ &\geq f(x, w(x)) + \varepsilon(D_{a^+}^{\alpha} \lambda)(x) \\ &\geq f(x, w_\varepsilon(x)) - \varepsilon L\lambda(x) + \varepsilon(D_{a^+}^{\alpha} \lambda)(x). \end{aligned} \tag{24}$$

On the other hand, from Lemma 2, it is clear that

$$\begin{aligned} (D_{a^+}^{\alpha} \lambda)(x) &= 2L(\log \frac{x}{a})^{\alpha-1} E_{\alpha,\alpha}(2L(\log \frac{x}{a})^{\alpha}) \\ &= 2L\lambda(x). \end{aligned} \tag{25}$$

Substituting (25) into (24), we obtain that

$$\begin{aligned}(D^{\alpha}_{a^+}w_{\varepsilon})(x) &\geq f(x,w_{\varepsilon}(x)) - \varepsilon L\lambda(x) + \varepsilon 2L\lambda(x)\\ &= f(x,w_{\varepsilon}(x)) + \varepsilon L\lambda(x)\\ &> f(x,w_{\varepsilon}(x)).\end{aligned}$$

It can be shown that $w_{\epsilon} \in C_{1-\alpha,\log}[a,b]$ satisfies (16). So by Theorem 2, for any $\varepsilon > 0$

$$v(x) < w_{\varepsilon}(x),\ a < x \leq b.$$

Hence, taking $\varepsilon \to 0$ on both sides of this inequality, we have

$$v(x) \leq w(x),\ a < x \leq b,$$

which completes the proof.

The following theorems give the sufficient conditions for global attractivity of the zero solution of (1).

Theorem 3. Let $y_{eq} = 0$ be an equilibrium point of equation (1). Let $V(x,y(x)) \in C_{1-\alpha,\log}[a,b]$ satisfies the Lipschitz condition (20) and following conditions:

$$k_1 \|y\|^{d} \leq V(x,y(x)) \leq k_2 \|y\|^{dc} \tag{26}$$

$$(D^{\alpha}_{a^+}V)(x) \leq -k_3 \|y\|^{dc}, \tag{27}$$

where $\alpha \in (0,1)$ and k_1, k_2, k_3, c and d are arbitrary positive constants. Then, $y_{eq} = 0$ is globally attractive.

Proof. From (26) and (27) we get

$$(D^{\alpha}_{a^+}V)(x) \leq -\frac{k_3}{k_2}V(x,y(x)). \tag{28}$$

By Theorem 1, the initial value problem

$$\begin{gathered}(D^{\alpha}_{a^+}V)(x) + \frac{k_3}{k_2}V(x,y(x)) = 0,\\ (J^{1-\alpha}_{a^+}V)(x)\mid_{x=a} = V_0,\end{gathered} \tag{29}$$

has a unique solution

$$V(x,y(x)) = V_0(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{k_3}{k_2}(\log\frac{x}{a})^{\alpha}\right]. \tag{30}$$

Taking into account Lemma 4, (28), (29), and (30) we obtain that

$$V(x,y(x)) \leq V_0(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{k_3}{k_2}(\log\frac{x}{a})^{\alpha}\right]. \tag{31}$$

Substituting (31) into (26), we get

$$\|y(x)\| \leq \left(\frac{V_0}{k_1}(\log\frac{x}{a})^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{k_3}{k_2}(\log\frac{x}{a})^{\alpha}\right]\right)^{\frac{1}{d}}.$$

Since

$$E_{\alpha,\alpha}\left[-\frac{k_3}{k_2}(\log\frac{x}{a})^{\alpha}\right] \to 0 \ \text{ as } x \to \infty,$$

the proof is completed.

Theorem 4. Let $y_{eq} = 0$ be an equilibrium point of the equation (1). Let $V(x, y(x)) \in C_{1-\alpha,\log}[a, b]$ satisfies the Lipschitz condition (20) and θ be a class-K function satisfies

$$V(x, y(x)) \geq \theta(\|y\|), \tag{32}$$

$$(D^{\alpha}_{a^+}V)(x) \leq 0, \tag{33}$$

where $\alpha \in (0, 1)$. Then, $y_{eq} = 0$ is globally attractive.

Proof. The proof is similar to proof of Theorem 3. From Theorem 1, the linear fractional differential equation

$$(D^{\alpha}_{a^+}V)(x) = 0 \tag{34}$$

with the initial condition $(J^{1-\alpha}_{a^+}V)(x) |_{x=a}= V_0$ has a unique solution

$$V(x, y(x)) = \frac{V_0}{\Gamma(\alpha)}(\log\frac{x}{a})^{\alpha-1}.$$

Taking into account Lemma 4, (33) and (34), we obtain that

$$V(x, y(x)) \leq \frac{V_0}{\Gamma(\alpha)}(\log\frac{x}{a})^{\alpha-1}. \tag{35}$$

Substituting (35) into (32), we get

$$\|y(x)\| \leq \theta^{-1}(\frac{V_0}{\Gamma(\alpha)}(\log\frac{x}{a})^{\alpha-1}).$$

Since $0 < \alpha < 1$ and θ is a class-K function, we have

$$\|y(x)\| \to 0 \text{ as } x \to \infty,$$

which completes the proof.

References

[1] N. Aguila-Camacho, M. A. Duarte-Mermoud, J. A. Gallegos, Lyapunov functions for fractional order systems. Commun Nonlinear Sci. Numer. Simul., 19(9), 2951-2957, 2014.

[2] E. Ahmed, A. S. Elgazzar, On fractional order differential equations model for nonlocal epidemics. Physica A:Statistical Mechanics and Its Appl., 379(2), 607-614, 2007.

[3] E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models. J. Math. Anal. Appl., 325(1), 542-553, 2007.

[4] A. Ali, F. Rabiei, K. Shah, On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions. J. Nonlinear Sci. Appl., 10, 4760-4777, (2017).

[5] F. Chen, J. J. Nieto, Y. Zhou, Global attractivity for nonlinear fractional differential equations. Nonlinear Anal. Real World Appl., 13(1), 287-298, 2012.

[6] K. Diethelm, The Analysis of Fractional Differential Equations. Springer-Verlag, Berlin, 2010.

[7] J. D. Gabano, T. Poinot, Fractional modelling and identification of thermal systems. Signal Processing, 91(3), 531-541, 2011.

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam, 2006.

[9] V. Lakshmikantham, S. Leela, D. J. Vasundhara, Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge UK, 2009.

[10] Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. Comput. Math. Appl., 59, 1810-1821, 2010.

[11] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. John Willey & Sons Inc., New York, 1993.

[12] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$. C.R. Acad. Sci. Paris, 137, 554-558, 1903.
[13] K. B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press, San Diego, 1974.
[14] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.
[15] J. Sabatier, M. Aoun, A. Oustaloup, G. Grégoire, F. Ragot, P. Roy, Fractional system identification for lead acid battery state of charge estimation. Signal Processing, 86(10), 2645-2657, 2006.
[16] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives; Theory and Applications. Gordon and Breach Science Publishers, Amsterdam, 1993.
[17] M. Zamani, M. Karimi-Ghartemani, N. Sadati, M. Parniani, Design of a fractional order PID controller for an AVR using particle swarm optimization. Control Engineering Practice, 17, 1380-1387, 2009.
[18] F. Zhang, C. Li, Y. Chen, Asymptotical stability of nonlinear fractional differential system with Caputo Derivative. International Journal of Differential Equations, Volume 2011, Article ID 635165, 12 pages, doi: 10.1155/2011/635165.

Asli B. Özaydin
Ankara University, Faculty of Sciences, Department of Mathematics, Turkey

Fatma Karakoç
Ankara University, Faculty of Sciences, Department of Mathematics, Turkey
E-mail address: fkarakoc@ankara.edu.tr